Notes for 6 April (Thursday)

1 Recap

1. Proved Abel's theorem.
2. Proved that if a double series converges absolutely, then we can interchange the order of summation.
3. Proved that if two power series agree on an interval, then they are identical.
4. Defined the exponential and proved its usual properties. Defined ln as its inverse and proved that it is continuous. Reduced the proof of differentiability to differentiability at $x = 1$.

2 Exponentials and Logarithms

(*Cont'd...*) If we prove that the derivative exists at 1 (and the derivative is 1) then we will be done. Here is a way to prove it : We shall prove that $\lim_{h\to 0^+} \ln(1+h)/h = 1$ (the left-handed limit is similar). To do so we take an arbitrary sequence $h_n > 0$ converging to 0. We shall prove that lim sup and lim inf are both 1. Indeed, let's only consider lim sup (lim inf is similar). So

$$\limsup \frac{\ln(1+h_n)}{h_n} = \lim_{k\to\infty} \frac{\ln(1+h_{n_k})}{h_{n_k}} = M$$

$$w_n = \ln(1+h_n) \Rightarrow e^{w_n} = 1 + h_n \Rightarrow \frac{w_n}{h_n}(1 + \frac{w_n^2}{2!h_n} + \ldots) = 1$$

Note that since $h_n > 0$, so is $w_n > 0$. The above equation implies that M is definitely finite. Moreover, it is easy to see (by comparison) that the infinte series part goes to 1. Thus $M = 1$.
Since $\ln^{'}(y) = \frac{1}{y}$, this means that $\ln(y) = \displaystyle\int_1^y \frac{dx}{x}$. It is also easy to see from the properties of exponentials that $\ln(u^q) = q\ln(u)$ for rational q. By continuity and monotonicity (i.e. $\sup \ln u^q = \lim \ln u^q = \ln \lim u^q = \ln \sup u^q$) we see that this is true for all reals. This means that $x^\alpha = e^{\alpha \ln(x)}$ for all real α and $x > 0$. Therefore x^α is differentiable and its derivative is $\alpha x^{\alpha-1}$. Lastly, here is one more property of logarithms : Logs change slower than any polynomial, i.e., $\lim_{x\to\infty} x^{-\alpha}\ln(x) = 0$ for every $\alpha > 0$ by L'Hospital's rule.

3 Trigonometric functions

We finally define trigonometric functions. We shall only define sin and cos and prove that they correspond to their geometric definition.
Indeed, define $C(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $S(x) = \frac{e^{ix}-e^{-ix}}{2i}$. It is easy to see that indeed C and S are real. Also, $e^{ix} = C(x) + iS(x)$. Now $e^{ix}e^{-ix} = |e^{ix}|^2 = 1$. Therefore $C^2 + S^2 = 1$. Moreover, $C^{'} = -S$ and $S^{'} = C$.

We need to prove that C and S are periodic with period 2π for some real number π. To do this, let's prove first that C vanishes at a positive number somewhere. Indeed, $C(0) = 1$. So it C does not vanish anywhere on the positive line, then $C > 0$ everywhere. But $S^{'} = C > 0$. Thus $S(x) > 0$ when $x > 0$ and < 0 when $x < 0$. This means that if $x < y$ then $2 \geq C(x) - C(y) = -\int_y^x S(t)dt > -S(x)(x-y)$. This is a contradiction when y is large.

Let x_0 be the smallest positive number such that $C(x_0) = 0$. This number exists because $C(0) \neq 0$ and the set of zeroes is closed (so the infimum is attained). Define $\pi = 2x_0$. Therefore $C(\frac{\pi}{2}) = 0$ and $S(\pi/2) = 1$. Hence by the addition formulae, $e^{\pi i}+1 = 0$ and $e^{2\pi i} = 1$. Hence $e^{z+2\pi i} = e^z$. We have the following theorem.

Theorem 1. 1. *e^z is periodic with period $2\pi i$.*

2. *$C = \cos$ and $S = \sin$ are periodic with period 2π*

3. *If $0 < t < 2\pi$ then $e^{it} \neq 1$.*

4. *If z is a complex number such that $|z| = 1$ then there is a unique $t \in [0, 2\pi)$ such that $e^{it} = z$.*

Proof. 1. We just need to show that 2π is the smallest positive number t_0 such that $e^{z+t_0 i} = e^z$. Indeed if there was a smaller t_0, then $e^{t_0 i} = 1$. Hence $e^{t_0 i/2} = \pm 1$. Thus $S(t_0/2) = 0$. But $S(t_0/2) = 2S(t_0/4)C(t_0/4)$ and therefore $C(t_0/4) = 0$ (because on $(0, \frac{\pi}{2})$ we know that S is increasing). But this is a contradiction because $\frac{\pi}{2}$ is the smallest positive zero of C.

2. Ditto.

3. Since $0 < t < \pi/2$ this means that $e^{it} = x + iy$ satisfies $0 < x < 1$ and $0 < y < 1$. Assume that $e^{it_0} = x + iy$ is a real number. This means that $e^{i4t_0} = (x+iy)^4$ is also real. But $e4it_0 = (x+iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2)$. If $0 < t < \frac{\pi}{2}$ we showed that $0 < x < 1$ and $0 < y < 1$. Thus $x^2 - y^2 = 0$. Since $x^2 + y^2 = 1$ this means that $x^2 = y^2 = \frac{1}{2}$. Thus $e^{4it_0} = -1$. Thus $e^{it_0} \neq 1$.

4. If there are two such ts then we produce a contradiction to the previous assertion. Let's prove existence of such a t. If $1 \geq x \geq 0, 1 \geq y \geq 0$ and $x^2 + y^2 = 1$, then since $\cos : [0, \frac{\pi}{2}] \to [0, 1]$ is surjective because the endpoints go to the endpoints, it is continuous, and decreasing. Likewise, so is sin surjective in that region. Thus there exists a t such that $\cos(t) = x$. Since $\sin(t)^2 + \cos^2(t) = 1$ this means that $\sin(t) = y$. For the other regions we can reduce them to this region. For example,

if $x < 0$ and $y \geq 0$ then $-iz$ is in the correct region and hence $-iz = e^{it}$. Thus $z = e^{i(t+\frac{\pi}{2})}$ and so on.

$\square$

Finally, we show that indeed the parameter t in the closed differentiable simple curve $\gamma(t) = e^{it}$ where $t \in [0, 2\pi]$ is the angle, i.e., t is the length of the arc joining $\gamma(0)$ and $\gamma(t)$. Indeed, $\gamma^{'}(t) = ie^{it}$. Therefore $|\gamma^{'}| = 1$.
Finally, the length of the hypotenuse is 1, the adjacent side is the x-coordinate of γ, namely, cos and the opposite is sin.

4 Fundamental theorem of algebra

Theorem 2. *If $P(z) = a_0 + a_1 z + \ldots a_n z^n$ where $n \geq 1$ and $a_n \neq 0$. Then $P(z)$ has a complex root if a_i are complex.*

Proof. By scaling $P(z)$ we may assume that $a_n = 1$. Put $\mu = \inf |P(z)|$. Suppose that $\mu \neq 0$. We will derive a contradiction. The intuition is that for large z, $P(z)$ is huge and hence μ is actually attained on some closed disc. This will be shown to be a problem.

If $|z| = R$, then $|P(z)| \geq R^n [1 - |a_{n-1}|R^{-1} - \ldots$. Since the right hand side goes to ∞ as $R \to \infty$ this means that $|P(z)| > \mu$ if $|z| > R_0$. Since $|P|$ is continuous on the closed disc with radius R_0, $|P|$ attains its minimum on the disc.Thus $|P(z_0)| = \mu$.

Suppose $\mu \neq 0$. Then the function $Q(z) = \frac{P(z+z_0)}{P(z_0)}$ is a nonconstant polynomial such that $Q(0) = 1$ and $|Q(z)| \geq 1$ for all z. There is an integer k such that $b_k \neq 0$ and $Q(z) = 1 + b_k z^k + \ldots b_n z^n$. By a previous theorem there exists a θ such that $e^{ik\theta} b_k = -|b_k|$. If $r > 0$ and $r^k |b_k| < 1$ then $|1 + b_k r^k e^{ik\theta}| = 1 - r^k |b_k|$. Thus $|Q(re^{i\theta})| \leq 1 - r^k(|b_k| - r|b_{k+1}| - \ldots)$. If r is small then $|Q(re^{i\theta}) < 1$. A contradiction. $\square$