Notes for 7 April, Friday

1. (Rudin chapter 3 Problem 7) Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$ if $a_n \geq 0$.

   Answer : Since $a_n \geq 0$, because monotone increasing sequences converge if and only if they are bounded, $\sum a_n$ converges if and only if $\sum_{n=1}^{N} a_n < C$ for some C and all N. Likewise, we just need to prove that $\sum_{n=1}^{M} \frac{\sqrt{a_n}}{n}$ is bounded above independent of M.
   Indeed, $\sum_{n=1}^{M} \sqrt{a_n}\frac{1}{n} \leq \sqrt{\sum a_n}\sqrt{\sum \frac{1}{n^2}}$ by the Cauchy-Schwarz inequality. The right hand side is bounded above because $\sum a_n$ is bounded above and $\sum 1/n^2$ converges by the p-series test.

2. (Rudin chapter 3 problem 9) Find the radius of convergence of each of the following power series :

   (a) $\sum n^3 z^n$

   Answer : For all of these problems, $R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{\limsup n^{3/n}} = \frac{1}{\lim n^{3/n}} = 1$. In what follows, all the lim sups turn out to be limits and hence we can use limit laws without mention or apology.

   (b) $\sum \frac{2^n}{n!} z^n$

   Answer : $R = \lim \frac{2^n}{n!}\frac{(n+1)!}{2^{n+1}} = \lim \frac{n+1}{2} = \infty$.

   (c) $\sum \frac{2^n}{n^2} z^n$

   Answer : $R = \lim \frac{2^n}{n^2}\frac{(n+1)^2}{2^{n+1}} = \frac{1}{2}$.

   (d) $\sum \frac{n^3}{3^n} z^n$.

   Answer : $R = \lim \frac{n^3}{3^n}\frac{3^{n+1}}{(n+1)^3} = \lim 3\frac{n^3}{(n+1)^3} = 3$.

3. (Rudin chapter 3 problem 12 part a)) Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{m=n}^{\infty} a_m$. Prove that

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Answer : By the way, this problem can be solved using the Cauchy-Schwarz inequality too. But we will do it Rudin's way. In what follows, all the $|.|$ signs go away because $a_n > 0$.

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \ldots + \frac{a_n}{r_m} \geq \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m} \tag{1}$$

Now $\displaystyle\sum_{k=m}^{k=n} \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m}$. If $n \to \infty$ then $r_n \to 0$ because $\sum a_n$ supposedly converges. This means that $\displaystyle\sum_{k=m}^{\infty} \frac{a_k}{r_k} > 1$ for all m which is a problem because if it is supposed to converge, then as $m \to \infty$ this is supposed to go to 0.

4. (Rudin chapter 4 problem 6) If $f : E \subset (X, d_X) \to (Y, d_Y)$ is a function, then the *graph* of f is the set of points $(x, f(x))$ in the metric space $(X \times Y, d_{X\times Y}((x, y), (a, b)) = \sqrt{d_X(x, a)^2 + d_Y(a, b)^2})$. Suppose E is a compact subset of X. Prove that f is continuous on E if and only if its graph is a compact subset of $X \times Y$.

Answer : Firstly, note that the projection maps $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are continuous because $d_X(\pi_1(x, y), \pi_1(x_0, y_0)) = d_X(x, x_0) \leq \sqrt{d_X(x, x_0)^2 + d_Y(y, y_0)^2} \leq d_{X\times Y}((x, y), (x_0, y_0))$. So taking $\delta = \epsilon$ we are done. Likewise for π_2.

(a) If f is continuous on E :
To prove that a subset of a metric space is compact, we simply need to prove that every sequence a_n has a convergent subsequence a_{n_k} (Indeed, see the remarks after theorem 2.41 in Rudin. Compactness is equivalent to saying that every infinite set has a limit point. Since every infinite set contains a sequence, and that limit points are indeed limits of subsequences, this is the case). If $(x_n, f(x_n)$ is a sequence in the graph, then since x_n is a sequence in E, it has a convergent subsequence $x_{n_k} \to x$. Thus $f(x_{n_k}) \to f(x)$ by continuity in Y. This means that for k large enough, $d_Y(f(x), f(x_{n_k})) < \frac{\epsilon}{2}$ and $d_X(x, x_{n_k}) < \frac{\epsilon}{2}$. Thus $d_{X\times Y}((x_{n_k}, f(x_{n_k})), (x, f(x)) < \sqrt{\epsilon^2/2} < \epsilon$. Hence we are done.

(b) If the graph is a compact subset of $X \times Y$:
We just have to show that $f^{-1}(C)$ is closed where C is a closed subset of Y. Now $f^{-1}(C) = \pi_1^{-1}(\pi_2^{-1}(C) \cap \mathit{Graph})$ which is closed because the graph is compact and hence closed, and π_i are continuous.

5. (Rudin chapter 4 problem 12) A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.

Answer : $f : S \subset \mathbb{R} \to \mathbb{R}$ and $g : f(S) \subset E \subset \mathbb{R} \to \mathbb{R}$ are uniformly continuous, then $g \circ f : S \to \mathbb{R}$ is uniformly continuous. Indeed, given an $\epsilon > 0$, there is a $\delta_{1\epsilon}$

(depending only on ϵ) such that $|a-b| < \delta_{1\epsilon}$ implies that $|g(a)-g(b)| < \epsilon$. Now given $\delta_{1\epsilon}$ there exists a δ_ϵ (depending only on $\delta_{1\epsilon}$ which depends only on ϵ) such that $|f(x)-f(y)| < \delta_{1\epsilon}$ whenever $|x-y| < \delta_\epsilon$. Thus $|x-y| < \delta_\epsilon$ implies that $|g(f(x)) - g(f(y))| < \epsilon$.

6. (Rudin chapter 4 problem 15) Call a mapping of X to Y open if $f(V)$ is an open subset of Y whenever V is an open subset of X.
Prove that every open continuous map from $\mathbb{R}$ to itself is monotonic.

   Answer : Suppose f is not monotonic. That means that without loss of generality we may assume that there exist $a < c < b$ such that $f(a) \leq f(b)$ and $f(b) < f(c)$. Since $f : [a,b] \to \mathbb{R}$ is continuous, by the extreme value theorem, this means that f attains a maximum at a point $x_0 \in [a,b]$. Clearly $x_0 \in (a,b)$ because of the existence of c. Now $(x_0-\epsilon, x_0+\epsilon) \subset (a,b)$ for small enough ϵ. Since f is open, $f(x_0-\epsilon, x_0+\epsilon)$ ought to be an open subset of $\mathbb{R}$ containing $f(x_0)$. But this means that small values larger than $f(x_0)$ should also be in the image of f which is a contradiction.

7. (Rudin chapter 5 problem 11, See the hint in Rudin) Suppose f is defined in a neighbourhood of x and suppose $f^{''}(x)$ exists. Show that $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f^{''}(x)$ and show by a counterexample that the limit may exist even if $f^{''}(x)$ does not.

   Answer : By assumption $f^{'}$ exists in a neighbourhood of x. By the L'Hospital rule $\lim_{h\to 0^+} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim_{h\to 0^+} \frac{f^{'}(x+h)-f^{'}(x-h)}{2h}$ provided the right hand side exists. Indeed, by definition of the derivative

$$f^{''}(x) = \lim_{h\to 0} \frac{f^{'}(x+h)-f^{'}(x)}{h} = \lim_{h\to 0} \frac{f^{'}(x)-f^{'}(x-h)}{h} \tag{2}$$

   which means that the desired limit is indeed $f^{''}$. A similar argument holds for $h \to 0^-$.

   As for the counterexample, let $g(x) = -\frac{x^2}{2}$ when $x \leq 0$ and $x^2/2$ when $x \geq 0$. Thus $g^{'}(x) = |x|$. Hence $g^{''}$ does not exist at 0 but $\lim_{h\to 0} \frac{g(h)+g(-h)}{h^2} = 0$ and hence exists.

8. (Rudin chapter 5 problem 15, See the hint in Rudin) Suppose f is a twice-differentiable function on $\mathbb{R}$, and M_0, M_1, M_2 are the suprema of $|f|, |f^{'}|, |f^{''}$ respectively on (a,∞). Prove that $M_1^2 \leq 4M_0M_2$. (Note the small correction : The strategy outlined in Rudin can only help with this problem. If you have an arbitrary a as in Rudin, it may not work.)

   Answer : Since f is twice-differentiable, Taylor's theorem shows that there is some $\theta \in (x, x+2h)$ such that $f(x+2h) = f(x) + f^{'}(x)2h + f^{''}(\theta)2h^2$. Thus $f^{'}(x) = \frac{1}{2h}(f(x+2h)-f(x)) - nf^{''}(\theta)$. This means that $|f^{'}(x)| \leq hM_2 + \frac{M_0}{h}$. Choosing h appropriately (to saturate the the AM-GM inequality), $hM_2 + M_0/h = 2\sqrt{M_2M_0}$. Thus $M_1^2 \leq 4M_0M_2$.

9. (Rudin chapter 6 problem 4) If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that f is not Riemann integrable on $[a, b]$ for any $a < b$.

Answer : Indeed, if P is any partition, then $U(P, f) - L(P, f) = \sum(M_i - m_i)\Delta x_i = \sum \Delta x_i = b - a$ (because the rationals and irrationals are dense) and hence cannot be made less an ϵ for small ϵ.

10. (Rudin chapter 6 problem 7) Suppose $f : [0, 1] \to \mathbb{R}$ is Riemann integrable on $[c, 1]$ for every $c > 0$. Define $\displaystyle\int_0^1 f dx = \lim_{c \to 0^+} \int_c^1 f dx$ if the limit exists and is finite.

    (a) If f is Riemann integrable on $[0, 1]$ show that this definition of the integral agrees with the old one.

    (b) Construct a function such that the above limit exists, although it fails to exist with $|f|$ in the place of f.

    Answer :

    (a) Indeed, if f is Riemann integrable, then $|\int_c^1 f dx - \int_0^1 f dx| = |\int_0^c f dx| \leq Mc \to 0$.

    (b) Take $f(x) = (-1)^n n$ on $[\frac{1}{(n+1)}, \frac{1}{n}]$. Thus $\displaystyle\lim_{c \to 0^+} \int_c^1 f dx = \frac{-1}{2} + \frac{1}{3} \ldots$ which we know converges by the alternating series test. But $\displaystyle\int |f| dx$ is the Harmonic series which diverges.

11. (Rudin chapter 7 problem 6) Prove that $f_n(x) = \displaystyle\sum_{k=1}^{n} (-1)^k \frac{x^2 + k}{k^2}$ converges uniformly in every bounded interval but does not converge absolutely for any value of x.

    Answer : Suppose $x \in [a, b]$. Then $|f_n(x) - f_m(x)| = |\displaystyle\sum_{k=m+1}^{n} (-1)^k \frac{x^2 + k}{k^2}| \leq b^2 \sum_{m+1}^{n} \frac{1}{k^2} + |\sum \frac{(-1)^k}{k}| < \epsilon$ when $n, m > N$ because $\sum b^2 \frac{1}{k^2}$ and $\sum \frac{(-1)^k}{k}$ converge and hence are Cauchy. Thus f_n converges uniformly. If f_n converged absolutely, then so would have $\sum \frac{x^2+k}{k^2} - \sum \frac{x^2}{k^2} = \sum \frac{1}{k}$ which is a contradiction.

12. (Rudin chapter 7 problem 13 part a)) Assume that f_n is a sequence of monotonically increasing functions from $\mathbb{R} \to [0, 1]$. Prove that there is a function f and a sequence n_k such that $f(x) = \lim_{k \to \infty} f_{n_k}(x)$ for every $x \in \mathbb{R}$.

    Answer : Consider the countable set of all rationals. By a theorem in the class, f_n has a subsequence f_{n_k} converging pointwise to a function f on all rationals. Define $f(x) = \sup_{q \leq x} f(q)$. My claim is that f is increasing, i.e., $f(x) \leq f(y)$ when $x \leq y$. Note that $f_{n_k}(x) \leq f_{n_k}(y)$ for all k. Taking limits, this shows that the statement is trivial for rational x and y. Now suppose $q_1 \leq x \leq q_2 \leq y$

are such that $f(q_1) \leq f(x) < f(q_1) + \epsilon$ and $f(q_2) \leq f(y) < f(q_2) + \epsilon$. Thus $f(x) < f(q_1) + \epsilon \leq f(q_2) + \epsilon \leq f(y) + \epsilon$. Since this is true for all $\epsilon > 0$, indeed $f(x) \leq f(y)$. Thus f has at most countable many discontinuities.

Suppose x is a point where f is continuous. Then we claim that indeed $f(x) = \lim_{m\to\infty} f_{n_m}(x)$. Indeed, $f(q) \leq f(x) \leq f(q) + \epsilon$ when $q \leq x \leq q + \delta$. Thus $f_{n_m}(q) - \epsilon \leq f(x) \leq f_{n_m}(q) + 2\epsilon \leq f_{n_m}(x) + 2\epsilon$ when $m > M_q$. This means that $f(x) \leq \liminf_m f_{n_m}(x) + 2\epsilon$ for every ϵ. Thus $f(x) \leq \liminf_m f_{n_m}(x)$. Now $f(x) \geq f(q + \frac{1}{n}) - \epsilon$ if n is large keeping $q + \frac{1}{n} \geq x$. Thus $f(x) \geq f_{n_m}(q + \frac{1}{n}) - \epsilon \geq f_{n_m}(x) - \epsilon$ when m and n are large. Taking $\limsup$ on both sides, we see that $f(x) \geq \limsup f_{n_m}(x) - \epsilon$ for all ϵ. Hence $f(x) = \lim f_{n_m}(x)$.

Now on the set E consisting of countable many discontinuities of f, there is a further subsequence of f_{n_k} which we call g_m converging to a function g pointwise (where $g = f$ on rationals and places where f is continuous).