Notes for 9 Feb (Thursday)

1 Revision of earlier concepts (A very very long recap essentially)

1.1 Set theory, Natural numbers, and Cardinalty

From a practical perspective, all we did was "The set theory you know from high school is fine except that you need to be careful. Not everything is a set. Sets can only be constructed from pre-existing ones. Everything that you or I can think of usually turns out to be a set."
So the moral of the story is not to worry too much about set theory except to respect its place in mathematics.
We constructed the set of natural numbers and stated Peano's axioms. From a practical perspective, all this means is that you can use induction to prove things. Also, recursive definitions are all right. Don't worry too much about natural numbers.
The really new thing in this section is cardinality. Two set A and B are said to have the same cardinality (written as $\#A = \#B$) if there exists a bijection (1-1 onto) function from A to B. A set is said to be finite if $\#A = \#\{1, 2, \ldots, n\}$. In other words it has n elements. The set of natural numbers is infinite. Any set that has the cardinality of natural numbers is said to be countable.
We proved (or at least) stated some useful theorems about cardinality :

1. If there is a $1-1$ map from A to B and another $1-1$ map from B to A then $\#A = \#B$.

2. If there is an onto map from A to B and another onto map from B to A, then $\#A = \#B$.

3. Every infinite subset of a countable set is countable. (That is, "countable" is the smallest infinity there is.)

4. If A and B are countable, then so is $A \times B$ (and inductively, if $A_1, A_2, \ldots, A_n$ are countable, then so is $A_1 \times A_2 \times A_3 \times \ldots A_n$.

5. Countable unions of countable sets are countable. That is, if A_i are countable sets then $\cup_{i=1}^{\infty} A_i$ is countable.

6. $\mathcal{P}(X)$ is strictly larger than X.

1.2 Integers, Rational numbers, Real numbers

From a practical perspective, integers $\mathbb{Z}$ are formally simply numbers of the type $a - b$ where a and b are natural numbers. (In other words, you are artificially constructing negative numbers.) Integers are countable.
Rational numbers are formally speaking, simply "ratios" p/q ($q \neq 0$) of integers. (In words, you are artificially constructing fractions.) Rationals (like integers and naturals) have a notion or order $\leq$ which obeys all the usual properties that it should obey. You can also add, subtract, multiply, and divide rationals (and all of them respect the ordering). Such an object where you can add, subtract, multiply, and divide stuff, and have an ordering that is respected by all these operations is called an ordered field. The problem with rationals is that there is no rational x such that $x^2 = 2$. The core reason behind this problem is that given any rational x such that $x^2 < 2$, I can find another rational $y > x$ such that $y^2 < 2$. In other words, there is a set bounded from above but it does not have a "least upper bound".
Recall that the "least upper bound" or "sup" of a set E is a number y such that
y is an upper bound of E.
Any number $z < y$ fails to be an upper bound of E. That is, there exists an $x \in E$ such that $x \geq z$.
We (with great difficulty) constructed a set called Real numbers $\mathbb{R}$ such that it contains the rational numbers, it is an ordered field, and has the least upper bound property. The construction itself was complicated (equivalence classes of Cauchy sequences of rationals). But the good thing is that you do not need to ever care about the construction. All you need to know about the real numbers is that they form an ordered field having the least upper bound property, i.e. every set that is bounded above has a least upper bound. Also, between any two reals there exists a rational (Density of rationals) and given any two reals $0 < x < y$ there is an integer n such that $nx > y$ (The Archimedian property).

The notion of least upper bound or sup is a nice substitute for "maximum". In other words, if you take the interval $(0,1)$, it has no largest element. So $max(0,1)$ does not make sense. However, $\sup(0,1) = 1$.

Lastly, real numbers are not countable. In fact, $\#\mathcal{P}(\mathbb{N}) = \#\mathbb{R}$. This is proven using Cantor's diagonalisation.

Let's do the problems of HW -1 :

1. Prove that algebraic numbers are countable :
   Obviously there are infinitely many algebraic numbers. (All integers are algebraic.) If we prove that algebraic numbers form an infinite subset of a countable set we will be done. (Infinite subsets of countable sets are countable.)

   Given an n, consider the set E_n of all complex numbers z satisfying a degree n polynomial with integral coefficients. There are at most n (distinct) roots of any polynomial. Now the set of coefficients $(a_0, a_1, \ldots, a_n)$ is countable because it is $\mathbb{Z} \times \mathbb{Z} \times \ldots\ n+1$ times. The set E_n is the union over all $(a_0, a_1, a_2, \ldots, a_n)$ of n

roots. Since countable unions of countable sets is countable, E_n is countable.

Taking a union over all n, we get all algebraic numbers. Once again, a countable union of countable sets is countable.

Fix a real number $b > 1$.

(a) If m is an integer, then define b^m. If $m > 0$, define $b^{1/m}$.
Ans. $b^0 = 1$. Assuming b^m is defined, $b^{m+1} = b^m \times b$.
Since we proved in the class that positive m^{th} roots of positive reals exist and are unique, we define $b^{1/m}$ as the unique positive real x such that $x^m = b$.

(b) Prove that $(b^m)^{1/n} = (b^p)^{1/q}$ if $r = m/n = p/q \geq 0$ and m, n, p, q are integers such that $n, q \neq 0$. Thus we can define $b^r = (b^m)^{1/n}$.
Ans. The only way to prove the first statement is by definition. If $x = (b^m)^{1/n}$ and $y = (b^p)^{1/q}$ then $x^n = b^m$ and $y^q = b^p$. This means that $(x^n)^p = (b^m)^p$. Using induction on p (You have to complete this induction in your HW/exam) one can prove that $x^{np} = (x^n)^p$ when n, p are integers. Then $x^{np} = x^{mq} = b^{mp}$. Since m^{th} roots are unique, we may take the m^{th} root on both sides to conclude that $x^q = b^p$. Therefore $x = y$.

(c) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
Ans. Firstly, note that if $r < 0$ then b^r is still defined as $(b^m)^{1/n}$ where this time m is assumed to be negative and $n > 0$. If $r = \frac{p}{q}$ and $s = \frac{m}{n}$ then $b^{r+s} = b^{(pn+qm)/(qn)}$. Assume that $qn \geq 0$ (By absorbing any negative signs into the numerator if necessary.) Then $b^{r+s} = (b^{pn+qm})^{1/qn} = (b^{pn}b^{qm})^{1/qn}$. Since we proved in the class that $(xy)^{1/n} = x^{1/n}y^{1/n}$ we see that $b^{r+s} = (b^{pn})^{1/qn}(b^{qm})^{1/qn} = b^r b^s$.

(d) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that $b^r = \sup B(r)$ if r is rational. Hence it makes sense to define $b^x = \sup B(x)$ for every real x.
Ans. So if $t \leq x$ and t, x are rational, then we claim that $b^t \leq b^x$. Indeed, $b^x = b^{x-t}b^t$. Indeed $b^{x-t} \geq 1$ because $b^{p/q} = (b^p)^{1/q}$ and no integer ≥ 1 has a power that is < 1. So $b^r \geq \sup B(r)$. But $r \in B(r)$. Therefore $b^r = \sup B(r)$.

(e) Prove that $b^{x+y} = b^x b^y$ if x and y are real.
Ans. Note that if $r \leq x$ and $s \leq y$ then $b^{r+s} = b^r b^s \leq b^x b^y$ by definition. If t is a rational such that $t < x + y$ then then $t - x < y$ and by the density property there exists a rational $s \leq y$ such that $t - x < s$. Therefore, $t - s < x$. By the density property again, there exists $t - s < r \leq x$. This means that $b^x b^y \geq b^r b^s = b^{r+s} > b^t$ for every rational $t < x + y$. I claim that this means $b^x b^y \geq b^{x+y}$. Indeed, if not, then $b^x b^y < b^{x+y}$. Because b^{x+y} is the supremum of b^t such that $t \leq x + y$, there exists an $a < x + y$ such that $b^x b^y < b^a$. This is a contradiction. Therefore, $b^{x+y} \leq b^x b^y$.

Suppose $b^{x+y} < b^x b^y$. By definition of supremum, (for small $\epsilon > 0$) there exist rationals $r \leq x, s \leq y$ such that $b^r > b^x - \epsilon > 0$ and $b^s > b^y - \epsilon > 0$. Thus

$b^{r+s} = b^r b^s > (b^x - \epsilon)(b^y - \epsilon) \geq b^x b^y - \epsilon(b^x + b^y)$. Therefore, taking supremum we see that $b^{x+y} \geq b^x b^y - \epsilon(b^x + b^y)$. Taking ϵ to be smaller than $\frac{b^x b^y - b^{x+y}}{b^x + b^y}$ we get a contradiction.

1.3 Topology

Basically, our main object of study is a metric space (X, d). You should think of the example of $\mathbb{R}$ with the metric $d(x, y) = |x - y|$ whenever you think of metric spaces. (Metric spaces in general satisfy weird properties but in this course we will deal largely with $\mathbb{R}$.) Basically a metric d is a way of measuring distances between points in such a way that tthe triangle inequality is satisfied.

Without repeating the material, here are the key definitions you should keep in mind.

1. A set E is called bounded if there is a point $p \in X$ and a radius $R > 0$ such that $E \subset B_R(p)$, i.e., E is contained in a large finite ball.

2. An open set U is one where around every point p there is an open ball $B_r(p) \subset U$. A union of arbitrary number of open sets is open. An intersection of finite number of closed sets is closed.

3. If you have a set E, then an open cover $\{U_\alpha\}$ of E is a collection of open sets U_α (one for each α) such that $E \subset \cup_\alpha U_\alpha$. For example, $[0, 1]$ is covered by $(-1, \frac{1}{2})$ and $(0.01, 2)$.

4. If $A \subset E \subset X$ (like $(0, 1) \subset \mathbb{R} \subset \mathbb{C}$) then A is relatively open with respect to E iff $A = U \cap E$ where $U \subset X$ is open. For example, $(0, 1]$ is relatively open in $(-1, 1]$ because $(0, 1] = (-1, 1] \cap (0, 2)$. Note that $(0, 1]$ is NOT open as a subset of $\mathbb{R}$.

5. A relatively open subset of an open subset is open. Likewise, relatively closed in closed is closed.

6. A limit point p of a set E is one such that every neighbourhood $B_r(p)$ contains a point $q_r \in E$ not equal to p.

7. A closed set F is such that F^c is open. Alternatively, F contains all its limit points.

8. The closure $\bar{E}$ of a set E is the smallest closed set containing E. In other words, $\bar{E}$ is a closed set such that *any* closed set containing E contains $\bar{E}$. Also, $\bar{E}$ consists of E and the limit points of E. For example, what is the closure of $(0, 1]$? It is $[0, 1]$.

9. A subset $E \subset X$ is said to be dense if $\bar{E} = X$. In other words, every element in X is either a point of E or a limit point of E. For example, rationals are dense in real numbers.

10. A set is called perfect if it is closed and all its points are limit points. A perfect set is uncountable.

1.4 Compactness

A set K is called compact if every open cover has a finite subcover. Meaning that if I have an arbitrarily huge number of open sets whose union contains K, then I can choose a finite subcollection such that their union contains K.

User's guide to compactness (i.e. what are examples of compact sets, theorems about them, and how to use compactness to solve problems or to prove theorems).

1. So firstly, compact sets are closed and bounded. So is the set of all algebraic numbers compact ? Of course not ! All integers are algebraic. So surely it is not bounded.

2. Secondly, the only compact subsets of $\mathbb{R}^n$ are the closed and bounded ones. (Heine-Borel theorem.) So is $[0, 1]$ compact ? Yes it is closed and bounded.

3. Thirdly, in $\mathbb{R}^n$ every infinite subsets of a compact set K has a limit point in K. (We will recall sequences soon but compactness in $\mathbb{R}^n$ is the same as saying that every sequence has a convergent subsequence.)

4. Fourthly, the way you use compactness as an assumption in a problem is either to conclude that a sequence has a limit point (a convergent subsequence) or to say something like "Every point p is contained in a neighbourhood satisfying some good properties. Since these neighbourhoods cover the set, and the set is compact, you only need finitely many such neighbourhoods."

Examples of problems :

1. (HW 2) Give an example of an open cover of $(0, 1)$ that does not have a finite subcover.
   Ans) Take $U_n = (0, 1 - \frac{1}{n})$. These sets are open. Of course $\cup_n U_n = (0, 1)$. Indeedm given any $x \in (0, 1)$ surely there exists an integer n (by the Archimedian property) such that $1 - x > \frac{1}{n}$. Therefore $x \in U_n$. But there is no finite subcover. Suppose there is, i.e., suppose $(0, 1) \subset U_{n_1} \cup U_{n_2} \ldots U_{n_k}$. Then let N be the maximum of all the n_is. Take $1 - \frac{1}{N+1} \in (0, 1)$. This does not belong to any of the U_{n_i}. Contradiction.