Notes for 8 Feb (Wednesday)

1 Recap

1. Did examples of lim sup and lim inf. (Use the "mechanical" formula if you don't want to think too much.)

2. Proved Cauchy's theorem about the convergence of a monotonically decreasing non-negative series. Did the p-series.

3. Defined the number e, proved the usual formula for it, and proved that it is irrational.

4. Did the Ratio and Root tests. Concluded that the Root test is more powerful (at least in principle).

2 Power series

A power series is a series of the type $\displaystyle\sum_{k=0}^{\infty} c_k z^k$ where c_k and z are complex numbers. Note that the notion of series, sequences, etc. go through for complex numbers as well. Also, the complex numbers are complete. (After all they are just $\mathbb{R}^2$ jazzed up with multiplication.) So it is easy to prove that if $\displaystyle\sum_{k=0}^{\infty} |a_k|$ converges then so does $\sum_{k=0}^{\infty} a_k$. Thus we can use our machinery of ratio and root tests for power series.

Indeed, let $L = \limsup |c_n|^{1/n}|z|$. By the root test, if $L < 1$ the power series converges, i.e., if $|z| < \frac{1}{\limsup |c_n|^{1/n}}$. In other words, there is a number $R = \frac{1}{\limsup |c_n|^{1/n}}$ called the radius of convergence such that if $|z| < R$ the series converges, and outside the disc it diverges. On the circle $|z| = R$ we have no idea.

1. The series $\sum \frac{z^n}{n!}$ converges for all z by the ratio test.

2. $\sum \frac{z^n}{n^2}$ has radius of convergence 1. Actually the series converges on the circle of convergence too because $\sum \frac{1}{n^2}$ does.

3. $\sum \frac{z^n}{n}$ converges on the open unit disc and diverges outside. It diverges when $z = 1$ but on all other points of the unit circle it converges. This will be proved later on.

3 Absolute convergence, Summation-by-parts

As discussed before, a series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. As proven earlier, absolute convergence implies convergence in the usual sense. The converse is NOT true. The classic example of such a "conditionally" convergent series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$. If you replace everything with its absolute value you get the Harmonic series. However, as given, the series converges (to $\ln(2)$ actually). We will see later on that if a series is not absolutely convergent, then weird weird things can happen. (Riemann's rearrangement theorem.)

We need a technique to handle non absolutely convergent series. Viewing a_n as $|a_n|$ multiplied with ± 1, we need a way to estimate $\sum a_n b_n$ in terms of the individual series. One such technique is called "summation by parts" (akin to integration-by-parts). Indeed,

Theorem 1. *Given $\{a_n\}, \{b_n\}$ put $A_n = \sum_{k=0}^{n} a_k$ if $n \geq 0$. Also put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have*

$$\sum_{n=p}^{q} a_n b_n = A_q b_q - A_{p-1} b_p - \sum_{n=p}^{q-1} A_n (b_{n+1} - b_n). \tag{1}$$

Proof. The proof is straightforward.

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{\tilde{n}=p-1}^{q-1} A_{\tilde{n}} b_{\tilde{n}+1} \tag{2}$$

which is equal to the right-hand side of the theorem. □

We now apply this to get the following result.

Theorem 2. *Suppose a_i are complex numbers and b_i real, and*

1. *The partial sums A_n of $\sum a_k$ form a bounded sequence.*
2. *$b_0 \geq b_1 \geq b_2 \ldots$.*
3. *$\lim_{n\to\infty} b_n = 0$.*

Then $\sum a_n b_n$ converges.

Proof. Indeed, we just have to show that the partial sums form a Cauchy sequence. Indeed, by summation-by-parts,

$$\begin{aligned} |\sum_{n=p}^{q} a_n b_n| &\leq |A_q| b_q + |A_{p-1}| b_p + \sum_{n=p}^{q-1} |A_n| (b_{n+1} - b_n) \\ &\leq C(b_q + b_p + b_q - b_p) = b_p + b_q < \epsilon, \end{aligned} \tag{3}$$

when $p, q > N$ for some N. □

As a corollary, we have the so-called alternating series test.

Theorem 3. *If*

1. $|c_1| \geq |c_2| \geq \dots,$

2. $\lim_{n\to\infty} c_n = 0,$ *and*

3. $c_n = (-1)^n |c_n|,$

then $\sum c_n$ *converges.*

Another corollary is the following theorem about convergence of power series.

Theorem 4. *Suppose the radius of convergence of* $\sum c_n z^n$ *is* 1, $c_0 \geq c_1 \dots$, $\lim c_n = 0$. *Then* $\sum c_n z^n$ *converges at every point on the unit circle except perhaps at* $z = 1$.

Proof. Apply theorem 2 with $a_n = z^n$ and $b_n = c_n$ to conclude the result. □

4 Multiplication, and addition of series

So how does one add and multiply series ?
Addition is trivial. You just add the individual terms. It is also easy to prove that $\sum (a_n + b_n) = \sum a_n + \sum b_n$ if $\sum a_n$ and $\sum b_n$ converge. Likewise, $\sum \alpha a_n = \alpha \sum a_n$.

The real challenge comes with multiplication. There are many ways to define multiplication of series. The most natural is the so-called Cauchy product. Indeed, if you take two polynomials $p(z) = \displaystyle\sum_{n=0}^{p} a_n z^n$ and $q(z) = \displaystyle\sum_{m=0}^{q} b_m z^m$, what is $p(z)q(z) =$? Indeed $p(z)q(z) = \sum_{n=0}^{p+q} c_z z^n$ where $c_n = \displaystyle\sum_{m=0}^{n} a_m b_{n-m}$. Therefore we define
$\sum a_n \sum b_n$ to be $\displaystyle\sum_{n=0} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

It is not clear that just because $A = \sum a_n$ and $B = \sum b_n$ converge that their Cauchy product $C = \sum a_n \sum b_n = \sum c_n$ converges. Indeed, here is a counterexample -
Let $A = \sum a_n = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. This converges conditionally. (Of course it diverges absolutely by the comparison and the p-tests.) Indeed, apply the alternating series test. Shockingly enough, $C = A^2$ diverges. Indeed, $c_n = \displaystyle\sum_{k=0}^{n} a_k a_{n-k} = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}\sqrt{n-k+1}}$.
We claim that $\lim_{n\to\infty} |c_n| \neq 0$ thus proving that the series $\sum c_n$ does not converge (the divergence test if you will). Indeed, $\displaystyle\lim_{n\to\infty} |c_n| \geq 2 \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n+2} = 2 \lim_{n\to\infty} \frac{n+1}{n+2} = 2.$

Note that in the above counterexample, somehow conditional convergence seems to screw things up. So if require that one of the series is absolutely convergent, we should be in good shape. Indeed, we have the following theorem due to Mertens.

Theorem 5. *Suppose* $A = \sum a_n$ *is absolutely convergent and* $B = \sum b_n$ *is convergent (perhaps just conditionally), then* $C = AB = \sum(\sum_{k=0}^{n} a_k b_{n-k})$ *is convergent and* $C = AB$.

We will do the proof the next time (thanks to my bungling up by not looking at Rudin).

Another question one can ask is - If C, A, and B converge then does C converge to AB ? Thanks to Abel, this is true. We shall prove this later on.

5 Rearrangements

In this section we prove a shocking and clever theorem due to Riemann. Firstly, note this strange phenomenon : The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ converges whereas to a number $< 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ (why? because each term you add after that is either negative (pushes the sum down) or is positive but small (so the pushed down sum does not rise beyond its original value)). However, consider the "rearranged series" $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$. Since $\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$, we see that the new partial sums satisfy $s_3^{'} < s_6^{'} < s_9^{'} \ldots$ and hence $\limsup s^{'}n > s_3^{'} = \frac{5}{6}$. In other words, if you rearrange the terms, you will no longer converge to the same sum (if you do converge at all).

Definition of rearrangement : Suppose $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a bijection, then $a_n^{'} = a_{f(n)}$. Then the series $\sum a_n^{'}$ is called a rearrangement of $\sum a_n$.