HW 1 (due on Jan 13, Friday, in the class)

1. (Rudin Problem 2, chapter 2) Assume the fundamental theorem of algebra for this problem - "Every polynomial of degree n having complex coefficients has exactly n complex roots."
   A complex number z is said to be *algebraic* if there are integers $a_0, a_1, \ldots a_n$, not all zero, such that $a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$. Prove that the set of algebraic numbers is countable. (Hint: First prove that for every positive integer N there are only finitely many equations with $n + |a_0| + |a_1| + \ldots + |a_n| = N$.)

2. (Rudin Problem 6, chapter 1) Fix a real number $b > 1$.
   (a) If m is an integer, then define b^m. If $m > 0$, define $b^{1/m}$.
   (b) Prove that $(b^m)^{1/n} = (b^p)^{1/q}$ if $r = m/n = p/q \geq 0$ and m, n, p, q are integers such that $n, q \neq 0$. Thus we can define $b^r = (b^m)^{1/n}$.
   (c) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
   (d) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that $b^r = \sup B(r)$ if r is rational. Hence it makes sense to define $b^x = \sup B(x)$ for every real x.
   (e) Prove that $b^{x+y} = b^x b^y$ if x and y are real.