Notes for 5 April (Wednesday)

1 Recap

1. Proved the Weierstrass approximation theorem by approximating the Dirac delta by polynomials.
2. Proved a corollary that $|x|$ can be approximated by polynomials vanishing at 0.
3. Proved that power series converge uniformly within their radius of convergence, and that you can integrate and differentiate term-by-term within the radius of convergence.
4. Wanted to prove Abel's theorem : If $\sum a_n = A$, $\sum b_n = B$ and $\sum c_n = C$ where c_n is the Cauchy product of a_n and b_n, then $C = AB$. Our strategy was to write $f(x) = \sum a_n x^n$, $g(x) = \sum b_n x^n$ and $h(x) = \sum c_n x^n$ and use the theory of power series to attack this.s

2 Power series

Theorem 1. *Suppose $\sum c_n$ converges. Put $f(x) = \sum_{n=0}^{\infty} c_n x^n$ when $-1 < x < 1$. Then $\lim_{x\to 1^-} f(x) = \sum c_n$.*

Proof. We have to somehow introduce the partial sums $s_n = c_0 + c_1 \ldots c_n$ (by the way put $s_{-1} = 0$). Now $\displaystyle\sum_{n=0}^{m} c_n x^n = \sum (s_n - s_{n-1}) x^n = (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m$. Take $m \to \infty$ to get $f(x) = (1-x)\sum s_n x^n$. Let $s = \lim_{n\to\infty} s_n$. Now $|f(x) - s| = (1-x)\sum (s_n - s) x^n$. If $n > N$ then $|s_n - s| < \frac{\epsilon}{2}$. Thus $|f(x) - s| < \frac{\epsilon}{2} + (1-x)\sum_{n=0}^{N} |(s_n - s)||x|^n < \epsilon$ if $1 - \delta < x < \delta$. □

We now prove a theorem regarding interchange of the order of summation :

Theorem 2. *Given a double sequence a_{ij} suppose that $\displaystyle\sum_{j=1}^{\infty} |a_{ij}| = b_i$ and that $\sum b_i$ converges. Then $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.*

Proof. Here is an unconventional way to prove this. (By the way, if you take a course on measure theory, this theorem is a trivial consequence of the so-called dominated convergence theorem.)
Let E be a countable set consisting of $x_0, x_1, \ldots$ where $x_n \to x_0$ as $n \to \infty$. Define

$f_i(x_0) = \sum_{j=1}^{\infty} a_{ij}$ and $f_i(x_n) = \sum_{j=1}^{n} a_{ij}$. Also let $g(x) = \sum_{i=1}^{\infty} f_i(x)$ where $x \in E$. This means that each f_i is continuous at x_0. Since $|f_i(x)| \leq b_i$ and $\sum b_i$ converges, by the Weierstrass M-test, $\sum f_i$ converges uniformly to $g(x)$. But for uniform convergence, limits and sums can be interchanged ($g(x)$ is continuous at x_0), i.e.,

$$\sum_i \sum_j a_{ij} = \sum_i f_i(x_0) = g(x_0)$$

$$= \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n) = \lim_{n\to\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{n} a_{ij} = \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{i} a_{ij} = \sum_j \sum_i a_{ij}.$$

□

If two power series converge to the same function on an open interval $(-R, R)$, then are they identical ? The following theorem answers this question in the affirmative. Actually a much weaker condition is enough but we will not pursue that.

Theorem 3. *Suppose* $f(x) = \sum_{n=0}^{\infty} c_n x^n$ *converges in* $|x| < R$. *If* $-R < a < R$ *then* f *can be expanded in a power series about* $x = a$ *which converges in* $|x - a| < R - |a|$ *and* $f(x) = \sum_n \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Proof. Note that $f(x) = \sum c_n(x-a+a)^n = \sum_n \sum_{m=0}^{n} c_n \binom{n}{m} a^{n-m}(x-a)^m$ which we hope equals $\sum_{m=0}^{\infty}[\sum_{n=m}^{\infty}\binom{n}{m}c_n a^{n-m}](x-a)^m$. This hope can be justified using the previous theorem. Indeed the previous theorem shows that this is possible if $\sum_{n=0}^{\infty}\sum_{m=0}^{n}|c_n\binom{n}{m}a^{n-m}(x-a)^m|$ converges. But this is the same as $\sum_n |c_n|(|x-a|+|a|)^n$ which converges if $|x-a|+|a| < R$. The formula for the coefficients follows from differentiating term-by-term. □

3 Exponentials and Logarithms

Define $E(z) = \sum \frac{z^n}{n!}$. The ratio test shows that this converges absolutely for all $z \in \mathbb{C}$. Therefore, $E(z)E(w)$ is the Cauchy product of the series which is $\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}z^k w^{n-k} = \sum \frac{(z+w)^n}{n!}z^n = E(z+w)$. Clearly $E(0) = 1$. Let us calculate the derivative : Since it is a power series, we may differentiate term-by-term to see that indeed $E^{'}(z) = E(z)$. Also, $E(1) = e$ (the number we defined earlier as a limit). Now the multiplication formula (by induction) implies that $E(n) = e^n$. If $q = n/m$ then $(E(q))^m = E(qm) = E(n) = e^n$. Therefore for positive rational q, $E(q) = e^q$. Since $E(q)E(-q) = 1$ this holds for all rationals.

Since $E(z)E(-z) = 1$ for all z, this means that $E(z) \neq 0$ for any z. Moreover, if z is a positive real, clearly the power series is positive (it has positive terms). If z is a negative real, then of course $E(z) = 1/(E(-z))$ is positive. Therefore $E^{'} > 0$ (and $E(z) \to \infty$ as $z \to \infty$). This means that E is increasing. If we define $e^x = \sup e^p$ where $p < x$ and p is rational, then the properties of $E(z)$ (continuity and monotonicity) so far, show that $E(x) = e^x$.

Lastly, e^x changes faster than any polynomial. Indeed, L'Hospital's rule easily proves that $\lim_{n\to\infty} x^n e^{-x} = 0$ for every n and $x \in \mathbb{R}$.

Since e^x is strictly increasing (it is 1-1) and its range is $(0, \infty)$ on $\mathbb{R}$, it has an inverse function $\ln(x) : (0, \infty) \to \mathbb{R}$ which is $1-1$. If $a < b$ then $e^x : [a, b] \to [e^a, e^b]$ is continuous and $1 - 1$ on a compact set and therefore $\ln : [e^a, e^b] \to [a, b]$ is continuous for every a, b. Thus ln is a continuous function. Now if $w = \ln(uv)$, then $e^w = uv$. If $u = \ln(x)$, $v = ln(y)$, then $e^w = e^x e^y = e^{x+y}$. Thus $\ln(uv) = \ln(u) + \ln(v)$. Now we shall prove that $\ln(x)$ is differentiable on its domain and $\ln^{'} = \frac{1}{x}$. Indeed,

$$\begin{aligned} \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} &= \lim_{h\to 0} \frac{\ln(x(1+h/x)) - \ln(x)}{h} = \lim_{h\to 0} \frac{\ln(1+h/x)}{h} \\ &= \frac{1}{x} \lim_{h\to 0} \frac{\ln(1+h/x)}{h/x}. \qquad (1) \end{aligned}$$

So if we prove that the derivative exists at 1 (and the derivative is 1) then we will be done. (*Cont'd....*)