Notes for 3 Mar (Friday)

1 Recap

1. Proved that monotonic functions have only jump discontinuities and that it implied that monotonic functions have at most countably many discontinuities (but they can occur on any countable set).
2. Defined differentiability and proved the usual rules for taking derivatives.
3. Proved the mean value theorems. (The most general being Cauchy's.)

2 Continuity of derivatives

Here is a cool result about derivatives. They satisfy the intermediate value property. In particular, they cannot have jump discontinuities.

Theorem 1. *Suppose $f : (a, b) \to \mathbb{R}$ is a differentiable function which is continuous on $[a, b]$. Let $f^{'}(a) < \lambda < f^{'}(b)$. Then there is a point $x \in (a, b)$ such that $f^{'}(x) = \lambda$.*

Proof. Let $g(x) = f(t) - \lambda t$. Since g is continuous on $[a, b]$ it attains its maximum somewhere. It cannot the maximum on a or b because $g^{'}(a) < 0$ and $g^{'}(b) > 0$. Therefore it attains it on $x \in (a, b)$. Therefore $g^{'}(x) = 0$. □

3 L'Hospital's rule

Basically we know that in order to evaluate limits of the form $0/0$ or ∞/∞, we replace the limit by the ratio of the limit of the derivatives. But we need a more accurate statement.

Theorem 2. *Suppose f and g are real and differentiable in (a, b), $g(x) \neq 0$ on (a, b), and $g^{'}(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$. Suppose A is an* extended *real number such that*

$$\lim_{x \to a^+} \frac{f^{'}(x)}{g^{'}(x)} \to A.$$

If $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$ or if $\lim_{x\to a^+} g(x) = \infty$ then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = A$.

Proof. We will prove the theorem when a and L are finite. The general case is not much different. (See Rudin.)
Note that (exercise) saying $\lim_{x\to a^+} f(x) = L$ is equivalent to saying that for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $a < x < a + \delta$ implies that $|f(x) - L| < \epsilon$. Similar statements hold even if L and a are extended reals.
Suppose $x_n \to a$ and $x_n > a$. We will prove everything for this sequence. (Since the sequence is arbitrary, we will be done.)
Choose an N such that $|x_n - a| < \frac{\delta}{2}$ for all $n > N$. Now for any $m, l > N$, we have $|x_m - x_l| < \delta$ by the triangle inequality. Let us choose δ to be so small that $|\frac{f^{'}(y)}{g^{'}(y)} - A| < \epsilon$ for all $0 < y - a < \delta$. Now by Cauchy's MVT

$$\frac{f(x_m) - f(x_l)}{g(x_m) - g(x_l)} = \frac{f^{'}(\theta)}{g^{'}(\theta)} \tag{1}$$

Therefore

$$A - \epsilon \leq \frac{f(x_m) - f(x_l)}{g(x_m) - g(x_l)} \leq A + \epsilon \tag{2}$$

There are two cases now :

1. $g(x_l) \to 0$ and $f(x_l) \to 0$
   In this case simply take the limit as $l \to \infty$ on both sides. (Note that because $g^{'}(x) \neq 0$, $g(x_m) \neq g(x_l)$.) Then we will get

$$A - \epsilon \leq \frac{f(x_m)}{g(x_m)} \leq A + \epsilon \tag{3}$$

   Now fix ϵ and take lim sup and lim inf to see that $A - \epsilon \leq \liminf \frac{f(x_m)}{g(x_m)} \leq \limsup \frac{f(x_m)}{g(x_m)} \leq A + \epsilon$. Since this is true for all $\epsilon > 0$, we are done.

2. $g(x_l) \to \infty$. Suppose L is so large that $g(x_l) > g(x_m)$ for all $l > L$. Therefore

$$(A - \epsilon)\frac{g(x_l) - g(x_m)}{g(x_l)} \leq \frac{f(x_l) - f(x_m)}{g(x_l)} \leq (A + \epsilon)\frac{g(x_l) - g(x_m)}{g(x_l)} \tag{4}$$

$$\Rightarrow (A - \epsilon)\frac{g(x_l) - g(x_m) + f(x_m)}{g(x_l)} \leq \frac{f(x_l)}{g(x_l)} \leq (A + \epsilon)\frac{g(x_l) - g(x_m) + f(x_m)}{g(x_l)} \tag{5}$$

   Taking lim sup and lim inf

$$(A - \epsilon) \leq \liminf \frac{f(x_l)}{g(x_l)} \leq \limsup \frac{f(x_l)}{g(x_l)} \leq (A + \epsilon) \tag{6}$$

   Taking $\epsilon \to 0$ we get the result.

□

A similar statement holds for a limit at b.
Here are some counterexamples where L'Hospital cannot be applied.

1. $\lim_{x\to\infty} \frac{x+\sin(x)}{x}$. The limit (by the limit laws and the squeeze rule) is 1. But if we naively apply L'Hospital we will conclude that it ought to be $\lim_{x\to\infty}(1+\cos(x))$ which does not exist. (The point is that the limit of the ratio of the derivatives must exist.)

2. $\lim_{x\to 0} \frac{x^2\sin(x^{-4})}{x}$. The limit by the squeeze rule is 0. If we apply L'Hospital, then we will say that it is equal to $\lim_{x\to 0}(2x\sin(x^{-4}) - 4x^{-3}\cos(x^{-4}))$ which does not exist.

Here is a nice corollary of L'Hospital :
If f is continuous at a, and $f^{'}$ exists in some interval containing a except perhaps a itself. Suppose $\lim_{x\to a} f^{'}(x)$ exists, then $f^{'}(a)$ exists and equals the limit.

Proof. Let $h(x) = \frac{f(x)-f(a)}{x-a}$. Consider $h(a+)$ and $h(a-)$. Both of them can be evaluated using L'Hospital to be $\lim_{x\to a} f^{'}(x)$. Thus $f^{'}(a) = \lim_{x\to a} f^{'}(x)$. □

4 Higher order derivatives and Taylor's theorem

If f is defined on an interval containing x, then we can ask whether $f^{'}$ exists. Inductively, if $f^{(n)}$ is defined on an interval, then we ask whether the next higher derivative $f^{(n+1)}$ exists. The point of higher derivatives is Taylor's theorem.

Theorem 3. *Suppose $f : [a,b] \to \mathbb{R}$ is continuous such that f^{n-1} is also continuous on (a,b) and $f^n(t)$ exists for every $t \in (a,b)$. Let α and β be distinct points of (a,b). Define $P(x) = \displaystyle\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k$. Then there exists a point $c \in [\alpha,\beta]$ such that $f(\beta) = P(\beta) + \frac{f^{(n)}(c)}{n!}(\beta-\alpha)^n$.*

Proof. Firstly, notice that we may without loss of generality assume that $\alpha = 0$. Indeed, if we manage to prove for $\alpha = 0$, then consider the function $g(x) = f(x+\alpha)$ and apply Taylor's theorem to this function to recover the original statement.
Suppose we manage to prove Taylor assuming that $f^{(k)}(0) = 0$ for $0 \le k \le n-1$, we will be done by simply considering $g(x) = f(x) - P(x)$ and applying the result.
So we just have to prove that $f(x) = \frac{f^{(n)}(c)}{n!}x^n$ for some $0 \le c \le x$ assuming the first $n-1$ derivatives (along with the function itself) vanish at 0. Consider $g(t) = f(t) - \frac{f(x)}{x^n}t^n$. Note that if prove that $g^{(n)}(c) = f^{(n)}(c) - n!\frac{f(x)}{x^n} = 0$ for some n then we will be done. Indeed, since $g(0) = 0 = g(x)$ there is a $c_1 \in [0,x]$ such that $g^{'}(c_1) = 0$. Note that $g^{'} = f^{'} - \frac{f(x)}{x^n}nt^{n-1}$. Since $g^{'}(0) = 0 = g^{'}(c_1)$ there is a c_2 such that $g^{''}(c_2) = 0$. Likewise, we can proceed to show that $g^{(n)}(c) = 0$ for some $c \in [0,x]$. □