Notes for 9 Mar (Thursday)

1 Recap

1. Defined vector-valued functions and their derivatives. Gave counterexamples to show that they do not necessarily satisfy the MVT and the L'Hospital rule.

2. But something similar to the MVT is true.

3. Went over the Riemann integral. Motivated and defined the Riemann-Stieltjes integral. Defined partitions, refinements.

4. The RS integral exists if and only for every $\epsilon > 0$ there exists a partition P such that $U(P) - L(P) < \epsilon$.

5. Showed that continuous functions are RS integrable (using uniform continuity).

2 RS integrability

Now we prove that if f is monontonic and α is continuous and monotonically increasing, then f is RS integrable.
Proof : Now since α is uniformly continuous, choose an n as before such that $|p-q| < \frac{b-a}{n}$ implies that $|\alpha(p) - \alpha(q)| < \frac{\epsilon}{|f(b)-f(a)|}$. Now $M_i - m_i = |f(x_i) - f(x_{i-1})|$ by monotonicity. Thus $U(P) - L(P) < \frac{\epsilon}{|f(b)-f(a)|} \sum |f(x_i) - f(x_{i-1})| = \epsilon$.

Lastly, we prove the following theorem.

Theorem 1. *Suppose f is bounded and has only finitely many points of discontinuity. Also assume that α is continuous at every discontinuity of f. Then f is RS integrable.*

Proof. Let $p_1, p_2, \ldots, p_k$ be the points of discontinuity of f. Cover p_i with a small intervale (α_i, β_i) such that $[\alpha_i, \beta_i] \subset [a, b]$ and α varies by at most ϵ on these intervals. On the rest of $[a, b]$, f is uniformly continuous. So the same arguments as before produce a partition P containing α_i, β_i along with other points such that either $M_i - m_i < \epsilon$ or $\Delta\alpha_i < \epsilon$. Thus $U(P)-L(P) < \epsilon(\alpha(b)-\alpha(\beta_k)+\alpha(\beta_{k-1})-\alpha(\beta_{k-2})+\ldots)+\epsilon(\sum M_i - m_i) < \epsilon(\alpha(b) - \alpha(a)) + \epsilon 2Mk$ (where $|f| \leq M$ on $[a, b]$) which can be made arbitrarily small by choosing ϵ small enough. □

Theorem 2. *Suppose f is RS integrable w.r.t α, $m \leq f \leq M$, and $g : [m, M] \to \mathbb{R}$ is continuous. Then $h = g \circ f$ is RS integrable.*

Proof. The intuition is that since $\sum(M_i - m_i)\Delta\alpha_i$ is small, either $M_i - m_i$ is small or the $\Delta\alpha_i$ is small. Either of these things should force a similar sum for h to be small.
Indeed, since g is uniformly continuous, choose a $\epsilon > \delta > 0$ such that $|g(y_2) - g(y_1)| < \epsilon$ whenever $|y_2 - y_2| < \delta$ for all $y_1, y_2 \in [m, M]$. Now assume that a partition P is chosen so that $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$. Therefore $\sum(M_i - m_i)\Delta\alpha_i < \delta^2$.
Let $M_i^* = \sup_{x\in[x_{i-1},x_i]} g(f(x))$ and likewise for m_i^*. Consider the set E of those i such that $M_i - m_i < \delta$. For such i, $M_i^* - m_i^* < \sup_{y\in[m_i,M_i]} g(y) - \inf_{y\in[m_i,M_i]} g(y) < \epsilon$. For all i not in E, $M_i - m_i \geq \delta$. So, $\delta(\sum_{i\notin E} \Delta\alpha_i) \leq \sum_{i\notin E}(M_i - m_i)\Delta\alpha_i < \delta^2$. Therefore $\sum_{i\notin E} \Delta\alpha_i < \delta$. Therefore

$$\begin{aligned}\sum(M_i^* - m_i^*)\Delta\alpha_i &< \epsilon\sum_{i\in E}\Delta\alpha_i + \sum_{i\notin E}(M_i^* - m_i^*)\Delta\alpha_i \\ &< \epsilon(\alpha(b) - \alpha(a)) + 2K\delta \\ &< \epsilon(\alpha(b) - \alpha(a) + 2K). \qquad (1)\end{aligned}$$

Since ϵ is arbitrary, we are done. □

This raises the question of which functions are Riemann integrable. The answer requires more knowledge than you will gain through this course. (The answer is : Functions that are continuous almost everywhere, i.e., if you throw a dart then you will almost surely not hit a discontinuity.)

3 Properties of the RS integral

1. If f, g are RS integrable, then $f+g$ and cf are so as well. Moreover, $\int_a^b (f+g)d\alpha = \int_a^b fd\alpha + \int_a^b gd\alpha$ and $\int_a^b cfd\alpha = c\int_a^b fd\alpha$.
2. If $f \leq g$ and both are RS integrable, then $\int_a^b fd\alpha \leq \int_a^b gd\alpha$.
3. If f is RS integrable on $[a, c]$ and on $[c, b]$ then f is RS integrable on $[a, b]$ and $\int_a^b fd\alpha = \int_a^c fd\alpha + \int_c^b fd\alpha$.
4. If $|f| \leq M$, and f is RS integrable then $|\int_a^b fd\alpha| \leq M(\alpha(b) - \alpha(a))$.
5. If f is RS integrable w.r.t to α_1, α_2 then it is so with respect to $\alpha_1+\alpha_2$ and $c\alpha_1$ where $c > 0$. Also $\int fd(\alpha_1 + \alpha_2) = \int fd\alpha_1 + \int fd\alpha_2$. Moreover, $\int fd(c\alpha_1) = c\int fd\alpha_1$.

Proof.

1. Choose partitions P_1 and P_2 such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$ and $U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2}$. By moving to a common refinement $P = P_1 \cup P_2$, we may assume that both of these hold true for P. Now $\sup_{x\in[x_{i-1},x_i]}(f+g)(x) \leq \sup_{x\in[x_{i-1},x_i]} f(x) + \sup_{x\in[x_{i-1},x_i]} g(x)$. Likewise for infimum. Thus $U(P, f+g) - L(P, f+g) < U(P, f) - L(P, f) + U(P, g) - L(P, g) < \epsilon$. Thus $f + g$ is RS integrable. Also $U(P, f + g) < U(P, f) + U(P, g)$. Now $U(P, f) < \int_a^b fd\alpha + \epsilon$ and likewise for g. Thus $U(P, f+g) < \int_a^b fd\alpha + \int_a^b gd\alpha + 2\epsilon$. Taking infimum we see that $\int_a^b (f + g)d\alpha \leq \int_a^b fd\alpha + \int_a^b gd\alpha$.

Likewise, applying the same argument to $L(P, f+g)$ we see that $\int(f+g)d\alpha = \int f d\alpha + \int g d\alpha$. The case of cf is even easier.

2. Since $f \leq g$, $\sup_{x\in[x_{i-1},x_i]} f(x) \leq \sup_{x\in[x_{i-1},x_i]} g(x)$. Thus $U(P,f,\alpha) \leq U(P,g,\alpha)$. Choosing P so that $\int_a^b g d\alpha + \epsilon > U(P,g,\alpha)$ (this will of course hold for all refinements of P), we see that $U(P,f) < \int_a^b g d\alpha + \epsilon$. Taking infimum over all refinements of P and $\int_a^b f d\alpha < \int_a^b g d\alpha + \epsilon$. Since ϵ is arbitrary, we are done.

3. Choose partitions P_1, P_2 such that $U(P_1, f, \alpha, [a,c]) - L(P_1, f, \alpha, [a,c]) < \frac{\epsilon}{2}$ and $U(P_1, f, \alpha, [a,c]) - \frac{\epsilon}{2} \leq \int_a^c f d\alpha < U(P_1, f, \alpha, [a,c])$ and likewise for P_2. Take $P = P_1 \cup P_2$. This is a partition of $[a,b]$. Now $U(P,f,\alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha)$ and likewise for L. Thus $U(P) - L(P) < \epsilon$. Thus f is RS integrable on $[a,b]$. Moreover, $U(P) - \epsilon \leq \int_a^c f d\alpha + \int_c^b f d\alpha < U(P)$. Since this is true for all P and ϵ, we are done.

4. $-M(\alpha(b) - \alpha(a)) \leq L(P,f) \leq U(P,f) \leq M(\alpha(b) - \alpha(a))$ for all partitions. Thus we are done.

5. The case with $c\alpha_1$ is easy. Suppose we choose a common partition P such that $U(P,f,\alpha_1) - L(P,f,\alpha_1) < \epsilon/2$ and likewise for α_2, then $U(P,f,\alpha_1+\alpha_2) - L(P,f,\alpha_1+\alpha_2) = \sum(M_i - m_i)(\Delta\alpha_1 + \Delta\alpha_2) < \epsilon$. Thus it is RS integrable. Arguments similar to those above show that $\int f d(\alpha_1+\alpha_2) = \int f d\alpha_1 + \int f d\alpha_2$.

□