Notes for 4th Jan (Wednesday)

1 Logistics

Please check the webpage (`http://www.math.iisc.ac.in/~vamsipingali/um204.html`) **regularly**. Check it like your life depends on it. The homeworks, exam dates, UP election results, and everything else will be on the darned webpage!

There are two teaching assistants for this course - S. Aiyyapan, and Nimisha Pahuja. If you have questions/doubts, first ask them, and then come me (after emailing me that is).

The grading policy is as follows :

1. HW - 10 %. There will be a homework due fortnightly. Please hand it in on Friday in the lecture to me (or to one of the teaching assistants). Late HWs are not acceptable.

2. Midterm - 30 %. The syllabus for the Midterm is everything we will have done till then except the Zermelo-Fraenkel axioms of set theory.

3. Final - 60% : The date and place will be announced later. The syllabus is everything we will have covered in this course. (From the beginning to the end except the Zermelo-Fraenkel axioms of set theory.)

The main textbook that we will follows is "Principles of Mathematical Analysis" by Walter Rudin (Third edition). It is fondly referred to as "Baby Rudin". Another very well-written text (but a bit verbose) is the 2 volume series "Analysis" by Terence Tao.

2 Trailer

Analysis is "Calculus done right". Back when Newton and Leibniz came up with calculus, they were quite informal. Later on, Cauchy et al realised that one has to be rigorous in order to avoid "paradoxes". So starting from the very definition of real numbers, they built the subject up. This is what we will do in this class. So many of the concepts might be familiar to you in varying levels of rigour. But, do not be complacent ! the subject can get hairy in some places.

At the end of this class I hope to teach you enough so as to appreciate the following.

1. What is a real number ? (What does this question even mean ? In terms of what ? What are we assumed to "know"?)

2. Are there as many rationals as irrationals ?

3. Can you come up with a number that is not the root of any polynomial with rational coefficients ?
4. What goes wrong when you apply L'Hôpital's rule to $\lim_{x \to 0} \frac{x^2 \sin(x^{-4})}{x}$?
5. Take an equilateral triangle of unit sides. Divide each side into three equal parts, throw the middle part out, and replace it with two sides of an equilateral triangle. Rinse and repeat. The area remains bounded whereas the length of the sides goes to infinity!

and more......

3 What is at the bottom of mathematics ?

Mathematics is a human construct. Clearly, one needs some "real world" element to even start talking about mathematics (either a human or a computer plugged into a socket). What lies at the interface of the real world and mathematics ? In other words, where must we begin ?

One might think that set theory is at the bottom of mathematics. After all, in high school we were plagued with nonsense like Venn Diagrams and so on. All that better have a point! Speaking of high school,, back then, everything was a set. The phrase "Set of all sets" wouldn't have raised eyebrows. But the philosopher Bertrand Russell came up with the following famous paradox (the "barber paradox") -

If there is only one barber in London who shaves all those and only those who do not shave themselves, then who shaves the barber? (In India the question is easy - Not everyone needs to be shaved! But we are talking of prissy London here.) If the barber shaves himself, he violates his constraint. If he does not shave himself, then by definition he has to shave himself!

More formally, consider the set A defined as the set of all sets that do not contain themselves as elements. This set cannot exist! The point is that not everything defined by an English sentence should be considered as a set. In other words, we need to have axioms that tells how to construct sets from "standard", "God-given" sets.

You might (rightly) have the following objection. If you want to axiomatize set theory itself, then you have to use phrases like "for all sets that something blah, there exists blah". But "for all" and "there exists" are themselves part of logic. But should we axiomatise logic itself ? If so, in terms of what ? (Normally when you do a course on logic, they use words like "functions" and "sets".) Don't we have a danger of circularity ? We do but can we mark clearly the interface between the real world and mathematics ?

The way out is the following. We (humans or computers) are assumed to be able to write "finite" strings from a "finite" alphabet consisting of the following symbols :
$\forall$, $\exists$, $\Rightarrow$, $\Leftarrow$, $\Leftrightarrow$, parantheses, comma, the constant ϕ (empty set), the binary predicates/relationships $=$ and $\in$, and an infinite collection of variables $x_1, x_2, \ldots$ (You can instead use $|, ||, ||| \ldots$ if you are uncomfortable with using numbers $1, 2, \ldots$)
whilst "following" some "rules" (the capabilies assumed are pattern matching, substitution into formulas, and recursion). Notice that already we are assuming that we understand what the terms in quotes mean. So we are already assuming a little bit of natural

numbers as "God-given" (as Kronecker would have said).

There are two kinds of strings we can write - Terms (i.e. just writing variables and constants) and formulas (things that have a truth value). There are some rules of deduction (like $P \Rightarrow Q$) that allow us to come up with new formulas from old ones. We need some formulae to begin with in order to find new ones. These formulae are the "axioms" of set theory. Now one can write every mathematical theorem as a formula in this language. (Read Yuri Mannin's book or ask Siddhartha Gadgil for further details.)

In other words, the capabilities of a computer language like C++ is enough to do mathematics. You can write all the axioms of set theory (and logic itself) in C++. When you run the program, you connect the abstraction to real life.

Rather than going in this formal route, we will follow Terence Tao (and Paul Halmos's book "Naive set theory") and simply write the axioms of set theory in simple English with the understanding that we know "intuitive logic". The axioms in Tao's book are somewhat redundant but convenient nonetheless. The axioms are called "Zermelo-Fraenkel" axioms (ZF). There is a controversial axiom called axiom of choice that every self-respecting mathematician believes. So the axioms will be ZFC (Zermelo-Fraenkel with choice).

4 Naive set theory done right - ZFC

"Definition" of a set - A set A is any "unordered collection" of objects. A "subset" A of B is a set all of whose elements are elements of B. Of course, all of this is too vague and dangerous. Basically, we will not define sets. Rather we will say that whatever sets are, they follow the following axioms and you can construct new sets from old ones in the such and such ways. (So already, the phrase "Set of all sets" should raise eyebrows.) Here are the "axioms" from Terence Tao's book :

1. If A is a set, then A is a valid object. In particular, given another set B you are allowed to ask whether A is an *element* of B. In fact, unlike Terence Tao (and like Paul Halmos) as far as we are concerned, *every* object is a set. (For example, the number 0 will be considered to be the empty set ϕ, the number one as the set containing the empty set $\{\phi\}$ and so on.)

2. Equality : Two sets are equal if and only if they contain the same elements (objects will be called elements from now onwards). (So order does not matter.)

3. Empty set : So far, we have no sets yet. So we postulate that there exists a set ϕ that does not contain any elements. (Of course by the previous axiom this set is unique.)
   By the way, this means (by trivial logic) that every non-empty set has some element x, i.e., we can "choose" an element from a non-empty set. In fact, we can prove that one can "choose" elements from *finitely* many non-empty sets (we need induction to do this). But can we "choose" elements from each set of a family of infinitely many non-empty sets ? That is the axiom of choice that we will discuss later on.

4. Pairing : If A and B are sets, there exists a set containing only A and B as elements.

5. Union : For any set (of sets because every object for us is a set anyway) $\mathcal{F}$ there exists a set containing every element of some set in $\mathcal{F}$. (The union of an arbitrary number of sets exists.)

6. Regularity : Every non-empty set X contains an element y such that X and y are disjoint. (So a set cannot be a member of itself.)

7. Specification : Ideally, we would like to construct a set by saying "X is a set consisting of things that satisfy some property". This axiom makes this "set builder" way of doing things precise.
   If A is a set, and $P(x)$ is a "property" that may be true or false for an element x in A, then there exists a subset X of A consisting of $x \in A$ such that $P(x)$ is true. Note that this means that we can only construct *subsets* of already existing sets this way. So no Russell paradox can occur.
   By the way, this axiom allows us to define the intersection of two sets and complement of a set. So we can do our usual De Morgan laws and everything we learnt in high school using Venn diagrams. (They are all *theorems* that can be proven using these axioms.)

8. Replacement : So far we cannot take a set and "transform" it into a new set using a "definable function". This axiom takes care of that.
   Suppose A is a set, and for any element $x \in A$ and an object (i.e. set) y there is a statement $P(x, y)$ such that for each $x \in A$, there is at most one y for which $P(x, y)$ is true. Then there exists a set B consisting of all y such that $P(x, y)$ is true. We abbreviate such y as $f(x)$.

9. Infinity ("To see a world in a grain of sand and heaven in a wild flower; To hold infinity in the palm of a hand and eternity in an hour" - Blake) : So far, we have the empty set ϕ, using pairing we can construct a set containing the empty set $\{\phi\}$, using it again, $\{\phi, \{\phi\}\}$, and so on. But these are only "finite" sets. We need a set containing all of these sets. (It will be a set containing the natural numbers, secretly speaking.)
   There exists a set S containing the empty set as an element, such that if $y \in S$, then $y \cup \{y\}$ is also a member of S. This clearly contains all natural numbers but how does one "extract" the set of natural numbers from this ? For that we need axioms of natural numbers. These axioms will be described later. (Peano's axioms.)

10. Power set : Given a set X, there exists a set $P(X)$ containing exactly all the subsets of X.

11. Axiom of choice : Let $\mathcal{F}$ be a set (of sets as always) containing an arbitrary number of elements each of which is a non-empty set. Let $U = \cup_i X_i$ be the union of all elements X_i of $\mathcal{F}$. Then there exists a "choice" function $f : \mathcal{F} \to U$ such that $f(i)$ is an element of X_i.

You can easily construct cartesian products of sets using the above axioms. Thus we can talk of relations and functions between sets.

5 Natural numbers and Peano axioms

For us, natural numbers will contain 0. We want to define natural numbers as that subset of the infinite set from ZFC containing $0 = \phi, 1 = \{\phi\} \ldots$. For this we need some axioms (regarding a "successor" function that we will denote as ++ motivated from $C++$). These axioms are called Peano's axioms. So finally the set of natural numbers will be the subset of the infinite set of ZFC containing all successors of $0 = \phi$. Here are Peano's axioms for a set $\mathbb{N}$ along with a function $++ : \mathbb{N} \to \mathbb{N}$.

1. 0 is a natural number, i.e., $\mathbb{N}$ is not empty.

2. $0 \neq n++$ for any natural number n (i.e. naturals do not loop around). (So 4 is not equal to 0 for instance.)

3. If $n++ = m++$ then $n = m$. This means that there is no "ceiling" for natural numbers.

4. Induction principle : Let $P(n)$ be a property pertaining to a natural number n. Suppose $P(0)$ is true, and suppose that whenever $P(n)$ is true, so is $P(n++)$. Then $P(n)$ is true for every natural number n.