# LECTURE NOTES ON CALABI'S CONJECTURES AND KÄHLER-EINSTEIN METRICS

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ABSTRACT. These are lectures notes of a mini course on Calabi's conjectures and Kähler-Einstein metrics, given as part of the Advanced Instructional School (AIS) in Riemannian geometry organized at Indian Institute of Science (IISc) in July 2019.

## CONTENTS



## 1. LECTURE-1: CALABI'S CONJECTURES

1.1. **A review of basic Kähler geometry.** Let $(M, \omega)$ be a compact, connected Kähler manifold. Then locally, the Kähler form is given by

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $\{g_{i\bar{j}}\}$ is a hermitian symmetric, positive definite matrix, such that $g_{i\bar{j};k} = g_{k\bar{j};i}$, where semi-colons denote derivatives. These conditions are of course equivalent to $\omega$ being a real, closed, positive $(1,1)$ form. Since $\omega$ is a closed, it determines a cohomology class $[\omega] \in H^2(M,\mathbb{R})$. We say that a cohomology class $\alpha \in H^2(M,\mathbb{R})$ is Kähler, and write $\alpha > 0$, if it contains a Kähler form. The set of all Kähler classes

$$\mathcal{C}_M := \{\alpha \in H^2(M,\mathbb{R}) \mid \alpha > 0\}$$

is an open, convex cone in the finite dimensional vector space $H^2(M,\mathbb{R})$, and is called the *Kähler cone* of $M$. We need the following fundamental result on the structure of the Kähler cone.

**Lemma 1.1** ($\sqrt{-1}\partial\overline{\partial}$-Lemma)**.** *Let $\omega_1$ and $\omega_2$ be real, closed $(1,1)$ forms such that $[\omega_2] = [\omega_1]$. Then there exists a $\varphi \in C^\infty(M,\mathbb{R})$ such that*

$$\omega_2 = \omega_1 + \sqrt{-1}\partial\overline{\partial}\varphi.$$

As a consequence, we can write

$$\mathcal{C}_M = \frac{\{d \text{ closed positive real } (1,1) \text{ forms}\}}{\text{Image}(\sqrt{-1}\partial\overline{\partial})}.$$

There is natural almost complex structure $J : TM \to TM$, $J^2 = -id$ given locally by

$$J\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial y^i},\ J\Big(\frac{\partial}{\partial y^i}\Big) = -\frac{\partial}{\partial x^i},$$

where $z^i = x^i + \sqrt{-1}y^i$. The pair $(J,\omega)$ determines a Riemannian metric on $M$ by

$$g(u,v) = \omega(u,Jv).$$

The Riemannian Ricci curvature is then given by $\text{Rc}_g(u,v) = \text{Ric}(\omega)(u,Jv)$, where

$$\text{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log\omega^n$$

is the Ricci form. Since $\omega^n$ defines a hermitian metric on the anti-canonical line bundle $K_M^*$, $\text{Ric}(\omega)$ being the corresponding curvature form is a representative of the first Chern class $c_1(M) := c_1(K_M^*)$.

**Conjecture 1.2.** *(Calabi) Conversely, every real $(1,1)$ form in $c_1(M)$ is the Ricci for of some Kähler form.*

1.2. **Prescribing Ricci curvature.** Calabi's conjecture was solved in 1978 by Yau.

**Theorem A** (Yau, [9])**.** *Given any $\rho \in c_1(M)$ and any $\alpha \in \mathcal{C}_M$, there exists an $\omega \in \alpha$ such that*

$$\text{Ric}(\omega) = \rho.$$

This immediately had some far reaching consequences. Due to the lack of space, we only mention one particular spectacular consequence. But first we state the following elementary consequence.

**Corollary 1.3.** *Let $M$ be a compact Kähler manifold. Then $M$ has a K—"ahler metric with positive (resp. zero and negative) Ricci curvature if and only if $c_1(M)$ is positive (resp. zero and negative).*

This in turn has various topological consequences. For instance, by an application of Bonnet-Myers, any manifold with $c_1(M)$ has a finite fundamental group.

**Corollary 1.4.** *There exist infinitely many non-flat Riemannian manifolds $(M,g)$ with $\text{Rc}_g \equiv 0$.*

**Remark 1.5.** Riemannian manifolds with Ricci curvature identically zero are called as *Ricci flat* manifolds. Before Yau's solution of the Calabi conjecture, not even a single example of a non-flat, compact Ricci flat manifold was known. Note that a flat manifold of dimension $m$ has $\mathbb{R}^m$ as it's universal cover. In particular, if a flat manifold is compact, it cannot be simply connected.

*Proof of Corollary 1.4.* Let $P$ be a homogenous polynomial in $\mathbb{C}^{n+1}$ of degree $n+1$, such that

$$\cap_{i=0}^{n}\Big\{\frac{\partial P}{\partial \xi^i} = 0\Big\} = \{0\}.$$

Then $M_P = \{[\xi^0, \cdots, \xi^n] \in \mathbb{P}^n \mid P(\xi^0, \cdots, \xi^n) = 0\}$ is a complex submanifold of $\mathbb{P}^n$ of dimension $n-1$. An infinite family of examples is provided by $P(\xi^0, \cdots, \xi^n) = (\xi^0)^{n+1} + \cdots + (\xi^n)^{n+1}$. If $\omega_P$ is the restriction of the Fubini-Study metric to $M_P$, then Lemma 1.6 below shows that

$$\mathrm{Ric}(\omega_P) = -\sqrt{-1}\partial\bar{\partial}\psi,$$

for a certain smooth function $\psi$ on $M_P$. In particular, $c_1(M_P) = 0$. By Theorem A, there exists a Ricci flat metric on $M_P$. Next, by Lefschetz hyperplane section theorem the fundamental group $\pi_1(M_P) \cong \pi_1(\mathbb{P}^n) = \{0\}$ if $n > 2$. Hence $M_P$ is simply connected whenever $n > 2$ (when $n = 2$, $M_P$ is simply an elliptic curve, and clearly not simply connected). But then by Remark 1.5, $Sec_g$ cannot be identically zero.

□

**Lemma 1.6.** *Let $P$ be a homogenous polynomial in $\mathbb{C}^{n+1}$ of degree $d$, such that*

$$\cap_{i=0}^{n}\Big\{\frac{\partial P}{\partial \xi^i} = 0\Big\} = \{0\}.$$

*Then*

1. $M_P = \{[\xi^0, \cdots, \xi^n] \in \mathbb{P}^n \mid P(\xi^0, \cdots, \xi^n) = 0\}$ *is a complex sub-manifold of $\mathbb{P}^n$ of dimension $n-1$.*
2. *If $\omega_P$ is the restriction of the Fubini-Study metric to $M_P$, then*

$$\mathrm{Ric}(\omega_P) = (n+1-d)\omega_P - \sqrt{-1}\partial\bar{\partial}\psi,$$

*where*

$$\psi := \log\Big(\frac{\sum_i \left|\frac{\partial P}{\partial \xi^i}\right|^2}{|\xi|^{2(d-1)}}\Big)$$

*is a smooth function on $\psi$.*

3. *In particular, $c_1(M) = (n+1-d)c_1(\mathcal{O}_{\mathbb{P}^n}(1)\Big|_{M_P})$ and hence is positive, zero or negative, depending on whether $d < n+1$, $d = n+1$ or $d > n+1$.*

**Definition 1.7.** A compact Kähler manifold is said to be

1. *Fano* if $c_1(M) > 0$,
2. *Calabi-Yau* if $c_1(M) = 0$ and
3. *General Type* if $c_1(M) < 0$.

**Remark 1.8.** Calabi-Yau manifolds are extensively studied by geometer and string theorists alike. We refer the interested reader to the excellent survey [10] article by Yau on the geometry of Calabi-Yau manifolds. Note that the vanishing of $c_1(M)$ implies that $K_M$ is topologically trivial. An interesting corollary to Theorem A is that some power $K_M^l$ is also holomorphically trivial.

## 1.3. **Kähler-Einstein metrics.**

**Theorem 1.9** (Uniformization theorem)**.** *Given any compact oriented Riemannian surface* $(\Sigma^2, g_0)$*, there exists a metric* $\tilde{g}$ *in the conformal class* $[g_0] = \{e^u g_0 \mid u \in C^\infty(\Sigma, \mathbb{R})\}$ *with constant Gauss curvature* $\mathrm{sgn}(\chi(M))$*, where* $\chi(M)$ *is the Euler characteristic.*

Recall that there exist isothermal coordinates with respect to which

$$g_0 = h(dx^2 + dy^2).$$

The isothermal coordiantes determine an integrable almost complex structure $J$ on $T\Sigma$ with holmorphic coordinate $z = x + \sqrt{-1}y$. Moreover the complex structure $J$ only depends on the conformal class $[g_0]$ and not on the particular metric $g_0$. The area element $\omega_g = dA_g$ of any $g \in [g_0]$ is then a Kähler metric on $M$, and the uniformization theorem is then equivalent to the statement that

$$\mathrm{Ric}(\omega_g) = \mathrm{sgn}(\chi(M)) \cdot \omega_g.$$

This motivates the next definition.

**Definition 1.10.** A Kähler metric $\omega$ on $M$ is said to be *Kähler-Einstein* if there exists a $\lambda > 0$ such that $\mathrm{Ric}(\omega) = \lambda\omega$.

**Remark 1.11.**
(1) (Rescaling) If $\omega$ is a Kähler-Einstein metric with $\lambda \neq 0$, then $\tilde{\omega} := |\lambda|\omega$ is also KE with

$$\mathrm{Ric}(\tilde{\omega}) = \frac{\lambda}{|\lambda|}\,\tilde{\omega},$$

and so without any loss of generality we can assume $\lambda = \pm 1$ or $0$.
(2) (Topological restriction) If $\omega$ is KE, then clearly $\lambda[\omega] = c_1(M)$, and hence the chern class must either vanish or must have a sign.

| $\lambda$ | $c_1(M)$ | $[\omega]$ |
|---|---|---|
| $-1$ | $< 0$ | $-c_1(M)$ |
| $0$ | $= 0$ | $\mathcal{C}_M$ |
| $1$ | $> 0$ | $c_1(M)$ |

**Theorem B.**
(1) *(Yau, [9]) If* $c_1(M) = 0$*, then for any* $\alpha \in \mathcal{C}_M$*, there exists an* $\omega \in \alpha$ *such that* $\mathrm{Ric}(\omega) = 0$*.*
(2) *(Aubin [1], Yau [9]) If* $c_1(M) < 0$*, then there exists a metric* $\omega$ *in* $-c_1(M)$ *such that* $\mathrm{Ric}(\omega) = -\omega$*.*

**Corollary 1.12.** *If* $P$ *is a homogenous polynomial* $\mathbb{C}^{n+1}$ *of degree* $d$*, such that*

$$\cap_{i=0}^{n}\Big\{\frac{\partial P}{\partial \xi^i} = 0\Big\} = \{0\}.$$

*Then* $M_P = \{[\xi^0, \cdots, \xi^n] \in \mathbb{P}^n \mid P(\xi^0, \cdots, \xi^n) = 0\}$ *admits a Kähler-Einstein metric with negative scalar curvature if* $d > n + 1$*.*

## 2. Lecture-2: Analytic preliminaries

For this lecture, we let $(M^n, \omega)$ be a compact Kähler manifold. At times, we will be concerned only with the underlying Riemannian structure $(M^m, g)$, where $m = 2n$.

**Definition 2.1.** Let $f \in C^\infty(M, \mathbb{R})$.

(1) The *complex gradient* $\nabla f : M \to \Gamma(M, T^{(1,0)}M)$ is defined to be

$$\nabla f := g^{i\bar{j}} \frac{\partial f}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}.$$

(2) The *complex Lapalcian* is defined to be

$$\Delta f := \nabla_i \nabla^i f = g^{i\bar{j}} \partial_i \partial_{\bar{j}} f.$$

(3) Given a $(1,1)$ form $\alpha = \sqrt{-1}\alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j$, we define the trace with respect to $\omega$ by

$$\mathrm{tr}_\omega \alpha = \Lambda_\omega \alpha := g^{i\bar{j}} \alpha_{i\bar{j}}.$$

**Remark 2.2.** (1) The Riemannian gradient is characterised by the property that

$$X(f) = g(\mathrm{grad} f, X)$$

for all vector fields $X$, and in fact is precisely half the real part of the complex gradient. More precisely,

$$\nabla f = \frac{1}{2}(\mathrm{grad} f - \sqrt{-1} J \mathrm{grad} f).$$

In particular $|\nabla f|^2 = |\mathrm{grad} f|^2/2$. We also have that the Laplace-Beltrami operator $\Delta_{LB}$ is twice the complex Laplacian.

(2) Our Laplacian is the so-called analysts Laplacian, and is a negative operator, as can be seen from the following integration by parts formula

$$\int_M \Delta f \varphi \omega^n = -\int_M \nabla f \cdot \nabla \varphi \ \omega^n.$$

(3) We have the identities

$$\Lambda_\omega \alpha = n \frac{\alpha \wedge \omega^{n-1}}{\omega^n}$$
$$\Delta f = n \frac{\sqrt{-1}\partial\bar{\partial} f \wedge \omega^{n-1}}{\omega^n}$$
$$|\nabla f|^2 = n \frac{\sqrt{-1}\partial f \wedge \bar{\partial} f \wedge \omega^{n-1}}{\omega^n}.$$

**Proposition 2.3.** *(Maximum principle)*

(1) *Let $f \in C^2(M, \mathbb{R})$. If $f$ has a local maximum at $p \in M$, then*

$$\sqrt{-1}\partial\bar{\partial} f(p) \leq 0.$$

*In particular, $\Delta f(p) \leq 0$.*

(2) *Moreover if $f \in C^2(M, \mathbb{R})$ such that $\sqrt{-1}\partial\bar{\partial} f \equiv 0$, then $f$ is a constant.*

*Proof.* (1) From the maximum principle from calculus, the real Hessian at a local max is non-positive. From this, proposition follows when $n = 1$, since if $t = u + iv$, then

$$\frac{\partial^2\psi}{\partial t\partial\bar{t}}(p) = \frac{1}{4}\Big(\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2}\Big)(p) \leq 0.$$

In general, let $\xi \in T^{1,0}_p M$, $\xi \neq 0$, and consider $\psi(t,\bar{t}) = f(p + t\xi, \overline{p + t\xi})$. Then $\psi$ has a local max at 0, and so

$$0 \geq \frac{\partial^2\psi}{\partial t\partial\bar{t}}(0) = \frac{\partial^2 f}{\partial z^i\partial\bar{z}^j}(p)\xi^i\overline{\xi^j}.$$

(2) Now suppose $\sqrt{-1}\partial\bar{\partial}f = 0$, and $\omega$ is any Káhler metric on $M$, then

$$0 = \int_M f \wedge \omega^{n-1} = -\int_M \sqrt{-1}\partial f \wedge \bar{\partial} f \wedge \omega^{n-1} = -n\int |\bar{\partial} f|^2\omega^n.$$

Hence $f$ is holomorphic, and thus a constant.

□

2.1. **Schauder estimates on $\mathbb{R}^m$.** For this section, we let $\Omega \subset \mathbb{R}^m$ be a bounded, connected, open set.

**Definition 2.4.** Let $f \in C^k(M,\mathbb{R})$. For any $\alpha \in (0,1)$, we define the Hölder semi norm by

$$[f]_{C^\alpha(\Omega)} := \sup_{x,y\in\Omega,\ x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

and the Hölder $C^{k,\alpha}$ norm by

$$||f||_{C^{k,\alpha}(\Omega)} := \sup_{|\beta|\leq k} ||D^\beta||_{C^0} + \sup_{|\beta|=k} [D^\beta f]_{C^\alpha(\Omega)},$$

where $\beta = (\beta_1, \cdots, \beta_m)$ is a multi-index, and $D^\beta$ is the corresponding partial derivative of order $|\beta| = \beta_1 + \cdots + \beta_m$. We then define the Hölder space $C^{k,\alpha}(\Omega)$ as

$$C^{k,\alpha}(M,\Omega) := \{f \in C^k(\Omega) \mid ||f||_{C^{k,\alpha}(\Omega')} < \infty \text{ for all } \Omega' \subset\subset \Omega\}.$$

Note that $C^{k,\alpha}(M,\Omega)$ is not a Banach space.

**Definition 2.5.** A linear second order partial differential operator is an operator $L : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ that can be written as

$$L[u] = a^{ij}\frac{\partial^2 u}{\partial x^i\partial x^j} + b^i\frac{\partial u}{\partial x^i} + cu,$$

where $a^{ij}, b^i, c \in C^0(M,\mathbb{R})$. We say that $L$ is *elliptic* if the matrix $a^{ij}$ is positive definite everywhere in $\Omega$.

**Example 2.6.** *If $g$ is any Riemannian metric on $\Omega$, then $\Delta_g$ is an elliptic operator. Indeed,*

$$\Delta_g = g^{ij}\frac{\partial^2}{\partial x^i\partial x^j} + \text{ lower order terms.}$$

The key result we need is as follows.

**Theorem 2.7** (Elliptic regularity and local Schuader estimates)**.** *Let $\Omega' \subset\subset \Omega$. Suppose $\alpha \in (0,1)$ and $k \in \mathbb{N}$. Suppose further that*

(1) *$L$ is an elliptic $2^{nd}$ order differential operator on $\Omega$ with $a^{ij}, b^i, c \in C^{k,\alpha}(\Omega)$.*

(2) *$u \in C^2(\Omega)$ satisfies $Lu \in C^{k,\alpha}$.*

*Then $u \in C^{k+2,\alpha}(\Omega)$, and moreover, we have the estimate*

$$||u||_{C^{k+2,\alpha}(\Omega')} \leq C\Big(||Lu||_{C^{k,\alpha}(\Omega)} + ||u||_{C^0(\Omega)}\Big),$$

*where $C$ depends only on the Hölder norms of the coefficients, the domains $\Omega$, $\Omega'$, and the constant of ellipticity $\Lambda$.*

**Remark 2.8.** The term $||u||_{C^0(\Omega)}$ is needed on the right, as can be seen by taking $L = \Delta$ and $u$ an arbitrarily big constant.

2.2. **Elliptic operators on compact Riemannian manifolds.** Now let $(M, g)$ be a compact Riemannian manifold of dimension $m$. We fix an open cover $M = \cap_{j=1}^N U_j$ such that

$$\frac{1}{2}\delta_{ij} \leq g_{ij}\Big|_{U_j} \leq 2\delta_{ij},$$

and a partition of unity $\{\rho_j\}$ subordinate to $\{U_j\}$.

**Definition 2.9.** For any $f \in C^k(M)$, we define

$$||f||_{C^{k,\alpha}(g)} := \sum_J ||\rho_j||_{C^{k,\alpha}(U_j)},$$

where Hölder norms on the right are the usual Hölder norms in Euclidean domains. We then define the $C^{k,\alpha}$-Hölder space by

$$C^{k,\alpha}(M) := \{f \in C^k(M) \mid ||f||_{C^{k,\alpha}(g)} < \infty\}.$$

**Remark 2.10.** Even though the Hölder norm depends on the metric, the Hölder spaces $C^{k,\alpha}(M)$ themselves are independent of the metric.

**Proposition 2.11.** (1) *For any $k \in \mathbb{N}$ and any $\alpha \in (0,1)$, $(C^{k,\alpha}(M) < || \cdot ||_{C^{k,\alpha}(g)})$ is a Banach space.*

(2) *For any $\alpha' < \alpha$, $C^{k,\alpha}(M) \subset C^{k,\alpha'}(M)$ is a compact embedding. That is, if $f_i$ is a sequence of functions in $C^{k,\alpha}(M)$ with $||f||_{C^{k,\alpha}(g)} \leq C$ for some uniform constant $C$, then there exists a sub-sequence $f_{i_j}$ converging in $|| \cdot ||_{C^{k,\alpha'}(g)}$ to a limiting function $f \in C^{k,\alpha'}(M)$.*

**Definition 2.12.** A $2^{nd}$ order differential operator is a map $L : C^\infty(M)^\infty(M)$ that can be written as

$$Lu = a^{ij}\nabla_i\nabla_j u + b^i\nabla_i + cu,$$

where $\nabla$ is the Levi-Civita connection, $a^{ij}$ and $b^i$ are continuous sections of $TM \otimes TM$ and $TM$ respectively and $u \in C^0(M)$. We say $L$ is *elliptic* if $a^{ij}$ is a positive definite form.

**Definition 2.13.** Given $L$, the adjoint $L^*$ is defined as the unique second order differential operator such that

$$\int_M Lf\varphi\, dV_g = \int_M fL^*\varphi\, dV_g,$$

for all $\varphi \in C^\infty(M)$.

**Lemma 2.14.** *If $L = a^{ij}\nabla_i\nabla_j + b^i\nabla_i + c$, then*

$$L^*\varphi = a^{ij}\nabla_i\nabla_j\varphi + (\nabla_j a^{ij} + \nabla_j a^{ji} - b^i)\nabla_i\varphi + (\nabla_i\nabla_j a^{ij} - \nabla_i b^i + c)\varphi.$$

*In particular, $L$ is elliptic if and only if $L^*$ is elliptic.*

**Example 2.15.** *The Laplace-Beltrami operator*

$$\Delta_g = g^{ij}\nabla_i\nabla_j$$

*is an example of an elliptic second order differential operator. In fact it is also self adjoint, that is* $\Delta_g^* = \Delta_g$.

**Definition 2.16.** (Weak solutions) We say that $u \in L^2(M)$ solves $Lu = f$ weakly, if for all $\varphi \in C^\infty(M)$,

$$\int )M f\varphi \, dV_g = \int uL^*\varphi \, dV_g.$$

Of course if $u \in C^2$ solves $Lu = f$, then it also solves this equation weakly.

**Theorem 2.17.** *(Regularity and Global Schauder estimates) Suppose* $f \in C^{k,\alpha}(M)$ *and* $u \in L^2(M)$ *satisfies* $Lu = f$ *weakly. Then* $u \in C^{k+2,\alpha}(M)$ *(or more precisely, this holds after possibly modifying* $u$ *on a measure zero set), and*

$$||u||_{C^{k+2,\alpha}(g)} \leq\leq C\Big(||f||_{C^{k,\alpha}(\Omega)} + ||u||_{C^0(\Omega)}\Big).$$

*In particular,*

$$\ker(L) = \{u \in L^2 \mid Lu = 0 \text{ weakly}\} \subset C^\infty(M).$$

**Theorem 2.18.** *(Existence) In the same setting as the above theorem, if* $f \in C^{k,\alpha}(M)$*, then there exists a unique* $u \in C^{k+2,\alpha}(M) \cap \ker(L)^\perp$ *solving* $Lu = f$ *if and only if* $f \in \ker(L^*)^\perp$*. In particular*

$$L : C^{k+2,\alpha}(M) \cap \ker(L)^\perp \to C^{k,\alpha}(M) \cap \ker(L^*)^\perp$$

*is an isomorphism.*

We need the following corollary in our proof of the Calabi conjecture.

**Corollary 2.19.** *Consider the equation*

$$\Delta_g u + \lambda u = f. \tag{*}$$

(1) *If* $\lambda = 0$*, then*

$$\Delta_g : \{u \in C^{k+2,\alpha}(M) \mid \int_M u \, dV_g = 0\} \to \{f \in C^{k,\alpha}(M) \mid \int_M f \, dV_g = 0\}$$

*is an isomorphism.*

(2) *If* $\lambda < 0$*, then* $\Delta_g + \lambda : C^{k+2,\alpha}(M) \to C^{k,\alpha}(M)$ *is an isomorphism.*

### 2.3. Poincare and Sobolev inequalities.

**Theorem 2.20.** *Let* $(M^m, g)$ *be a compact Riemannian manifold. There exists constants* $C_S$ *and* $C_P$ *such that for any* $f \in C^\infty(M, \mathbb{R})$,

(1) *(Soboelv ienquality)*

$$\Big(\int_M |f|^{\frac{2m}{m-2}} dV_g\Big)^{\frac{m-2}{m}} \leq C_S\Big(\int_M |\nabla f|^2 \, dV_g + \int_M f^2 \, dV_g\Big)$$

(2) *(Poincare inequality) If* $\hat{f} = \frac{1}{V}\int_M f \, dV_g$ *is the average of the function* $f$ *on* $M$*, then*

$$\int_M (f - \hat{f})^2 \, dV_g \leq C_P \int_M |\nabla f|^2 \, dV_g.$$

**Remark 2.21.** Here $C_S$ and $C_P$ denote the best possible constants for which the above inequalities hold, and are referred to as the Sobolev and Poincare constants of $(M, g)$ respectively. The Poincare constant $C_P$ is in fact precisely the inverse of the first non-zero eigenvalue $\lambda_1$ of $-\Delta_g$. This follows from the Rayleigh quotient characterisation of eigenvalues

$$\lambda_1 = \inf_{f \in C^\infty(M,\mathbb{R}),\ \hat{f}=0} \frac{\int_M |\nabla f|^2 \, dV_g}{\int_M f^2 \, dV_g}.$$

## 3. Lecture-3: Complex Monge-Ampere equations

Recall that our goal is to solve the following equations.

(1) Given $\rho \in c_1(M)$, real and $(1,1)$ form, and any Kähler class $\alpha \in \mathcal{C}_M$, find $\omega \in \alpha$ such that
$$\mathrm{Ric}(\omega) = \rho.$$

(2) Solve
$$\mathrm{Ric}(\omega) = \lambda\omega.$$

We can combine the equations into a single *twisted KE equation*

$$\mathrm{Ric}(\omega) = \lambda\omega + \rho, \tag{t-KE}$$

We seek solutions $\omega$ in a given Kähler class $\alpha \in \mathcal{C}_M$. Then necessarily $[\rho] \in c_1(M) - \lambda\alpha$. If $\hat{\omega} \in \alpha$ is a fixed background metric, by the $\sqrt{-1}\partial\overline{\partial}$-lemma, there exist functions $F, \varphi \in C^\infty(M, \mathbb{R})$ such that

$$\mathrm{Ric}(\hat{\omega}) = \lambda\hat{\omega} + \rho + \sqrt{-1}\partial\overline{\partial}F$$
$$\omega = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi.$$

Then (t-KE) is equivalent to the equation

$$-\sqrt{-1}\partial\overline{\partial}\log\Big(\frac{(\hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi)^2}{\hat{\omega}^n}\Big) = -\sqrt{-1}\partial\overline{\partial}(F - \lambda\varphi).$$

By Lemma 2.3, it follows that solving (t-KE) is then equivalent to solving the following complex Monge-Ampere equation.

$$\begin{cases} (\hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F-\lambda\varphi}\hat{\omega}^n \\ \omega := \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi > 0 \\ (\lambda = 0)\ \ \sup_M \varphi = 0 \end{cases} \tag{CMA}$$

When $\lambda = 0$, the extra normalisation is needed in order to obtain *a priori* estimates. For instance, we would need to obtain uniform $C^0$-estimates on solutions to the above equation (uniform in the sense of only depending on $||F||_{C^0}$ and $(M, \hat{\omega})$; cf. Proposition 4.3). Now, if $\varphi$ is a solution, $\varphi + c$ will also be a solution for any constant $c$, but with a $C^0$-norm that goes to infinity as $c \to \infty$. Moreover, if $\lambda = 0$, there is also an additional necessary condition for existence of solutions. To see this, integrating both sides of (CMA), we obtain

$$\int_M e^F\hat{\omega}^n = \int_M (\hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \int_M \hat{\omega}^n,$$

where we used Stokes' theorem in the second equality.

**Theorem C.** (1) *(Yau) If $\lambda = 0$, (CMA) has a solution $\varphi \in C^\infty(M, \mathbb{R})$ if and only if*
$$\int_M e^F\hat{\omega}^n = \int_M \hat{\omega}^n.$$

(2) *(Yau, Aubin) If $\lambda < 0$, then (CMA) has a solution $\varphi \in C^\infty(M, \mathbb{R})$.*

The uniqueness of such solutions was proved by Calabi.

**Theorem 3.1** (Calabi)**.** *If $\lambda \leq 0$, the solutions to (CMA) are unique.*

*Proof.* Suppose $\varphi_1$ and $\varphi_2$ are two solutions to (CMA). We let $\omega_i = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_i$ and $\psi = \varphi_2 - \varphi_1$.

- **Case-1.** $\lambda < 0$. Then $\psi$ solves the equation

$$(\omega_1 + \sqrt{-1}\partial\overline{\partial}\psi)^n = e^{-\lambda\psi}\omega_1^n.$$

If $p$ is a maxima of $\psi$, then by the maximum principle $\sqrt{-1}\partial\overline{\partial}\psi(p) \leq 0$, and so

$$e^{-\lambda\psi(p)}\omega_1^n = (\omega_1 + \sqrt{-1}\partial\overline{\partial}\psi)^n \leq \omega_1^n,$$

and hence $-\lambda\psi(p) \leq 0$ or equivalently (since $\lambda < 0$) $\psi(p) \leq 0$. This shows that $\varphi_2(x) \leq \varphi_1(x)$ for all $x \in M$. By symmetry, the reverse is also true and we obtain that $\varphi_1 \equiv \varphi_2$.

- **Case-2.** $\lambda = 0$. Now $\psi$ solves the equation

$$(\omega_1 + \sqrt{-1}\partial\overline{\partial}\psi)^n = \omega_1^n.$$

Subtracting the two sides, multiplying by $\psi$ and integrating, we obtain

$$\begin{aligned}
0 &= \int_M \psi[(\omega_1 + \sqrt{-1}\partial\overline{\partial}\psi)^n - \omega_1^n] \\
&= \int_M \psi\sqrt{-1}\partial\overline{\partial}\psi \wedge \sum_{j=0}^{n-1} \omega_2^j \wedge \omega_1^{n-1-j} \\
&= -\int_M \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \sum_{j=0}^{n-1} \omega_2^j \wedge \omega_1^{n-1-j} \\
&\leq -\int_M \sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega_1^{n-1} \\
&= -n^{-1}\int_M |\nabla_1\psi|^2\omega_1^n,
\end{aligned}$$

where we used Lemma 3.2 below in line 4 and Remark 2.2 in line 5. Since the right side is always non-positive, this forces $\psi$ to be a constant. But since $\sup_M \varphi_1 = \sup_M \varphi_2 = 0$, this means that $\varphi_1 \equiv \varphi_2$.

□

**Lemma 3.2.** *For any positive real $(1,1)$ forms $\omega$ and $\hat{\omega}$, and any $\psi \in C^\infty(M,\mathbb{R})$,*

$$\frac{\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega^j \wedge \hat{\omega}^{n-1-j}}{\hat{\omega}^n} \geq 0.$$

*Proof.* By applying a unitary transformation, we can assume that at a point $p$,

$$g_{i\bar{j}} = \lambda_i\delta_{i\bar{j}},\ \hat{g}_{i\bar{j}} = \delta_{i\bar{j}},$$

where each $\lambda_i > 0$. Then one can check that

$$\frac{\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega^j \wedge \hat{\omega}^{n-1-j}}{\hat{\omega}^n} = \sum_{i,i_1,\cdots,i_j} |\psi_i|^2\lambda_{i_1}\cdots\lambda_{i_j} \geq 0.$$

□

3.1. **Continuity method.** For $t \in [0,1]$, consider

$$(*_t) \qquad \begin{cases} (\hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = c_t e^{tF-\lambda\varphi}\hat{\omega}^n \\ \omega_t := \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_t > 0 \\ (\lambda = 0)\ \sup_M \varphi_t = 0, \end{cases}$$

where $c_t$ are the constants

$$c_t = \begin{cases} 0,\ \lambda < 0 \\ \Big(\int_M \hat{\omega}^n\Big)\Big(\int_M e^{tF}\hat{\omega}^n\Big)^{-1},\ \lambda = 0. \end{cases}$$

Note that $e^{-\sup_M F} \leq c_t \leq e^{-\inf_M F}$, and that $c_t$ is continuous in $t$. Hence the constants are pretty mild and do not cause any trouble in the estimates. We let

$$I := \{t \in [0,1] \mid (*_t) \text{ has a smooth solution}\}.$$

The proof of Theorem C then consists of three parts

- **Step-1.** $I$ is non-empty. This is trivial since $\varphi \equiv 0$ is a solution to $(*_t)$ at $t = 0$ and hence $0 \in I$.
- **Step-2.** $I$ is open. This is accomplished via an inverse function theorem and a perturbation argument.
- **Step-3.** $I$ is closed. That is, if $t_k \in I$ such that $t_k \to T$, then $T \in I$. This is done via obtaining *a priori* estimates and applying Arzela-Ascoli theorem.

3.2. **Openness.**

**Proposition 3.3.** *If $(*_t)$ has a smooth solution $\varphi_{t_0}$ at $t = t_0$, then there exists an $\varepsilon > 0$ such that $(*_t)$ has a smooth solution for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1]$.*

The proof relies on the following infinite dimensional inverse function theorem.

**Theorem 3.4** (Inverse function theorem (abbr. IFT))**.** *Let $X$ and $Y$ be Banach manifolds and let $\mathcal{M} : X \to Y$ be a $C^1$ Frechet differentiable map. Suppose*

(1) $\mathcal{M}(x_0) = y_0$.
(2) $D_{x_0}\mathcal{M} : \mathcal{T}_{x_0}X \to \mathcal{T}_{y_0}Y$ *is an isomorphism of Banach spaces.*

*Then there exist neighbourhoods $x_0 \in U \subset X$ and $y_0 \in V \subset M$ and a $C^1$ Frechet differentiable map $\mathcal{G} : V \to U$ such that $\mathcal{M}(\mathcal{G}(y)) = y$ for all $y \in V$. In particular, for every $y \in V$, there exists a solution to $\mathcal{M}(x) = y$ in $U$.*

Banach manifolds are essentially topological spaces that are locally homeomorphic to isomorphic Banach spaces. A Frechet derivative is also defined in the usual way. Even though we will not define these notions rigorously here, they will be self evident when we use them in the proof below.

*Proof of Proposition 3.3.* We will give the complete proof in the case of $\lambda = 0$. The other case is even easier, and we only indicate the main steps.

- **Case-1.** $\lambda = 0$. Then $\varphi_{t_0}$ satisfies
$$(\hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_{t_0})^n = c_{t_0} e^{t_0 F}\hat{\omega}^n.$$
We set $\omega_{t_0} = \hat{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_{t_0}$. For $|t - t_0| << 1$, we wish to find a solution to $(*_t)$ of the form $\varphi_t = \varphi_{t_0} + \psi_t$. Then $\varphi_t$ solves $(*_t)$ if and only if $\psi_t$ solves
$$(\omega_{t_0} + \sqrt{-1}\partial\overline{\partial}\psi_t)^n = \frac{c_t}{c_{t_0}} e^{(t-t_0)F}\omega_{t_0}^n.$$

So consider the mapping

$$\mathcal{M}(\psi) = \frac{(\omega_{t_0} + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\omega_{t_0}^n}.$$

Clearly $\mathcal{M}(0) = 1$, and our goal is to solve $\mathcal{M}(\psi_t) = f_t := \frac{c_t}{c_{t_0}} e^{(t-t_0)F}$ for $|t - t_0| << 1$. The function spaces we need are as follows

$$C_0^{k,\alpha}(M) := \{\psi \in C^{k,\alpha}(M) \mid \int_M \psi\omega_{t_0}^n = 0\}$$

$$C_1^{k,\alpha}(M) := \{f \in C^{k,\alpha}(M) \mid \int_M f\omega_{t_0}^n = \int \omega_{t_0}^n\}.$$

Since

$$\int_M \mathcal{M}(\psi)\omega_{t_0}^n = \int_M (\omega_{t_0} + \sqrt{-1}\partial\overline{\partial}\psi)^n = \int_M \omega_{t_0}^n,$$

it is clear that $\mathcal{M} : C_0^{3,\alpha}(M) \to C_1^{1,\alpha}(M)$. Next, we observe that $\mathcal{T}_\psi C_0^{3,\alpha}(M) = C_0^{3,\alpha}(M)$ and $\mathcal{T}_f C_1^{1,\alpha}(M) = C_0^{1,\alpha}(M)$. To apply the inverse function theorem, we need to compute the derivative.

**Claim-1.** $D_0\mathcal{M}(\eta) = \Delta_{\omega_{t_0}}\psi$.

*Proof.*

$$\begin{aligned} D_0\mathcal{M}(\eta) &= \frac{d}{ds}\Big|_{s=0}\mathcal{M}(s\eta) \\ &= \frac{d}{ds}\Big|_{s=0}\frac{(\omega_{t_0} + s\sqrt{-1}\partial\overline{\partial}\eta)^n}{\omega_{t_0}^n} \\ &= \frac{d}{ds}\Big|_{s=0}\frac{\omega_{t_0}^n + sn\sqrt{-1}\partial\overline{\partial}\eta \wedge \omega_{t_0}^{n-1} + O(s^2)}{\omega_{t_0}^n} \\ &= \frac{n\sqrt{-1}\partial\overline{\partial}\eta \wedge \omega_{t_0}^{n-1}}{\omega_{t_0}^n} \\ &= \Delta_{\omega_{t_0}}\psi. \end{aligned}$$

□

By Corollary **??**, $\Delta_{t_0} : \mathcal{T}_0 C_0^{3,\alpha}(M) = C_0^{3,\alpha}(M) \to \mathcal{T}_{f_1} C_1^{1,\alpha}(M) = C_0^{1,\alpha}(M)$ is an isomorphism, and so by the inverse function theorem there exist $C^{1,\alpha}$ neighbourhood $V$ of $f_t$ and $C^{3,\alpha}$ neighbourhood $U$ of $\psi$ such that for all $f \in V$, there exists a solution to $\mathcal{M}(\psi) = f$. In particular, there exists a solution $\psi_t$ to $\mathcal{M}(\psi_t) = f_t$ if $|t - t_0| << 1$. Moreover, since the inverse $\mathcal{G}$ is continuous, if we choose $|t - t_0| << 1$, we can also ensure that $\omega_{t_0} + \sqrt{-1}\partial\overline{\partial}\psi_t > 0$. We then put $\varphi = \varphi_{t_0} + \psi_t - \sup_M(\varphi_{t_0} + \psi_t)$. Clearly $\varphi_t$ is a $C^{3,\alpha}$ solution to $(*_t)$. We claim that it is in fact smooth by using the standard method of *bootstrapping*. Since this is a local issue, it suffices to prove that the restriction to a coordinate neighbourhood is smooth. Taking log on both sides of $(*_t)$ and differentiating with respect to $\partial_k$ we obtain

$$(g_t)^{i\bar{j}}\partial_i\partial_{\bar{j}}(\partial_k\varphi_t) = -(g_t)^{i\bar{j}}\hat{g}_{i\bar{j};k} + \partial_k H,$$

where $H$ is some smooth function. Since $\varphi_t$ is in $C^{3,\alpha}$, and $g_t > 0$, we also have that $g_t^{-1}$ is in $C^{1,\alpha}$. Hence the right hand side is also in $C^{1,\alpha}$. Then

$\partial_k \varphi_t$ satisfies and elliptic equation with coefficients in $C^{1,\alpha}$ and right hand side in $C^{1,\alpha}$. So by local Schauder estimates $\partial_k \varphi_t$ is in $C^{3,\alpha}$. Similarly $\partial_{\bar{l}} \varphi_t$ is in $C^{3,\alpha}$, and so $\varphi_t$ is in fact in $C^{4,\alpha}$. Going back, $g_t^{-1}$ is now in $C^{2,\alpha}$, and hence $\varphi_t$ in turn is in $C^{5,\alpha}$, and so on. This shows that $\varphi_t$ is in fact smooth.

- **Case-2.** $\lambda < 0$. In this case we can in fact work with the usual (unnormalised) Hölder spaces and define the map $\mathcal{M} : C^{3,\alpha} \to C^{1,\alpha}$ by
$$\mathcal{M}(\psi) = \log\Big(\frac{(\omega_{t_0} + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\omega_{t_0}^n}\Big) + \lambda\psi.$$
The linearisation is then $\Delta_{t_0} + \lambda$, which is invertible by Corollary **??** since $\lambda < 0$. The rest of the argument, including bootstrapping, is identical.

□

## 4. Lecture-4: A priori estimates

In the previous lecture, we set up a continuity method to prove Theorem C and proved that $I$ is an open set. To complete the proof of Theorem C, we need to prove some *a priori* estimates. For the sake of notational convenience we drop the parameter $t$, and instead consider the following Monge-Ampere equation:

$$\text{(CMA)} \qquad \begin{cases} (\hat{\omega}+\sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F-\lambda\varphi}\hat{\omega}^n \\ \omega := \hat{\omega}+\sqrt{-1}\partial\overline{\partial}\varphi > 0 \\ (\lambda=0)\ \sup_M \varphi = 0 \end{cases}$$

The estimates we need are as follows. All the geometric quantities with respect to $\hat{\omega}$ are denoted by a hat.

**Proposition 4.1.** *Let $(M,\hat{\omega})$ be a compact Kähler manifold. Suppose a Kähler metric $\omega$ satisfies*

$$Ric(\omega) = \lambda\omega + \rho \tag{1}$$

$$\Lambda^{-1}\hat{\omega} < \omega < \Lambda\hat{\omega}, \tag{2}$$

*for some $\Lambda > 0$. Then there exists a constant $C$ depending only on $(M,\hat{\omega})$, $\Lambda$ and $|\nabla\rho|^2_{\hat{\omega}}$ such that*

$$|\hat{\nabla}\omega|^2_{\hat{\omega}} \le C,$$

*where $\hat{\nabla}$ is the Levi-Civita connection with respect to $\hat{\omega}$.*

**Proposition 4.2.** *Suppose $\varphi \in C^\infty(M,\mathbb{R})$ solves (CMA). Then there exists constants $C, A > 0$ depending only on $(M,\hat{\omega})$, $\sup_M |F|$ and $\inf_M \hat{\Delta}F$ such that*

$$C^{-1}e^{-A(\varphi-\inf_M\varphi)}\hat{\omega} \le \omega \le Ce^{A(\varphi-\inf_M\varphi)}\hat{\omega}.$$

*In particular, if $||\varphi||_{C^0} \le c_0$, then $\omega$ satisfies (2) with $\Lambda = Ce^{Ac_0}$.*

**Proposition 4.3.** *If $\varphi \in C^\infty(M,\mathbb{R})$ solves (CMA) and $\lambda \le 0$, then there exists $C$ depending on $(M,\hat{\omega})$ and $||F||_{C^0}$ such that*

$$||\varphi||_{C^0(M)} \le C.$$

Assuming these, we can now complete the proof of Theorem C.

*Proof of Theorem C.*. Recall the continuity method

$$(*_t) \qquad \begin{cases} (\hat{\omega}+\sqrt{-1}\partial\overline{\partial}\varphi_t)^n = c_t e^{tF-\lambda\varphi_t}\hat{\omega}^n \\ \omega_t := \hat{\omega}+\sqrt{-1}\partial\overline{\partial}\varphi_t > 0 \\ (\lambda=0)\ \sup_M \varphi_t = 0. \end{cases}$$

We have already proved that $I := \{t \in [0,1] \mid (*_t) \text{ has a smooth solution}\}$ is non-empty and open. To complete the proof, we need to show that this set is also closed. So let $t_k \in I$ such that $t_k \to T$, and let $\varphi_k := \varphi_{t_k}$ solve the equation $(*_t)$ at $t = t_k$. Then Propositions 4.1-4.3 imply that there exists a constant $C$ such that $|\hat{\nabla}\omega|^2_{\hat{\omega}} \le C$. We then have

$$|\hat{\nabla}\hat{\Delta}\varphi_k|^2 \le |\hat{\nabla}\bar{\hat{\nabla}}\hat{\nabla}\varphi_k|^2 \le |\hat{\nabla}\omega|^2_{\hat{\omega}} \le C,$$

and hence,$if\alpha \in (0,1)$, we have $||\hat{\Delta}\varphi_k||_{C^\alpha(\hat{\omega})} \le C$. But then by Schauder estimates and Proposition 4.3 we have a uniform bound on $||\varphi_k||_{C^{2,\alpha}(\hat{\omega})}$.

**Claim.** $||\varphi_k||_{C^{3,\alpha}(\hat{\omega})}$ is uniformly bounded.

*Proof.* It is enough to obtain local bounds. Taking log and differentiating equation ($*_t$) with respect to $\partial_a$

$$(g_{t_k})^{i\bar{j}}\partial_i\partial_{\bar{j}}(\partial_a)\varphi_k + \lambda(\partial_a\varphi_k) = -(g_{t_k})^{i\bar{j}}\hat{g}_{i\bar{j};a} + \partial_a H,$$

where $H$ is a given smooth function, depending on $\hat{g}$ and $F$. Since $||\varphi_k||_{C^{2,\alpha}(\hat{\omega})}$ is uniformly bounded, and $g_{t_k} \geq \Lambda^{-1}\hat{g}$, $||g_{t_k}||_{C^{\alpha}(\hat{\omega})}$ is uniformly bounded. By Schauder estimates $\partial_a\varphi_k$ has uniformly bounded $C^{2,\alpha}(\hat{\omega})$ norm. One can argue similarly for $\partial_{\bar{b}}\varphi_k$, and hence $\varphi_k$ has uniformly bounded $C^{3,\alpha}(\hat{\omega})$ norms. □

Now, by Arzela-Ascoli, $\varphi_k \to \varphi_T \in C^{3,\alpha'}(\hat{\omega})$ for some $\alpha' < \alpha$. Clearly $\varphi_T$ satisfies ($*_t$) at $t = T$, and once again by a bootstrapping argument similar to the one in the proof of Proposition 3.3, we have that $\varphi_T \in C^\infty(M, \mathbb{R})$. □

4.1. **$C^2$-estimates.** Note that since $\omega$ and $\hat{\omega}$ are positive, $\omega \leq \Lambda\hat{\omega}$ if and only if $\mathrm{tr}_{\hat{\omega}}\omega \leq C$. Hence it is sufficient to estimate $\mathrm{tr}_{\hat{\omega}}\omega$ and $\mathrm{tr}_{\omega}\hat{\omega}$. The key estimate we need is the following.

**Lemma 4.4.** *There exists constants $B, C > 0$ depending on $(M, \hat{\omega})$, and $\inf_M \hat{\Delta}F$ such that*

$$\Delta \log \mathrm{tr}_{\hat{\omega}}\omega \geq -B\mathrm{tr}_{\omega}\hat{\omega} - C.$$

Assuming this we can complete the proof of the required $C^2$-estimate.

*Proof of Proposition 4.2.* Since $\mathrm{tr}_{\omega}\hat{\omega} = n - \Delta\varphi$, from the Lemma if we set $A = B+1$, we have

$$\Delta(\log \mathrm{tr}_{\hat{\omega}}\omega - A\varphi) \geq \mathrm{tr}_{\omega}\hat{\omega} - C,$$

for some constant $C$. Suppose $\log \mathrm{tr}_{\hat{\omega}}\omega - A\varphi$ takes the maximum value at $p \in M$, then by the maximum principle, $\mathrm{tr}_{\omega}\hat{\omega}(p) \leq C$. We can assume that $\hat{\omega}$ at $p$ is Euclidean, and $\omega$ at $p$ is diagonal with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} \leq C.$$

In particular, there exists a constant $c_0$ such that $\lambda_i > c_0$ for each $i$. From the equation $\omega^n = e^{F-\lambda\varphi}\hat{\omega}^n$, the determinant $\Pi_{i=1}^n\lambda_i$ is uniformly bounded in terms of $||F||_{C^0}$ and $e^{-\lambda\varphi}$, and hence there is uniform upper bound $\lambda_i \leq Ce^{-\lambda\varphi(p)}$. In particular, $\mathrm{tr}_{\hat{\omega}}\omega(p) \leq Cne^{-\lambda\varphi(p)}$.

But then, since $p$ is a point of maximum for $\log \mathrm{tr}_{\hat{\omega}}\omega - A\varphi$, we have

$$\log \mathrm{tr}_{\hat{\omega}}\omega \leq A\varphi + \log C - \lambda\varphi(p) - A\varphi(p),$$

and so possibly by increasing $A$ a bit,

$$\mathrm{tr}_{\hat{\omega}}\omega \leq Ce^{A(\varphi - \inf_M \varphi)}.$$

Again applying the same reasoning as above, and possibly increasing $A$, we also have the reverse inequality

$$\mathrm{tr}_{\omega}\hat{\omega} \leq Ce^{A(\varphi - \inf_M \varphi)}.$$

□

*Proof of Lemma 4.4.* We compute at a point $p \in M$ using normal coordinates with respect to $\hat{\omega}$. That is, at $p$

$$\hat{g}_{i\bar{j}} = \delta_{i\bar{j}},\ \hat{g}_{i\bar{j};k} = \hat{g}_{i\bar{j};\bar{l}} = 0.$$

By a unitary change of coordinates, we can also assume that $\omega$ at $p$ is given by the diagonal matrix with positive eigenvalues $\{\lambda_i\}$. Putting $u = \mathrm{tr}_{\hat{\omega}}\omega$ we have

$$\Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}.$$

We now compute

$$\begin{aligned} \Delta u &= g^{i\bar{j}}\partial_i\partial_{\bar{j}}(\hat{g}^{k\bar{l}}g_{k\bar{l}}) \\ &= g^{i\bar{j}}\partial_i(-\hat{g}^{k\bar{b}}\hat{g}_{a\bar{b};\bar{j}}\hat{g}^{a\bar{l}}g_{k\bar{l}} + \hat{g}^{k\bar{l}}g_{k\bar{l};\bar{j}}) \\ &= -g^{i\bar{j}}\hat{g}^{k\bar{b}}\hat{g}_{a\bar{b};i\bar{j}}\hat{g}^{a\bar{l}}g_{k\bar{l}} + g^{i\bar{j}}\hat{g}^{k\bar{l}}g_{k\bar{l};i\bar{j}} \end{aligned}$$

For the first term on the right,

$$\begin{aligned} -g^{i\bar{j}}\hat{g}^{k\bar{b}}\hat{g}_{a\bar{b};i\bar{j}}\hat{g}^{a\bar{l}}g_{k\bar{l}} &= -g^{i\bar{i}}\hat{g}_{k\bar{k};i\bar{i}}g_{k\bar{k}} \\ &\geq -B\sum_{i,k} g^{i\bar{i}}g_{k\bar{k}} \\ &= -B(\mathrm{tr}_{\hat{\omega}}\omega)(\mathrm{tr}_{\omega}\hat{\omega}). \end{aligned}$$

Here $B$ is an upper bound on the numbers $\hat{g}_{k\bar{k};i\bar{i}}$, or more intrinsically, an upper bound on the holomorphic sectional curvature of $\hat{g}$, that is,

$$\hat{R}_{i\bar{j}k\bar{l}} \leq B(\hat{g}_{i\bar{j}}\hat{g}_{k\bar{l}} + \hat{g}_{i\bar{l}}\hat{g}_{k\bar{j}}).$$

For the second term, we recall that

$$R_{i\bar{j}k\bar{l}} = -g_{k\bar{l};i\bar{j}} + g^{a\bar{b}}g_{k\bar{b};i}g_{a\bar{l};\bar{j}},$$

and so

$$\begin{aligned} g^{i\bar{j}}\hat{g}^{k\bar{l}}g_{k\bar{l};i\bar{j}} &= -g^{i\bar{j}}\hat{g}^{k\bar{l}}R_{k\bar{l};i\bar{j}} + g^{i\bar{i}}g^{a\bar{a}}g_{k\bar{a};i}g_{a\bar{k};\bar{i}} \\ &= -\hat{g}^{k\bar{l}}R_{k\bar{l}} + g^{i\bar{i}}g^{a\bar{a}}g_{k\bar{a};i}g_{a\bar{k};\bar{i}}. \end{aligned}$$

Taking log and $\sqrt{-1}\partial\bar{\partial}$ on both sides of (CMA), we have that

$$-R_{k\bar{l}} = F_{k\bar{l}} - \lambda\varphi_{k\bar{l}} - \hat{R}_{k\bar{l}},$$

and so

$$\begin{aligned} -\hat{g}^{k\bar{l}}R_{k\bar{l}} &= \hat{\Delta}F - \lambda\hat{\Delta}\varphi - \hat{S} \\ &= \hat{\Delta}F - \lambda\hat{\Delta}\varphi - \hat{S} \\ &= \hat{\Delta}F - \lambda\mathrm{tr}_{\hat{\omega}}\omega + \lambda n - \hat{S} \\ &\geq -C\mathrm{tr}_{\hat{\omega}}\omega - C. \end{aligned}$$

Putting all of this together, we have

$$\Delta u \geq -B(\mathrm{tr}_{\hat{\omega}}\omega)(\mathrm{tr}_{\omega}\hat{\omega}) - C\mathrm{tr}_{\hat{\omega}}\omega - C + g^{i\bar{i}}g^{a\bar{a}}|g_{k\bar{a};i}|^2. \tag{3}$$

Finally, using the Cauchy Schwarz inequality

$$(\mathrm{tr}_{\hat{\omega}}\omega)(\mathrm{tr}_{\omega}\hat{\omega}) \geq n^2,$$

we have

$$\begin{aligned}\Delta \log u &\geq -B\mathrm{tr}_{\omega}\hat{\omega} - C - \frac{C}{\mathrm{tr}_{\hat{\omega}}\omega} + \frac{g^{i\bar{i}}g^{a\bar{a}}g_{k\bar{a};i}g_{a\bar{k};\bar{i}}}{u} - \frac{|\nabla u|^2}{u^2}\\ &\geq -(B + C/n^2)\mathrm{tr}_{\omega}\hat{\omega} - C + \frac{g^{i\bar{i}}g^{a\bar{a}}g_{k\bar{a};i}g_{a\bar{k};\bar{i}}}{u} - \frac{|\nabla u|^2}{u^2}.\end{aligned}$$

**Claim.** $\frac{g^{i\bar{i}}g^{a\bar{a}}|g_{k\bar{a};i}|^2}{u} - \frac{|\nabla u|^2}{u^2} \geq 0.$

*Proof.* This follows from two applications of the Cauchy Schwarz inequality. First, we compute

$$\begin{aligned}|\nabla u|^2 &= \sum_i g^{i\bar{i}}\partial_i(\mathrm{tr}_{\hat{\omega}}\omega)\overline{\partial_i(\mathrm{tr}_{\hat{\omega}}\omega)}\\ &= \sum_i g^{i\bar{i}}\partial_i(\hat{g}^{k\bar{l}}g_{k\bar{l}})\overline{\partial_i(\hat{g}^{a\bar{b}}g_{a\bar{b}})}\\ &= \sum_{i,k,a} g^{i\bar{i}}g_{k\bar{k};i}g_{a\bar{a};\bar{i}}\end{aligned}$$

Recall that $g_{i\bar{j}} = \lambda_i\delta_{i\bar{j}}$, and so in particular $g^{i\bar{i}} = \lambda_i^{-1}$, and

$$\begin{aligned}|\nabla u|^2 &= \sum_{i,k,a}\lambda_i^{-1}g_{k\bar{k};i}g_{a\bar{a};\bar{i}}\\ (g^{i\bar{i}}g^{a\bar{a}}|g_{k\bar{a};i}|^2)u &= \Big(\sum_{i,k,a}\lambda_i^{-1}\lambda_a^{-1}|g_{k\bar{a};i}|^2\Big)\Big(\sum_p\lambda_p\Big).\end{aligned}$$

We now estimate

$$\begin{aligned}|\nabla u|^2 &= \sum_{k,a}\sum_i\frac{g_{k\bar{k};i}}{\sqrt{\lambda_i}}\frac{g_{a\bar{a};\bar{i}}}{\sqrt{\lambda_i}}\\ &\leq \sum_{k,a}\Big(\sum_i\lambda_i^{-1}|g_{k\bar{k};i}|^2\Big)^{1/2}\Big(\sum_j\lambda_j^{-1}|g_{a\bar{a};j}|^2\Big)^{1/2}\\ &= \Big(\sum_a\Big(\sum_j\lambda_j^{-1}|g_{a\bar{a};j}|^2\Big)^{1/2}\Big)^2\\ &= \Big(\sum_a\lambda_a^{1/2}\Big(\sum_j\lambda_j^{-1}\lambda_a^{-1}|g_{a\bar{a};j}|^2\Big)^{1/2}\Big)^2\\ &\leq \Big(\sum_p\lambda_p\Big)\Big(\sum_a\sum_j\lambda_j^{-1}\lambda_a^{-1}|g_{a\bar{a};j}|^2\Big)\\ &\leq \Big(\sum_p\lambda_p\Big)\Big(\sum_{a,j,k}\lambda_j^{-1}\lambda_a^{-1}|g_{k\bar{a};j}|^2\Big)\end{aligned}$$

This completes the proof of the claim, and hence the proof of the lemma.

□

4.2. $C^3$**-estimate.** Just as in the case of the $C^2$ estimate, the proof of the $C^3$ estimate relies on the following differential inequality.

**Lemma 4.5.** *There exists a constant $C$ depending only on $(M,\hat{\omega})$, $||\nabla\rho||_{\hat{\omega}}$ and $\Lambda$ such that if $S = |\hat{\nabla}\omega|^2_{\omega}$, then*

$$\Delta S \geq -CS - C.$$

*Proof of Proposition 4.1.* Recall that we are assuming $\omega$ solves

$$Ric(\omega) = \lambda\omega + \rho$$
$$\Lambda^{-1}\hat{\omega} < \omega < \Lambda\hat{\omega},$$

As in the case of the $C^2$ estimate, we need a barrier function $u$ such that $\Delta u$ is roughly $S$ and $u$ is bounded. We take $u = \mathrm{tr}_{\hat{\omega}}\omega$. In the previous lecture we proved the following differential ienquality

$$\begin{aligned}\Delta u &\geq -B(\mathrm{tr}_{\hat{\omega}}\omega)(\mathrm{tr}_{\omega}\hat{\omega}) - C\mathrm{tr}_{\hat{\omega}}\omega - C + g^{i\bar{i}}g^{a\bar{a}}|g_{k\bar{a};i}|^2 \\ &\geq -C + g^{i\bar{i}}g^{a\bar{a}}|g_{k\bar{a};i}|^2.\end{aligned}$$

The second term is almost $S$. In fact in normal coordinates for $\hat{\omega}$, and diagonalising $\omega$, we have

$$S = g^{i\bar{i}}g^{a\bar{a}}g^{k\bar{k}}|g_{k\bar{a};i}|^2 \leq \Lambda g^{i\bar{i}}g^{a\bar{a}}|g_{k\bar{a};i}|^2,$$

and so

$$\Delta u \geq -C + \Lambda^{-1}S.$$

Then by Lemma 4.5

$$\Delta(S + (C+1)\Lambda u) \geq S - C.$$

An application of maximum principle then gives a uniform upper bound on $S$. This in turn gives a uniform upper bound on $|\hat{\nabla}\omega|^2_{\hat{\omega}} \leq \Lambda^3 S$, completing the proof of the Proposition. □

## 5. Lecture-5: $C^0$-estimate

We prove Proposition 4.3 in this lecture, thereby completing the proof of Calabi conjecture.

*Proof of Proposition 4.3.* The proof is a simple application of the maximum principle for the case $\lambda < 0$, while it is much more involved for the $\lambda = 0$.

- **Case-1 : $\lambda < 0$.** In this case $\omega$ solves
$$(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F-\lambda\varphi}\hat{\omega}^n.$$
Let $p \in M$ such that $\varphi(p) = \sup_M \varphi$. Then by the maximum principle we have $\sqrt{-1}\partial\bar{\partial}\varphi(p) \leq 0$, and so
$$\hat{\omega}^n(p) \geq e^{F-\lambda\varphi(p)}\hat{\omega}^n(p).$$
Thus $e^{-\lambda\varphi(p)} \leq e^{-\inf_M F}$ or $\varphi(p) \leq ||F||_{C^0}/(-\lambda)$. Similarly, we can obtain a lower bound for $\varphi$.
- **Case-2 : $\lambda = 0$.** For simplicity, we rescale $\hat{\omega}$ so that $\int_M \hat{\omega}^n = 1$, and set
$$\psi = \varphi - \int_M \varphi\hat{\omega}^n.$$
Then $\psi$ still satisfies
$$(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^F\hat{\omega}^n. \tag{4}$$
The proof relies on the elementary fact that
$$\sup_M |\psi| = \lim_{p\to\infty} ||\psi||_{L^p(\hat{\omega})},$$
and that the sequence of $L^p$ norms on the right is increasing. Multiplying (4) by $\psi|\psi|^{\alpha-1}$ and integrating by parts
$$\begin{aligned}\int_M \psi|\psi|^{\alpha-1}(\hat{\omega}^n - \omega^n) &= -\int_M \psi|\psi|^{\alpha-1}\sqrt{-1}\partial\bar{\partial}\psi \wedge \sum_{j=0}^{n-1}\omega_\psi^j \wedge \hat{\omega}^{n-1-j}\\ &= \alpha\int_M |\psi|^{\alpha-1}\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \sum_{j=0}^{n-1}\omega_\psi^j \wedge \hat{\omega}^{n-1-j}\\ &\geq \alpha\int_M |\psi|^{\alpha-1}\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \hat{\omega}^{n-1}\\ &= \alpha n\int_M |\psi|^{\alpha-1}|\hat{\nabla}\psi|^2\hat{\omega}^n\\ &= \frac{4n\alpha}{(\alpha+1)^2}\int_M |\hat{\nabla}\psi|\psi|^{\frac{\alpha-1}{2}}|^2\hat{\omega}^n\end{aligned}$$
Note that, just as in the uniqueness proof, we again used Lemma 3.2 in the third line. Now, the left hand side above can be bounded using the equation, and so we obtain, the inequality
$$\int_M |\hat{\nabla}\psi|\psi|^{\frac{\alpha-1}{2}}|^2\hat{\omega}^n \leq C\frac{(\alpha+1)^2}{4\alpha}\int_M |\psi|^\alpha\hat{\omega}^n. \tag{5}$$

By Sobolev inequality, if we let $p = \alpha + 1$, and $\beta = n/n - 1$, we get that if $p > 1$,

$$\begin{aligned}\Big(\int_M |\psi|^{p\beta}\hat{\omega}^n\Big)^{\frac{1}{\beta}} &\leq Cp\Big(\int_M |\psi|^{p-1}\hat{\omega}^n + \int_M |\psi|^p\hat{\omega}^n\Big)\\ &\leq Cp\Big(\Big(\int_M |\psi|^p\hat{\omega}^n\Big)^{\frac{p-1}{p}} + \int_M |\psi|^p\hat{\omega}^n\Big)\\ &\leq Cp\max\Big(1, \int_M |\psi|^p\hat{\omega}^n\Big),\end{aligned}$$

Taking the $p^{th}$ root,

(6) $$\max(1, ||\psi||_{L^{p\beta}(\hat{\omega})}) \leq (Cp)^{1/p}\max(1, ||\psi||_{L^p(\hat{\omega})}).$$

Let $p_0 = 2$, $p_{k+1} = p_k\beta$, and $A_k = \max(1, ||\psi||_{L^{p_k}(M,\hat{\omega})})$.

$$\begin{aligned}\log A_{k+1} &\leq \frac{\log C}{p_k} + \frac{\log p_k}{p_k} + \log A_k\\ &\leq \Big(\frac{\log C}{2} + \log 2\Big)\sum_{i=0}^{k}\frac{1}{\beta^i} + \log\beta\sum_{i=0}^{k}\frac{k}{\beta^k} + \log A_0\\ &\leq C + A_0,\end{aligned}$$

since all the series are convergent. Exponentiating and taking limit

$$\max(1, \sup_M |\psi|) = \lim_{k\to\infty} A_{k+1} \leq CA_0 = C\max\Big(1, ||\psi||_{L^2(\hat{\omega})}\Big).$$

To control the $L^2$ norm, by the Poincare inequality, inequality (5) with $\alpha = 1$, and Holder inequality

$$\int_M \psi^2\hat{\omega}^n \leq \int_M |\hat{\nabla}\psi|^2\hat{\omega}^n \leq C\int_M \psi\hat{\omega}^n \leq C\Big(\int_M \psi^2\hat{\omega}^n\Big)^{\frac{1}{2}},$$

and so $||\psi||_{L^2(\hat{\omega})} \leq C$. Going back to $\varphi$, we then have

$$\int_M \varphi\hat{\omega}^n - C \leq \varphi \leq \int_M \varphi\hat{\omega}^n + C.$$

Since $\sup_M \varphi = 0$, the right side gives a lower bound for integral of $\varphi$, and then the left side gives a lower bound on $\inf_M \varphi$, and we are done.

□

## 6. Lecture-6: The Fano case

We saw in the proof of Theorem C, there are two notable problems when $\lambda > 0$. Firstly, the openness argument does not work, since $\Delta + \lambda$ might have a kernel if $\lambda > 0$, and hence need not be invertible. In fact if $M$ is Fano and $\omega$ is Kähler-Einstein, then the kernel of $\Delta + 1$ corresponds precisely to holomorphic vector fields. The second problem is that the $C^0$-estimate does not go through. The openness issue is easy to fix, by simply choosing a different continuity method. So for the rest of the lecture, we will assume that $M$ is Fano, that is $c_1(M) > 0$, and attempt to solve the following equation:

(7) $$\mathrm{Ric}(\omega) = \omega.$$

If we now take a reference form $\alpha \in c_1(M)$, then by the $\sqrt{-1}\partial\overline{\partial}$-lemma, there exists a function $F$ such that

$$\mathrm{Ric}(\alpha) = \alpha + \sqrt{-1}\partial\overline{\partial}F.$$

Then $\omega = \alpha + \sqrt{-1}\partial\overline{\partial}\varphi$ solves (7) if and only if

$$\begin{cases}(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F-\varphi}\alpha^n \\ \omega := \alpha + \sqrt{-1}\partial\overline{\partial}\varphi > 0.\end{cases} \tag{8}$$

We consider the following continuity method

$$\begin{cases}(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = e^{F-t\varphi}\alpha^n \\ \omega_t := \rho + \sqrt{-1}\partial\overline{\partial}\varphi_t > 0.\end{cases} \tag{$**_t$}$$

At the level of Ricci curvature, the corresponding equation is

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\alpha. \tag{$***_t$}$$

We let

$$I = \{t \in [0,1] \mid (8) \text{ has a smooth solution}\}.$$

Then by Theorem C, $0 \in I$. For openness, we proceed as before. Suppose there is a solution $\varphi_{t_0}$ to ($**_t$) at $t = t_0$, then we consider the following map $\mathcal{M} : C^{3,\alpha} \to C^{1,\alpha}$:

$$\mathcal{M}(\psi) = \log \frac{(\omega_{t_0} + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\omega_{t_0}^n} + t\varphi.$$

Clearly $\mathcal{M}(0) = 0$. Then it is easy to check that

$$D_0\mathcal{M}(\psi) = \Delta_{\omega_{t_0}}\psi + t_0\psi.$$

To prove openness, we only have to show that the kernel of this map is trivial, and this is accomplished by the following Lemma.

**Lemma 6.1.** *If $\omega$ is a Kähler metric such that $\mathrm{Ric}(\omega) > t\omega$ and $\lambda_1$ is the first non-zero eigenvalue of the Laplacian, then $\lambda_1 > t$.*

*Proof.* By the standard Bochner formula (see Week-2 assignment 7(c)), and our assumption that $\mathrm{Ric}(\omega) > t\omega$,

$$\Delta|\nabla f|^2 = |\nabla\overline{\nabla} f|^2 + \langle\nabla\Delta f, \nabla f\rangle + R_{i\bar{j}}\nabla^i f\overline{\nabla^j f} \geq \langle\nabla\Delta f, \nabla f\rangle + t|\nabla f|^2.$$

If $f$ is any eigenfunction, that is $\Delta f + \lambda f = 0$ with $\lambda > 0$, then

$$\Delta|\nabla f|^2 \geq (t-\lambda)|\nabla f|^2.$$

The integral of the left hand side is zero, and hence $(t-\lambda)|\nabla f|^2 \leq 0$. Since $\lambda > 0$, $f$ is not a constant, and hence $\lambda > t$. □

Now, by ($***_t$), if $t_0 < 1$, $\mathrm{Ric}(\omega_{t_0}) > t_0\omega_{t_0}$ since $\alpha > 0$, and so by the Lemma $\Delta_{\omega_{t_0}} + t_0$ has not kernel, and hence is invertible. The following proposition is a consequence of the above discussion and propositions 4.2 and 4.1.

**Proposition 6.2** (Aubin, Yau)**.** *There exists a solution to (8) if there exists a constant $C$ such that for any solution of ($**_t$),*

$$||\varphi_t||_{C^0} \leq C.$$

**Remark 6.3.** Uniqueness of Kähler-Einstein metrics is obviously false, since Fano manifolds have plenty of holomorphic vector fields. So if $\omega$ is a KE metric and $\psi$ is a biholomorphism generated by a holomorphic vector field, then $\psi^*\omega$ is also a KE metric. In the late 1980s Bando-Mabuchi that this is the only obstruction to uniqueness. In particular, they showed that if $\omega_1$ and $\omega_2$ are two solutions to (7), then there exists a biholomorphism $\Psi$ such that $\omega_2 = \Psi^*\omega_1$. There proof actually involved solving the continuity backwards, and as a consequence one obtains a converse to the above Proposition. Namely, that if there exists a KE, then one can obtain a $C^0$ bound along the continuity method.

6.1. **Obstructions of Futaki and Matsushima and the YTD conjecture.** We denote the space of holomorphic vector fields on $M$ by $\eta(M)$. Locally any $\xi \in \eta(M)$ looks like

$$\xi = \xi^i \frac{\partial}{\partial z^i},$$

where each $\xi^\alpha$ is a local holomorphic function. We denote the space of biholomorphisms of $M$ by $\mathrm{Aut}(M)$, and it's identity component by $\mathrm{Aut}_0(M)$. One of the earliest obstructions to the existence of KE metrics on Fano manifolds was found by Matsushima in the 1950s.

**Proposition 6.4** (Matsushima)**.** *If $M$ is Fano and admits a Kähler-Einstein metric, then $\mathrm{Aut}_0(M)$ is reductive.*

**Corollary 6.5.** *$\mathbb{P}^2$ blown up at one or two points do not admit a KE.*

To describe the obstruction discovered by Futaki, we first observe that if $\omega$ is a Kähler metric, then $i_\xi\omega := g_{i\bar{j}}\xi^i$ is a $\overline{\partial}$-closed $(0,1)$ form, and since every Fano manifold has finite fundamental group (a consequence of Calabi conjecture), $H^{0,1}_{\overline{\partial}}(M,\mathbb{C}) = \{0\}$, and hence there exists a function $\theta_\xi \in C^\infty(M,\mathbb{C})$ (unique upto a constant) such that

$$\sqrt{-1}\overline{\partial}\theta_\xi = i_\xi\omega.$$

We then define the *Futaki invariant* by

$$\mathrm{Fut}_\omega(\xi) = \int_M \theta_\xi(\mathrm{Ric}(\omega) - \omega)\wedge\omega^{n-1}. \tag{9}$$

It turns out that the Futaki invariant is in fact independent of the metric chosen in $c_1(M)$.

**Lemma 6.6.** *If $\omega_1$ and $\omega_2$ are two metrics in $c_1(M)$, then for all $\xi \in \eta(M)$,*

$$\mathrm{Fut}_{\omega_2}(\xi) = \mathrm{Fut}_{\omega_1}(\xi).$$

*Hence we simply denote the Futaki invariant as $\mathrm{Fut}(\xi)$ without any reference to a particular Kähler metric, and as a consequence if $M$ admits a KE, then $\mathrm{Fut}(\cdot) \equiv 0$.*

**Conjecture 6.7.** *(Calabi) If $M$ is Fano and has no non-trivial holomorphic vector field, then it admits a Kähler-Einstein metric.*

The conjectured was prove for Kähler surfaces by Tian [7] in the late eighties. Unfortunately the above conjecture turned out to be false.

**Remark 6.8.** In 1997, Tian [8] proved that a certain Fano three-fold $M$, studied by Mukai-Umemera, admits complex structures with no holomorphic vector fields and also no KE. Enroute to proving this, he introduced the notion of $K$-stability which involved allowing the manifold to degenerate and computing a Futaki invariant on a possibly singular normal $\mathbb{Q}$-Fano variety. Since mid 80s, Yau had already been advocating that the obstruction to existence of KEs on Fano manifolds must be related to some algebro-geometric stability. Tian in his 1997 paper proceeded to conjecture that a Fano manifold admits a Kähler-Einstein metric if and only if it is $K$-stable. In early 2000's Donaldson extended the definition of $K$-stability to all pairs $(M, L)$ of complex manifolds polarised with ample line bundles, and conjectured that the existence of a constant scalar curvature Kähler metric in $c_1(L)$ (of which KE metrics are special cases where $L = K_M^*$) should be equivalent to $K$-stability. This (still open) central conjecture in the field is called the Yau-Tian-Donaldson conjecture. The interested reader can refer to the excellent book [5] for an introduction for this circle of ideas.

**Definition 6.9.** Let $M$ be a Fano manifold. A special degeneration of $M$ consists of an embedding $M \hookrightarrow \mathbb{P}^{N_r}$ by sections of $K_M^{-r}$ and a $C^*$ subgroup of $PGL(N, \mathbb{C})$ such that the limit $W := \lim_{t\to 0} M$ is a normal variety. The $\mathbb{C}^*$ action fixes $W$ and induces a holomorphic vector field $w$ on $W$. We say $M$ is $K$-stable if for all such special degenerations

$$\mathrm{Fut}(W, w) \geq 0,$$

with equality if and only if $W = g \cdot M$ for some $g \in PGL(N, \mathbb{C})$.

In 2012 the YTD conjecture was finally settled for Fano manifolds.

**Theorem 6.10** (Chen-Donaldson-Sun [2])**.** *A Fano manifold $M$ admits a Kähler-Einstein metric if it is $K$-stable.*

6.2. **Kähler-Einstein metrics along the smooth continuity method.** The only if part had already been established by Tian for manifolds with no holomorphic vector fields (in the same 1997 paper discussed above) and by Robert Berman in general. The method of Chen-Donaldson-Sun used a continuity method through Kähler-Einstein metrics with cone singularities. The main idea is to begin with a conical Kähler-Einstien metric with small cone angle $2\pi\beta$ along an anti-canonical divisor, and then to deform this cone angle to $2\pi$, thereby obtaining a *smooth* KE metric in the limit. After their paper appeared, there still remained a question as to whether one could prove Theorem 6.10 using the continuity method $(**_t)$. This program was completed in 2015 by Gabor Szekelyhidi and the author [3], by adapting the techniques developed by Chen-Donaldson-Sun. The advantage of using the smooth continuity method, as opposed to the conical continuity method is that we were able to obtain an equivariant version of the theorem, which has been particularly useful in finding new examples of Kähler-Einstein manifolds.

**Theorem 6.11.** *If $M$ is equivariantly $K$-stable, then there exists a solution to $(**_t)$ for all $t \in [0, 1]$. In particular, there exists a Kähler-Einstein metric.*

*A broad overview of the proof.* Suppose the continuity method fails for the first time at some time $T$. Then there exists a sequence $\varepsilon_0 < t_k \to T$ such that $\omega_k := \omega_{t_k}$ solve

$$\mathrm{Ric}(\omega_k) = t_k\omega_k + (1 - t_k)\alpha.$$

In particular, $\mathrm{Ric}(\omega_k) > \varepsilon_0\omega_k$. Since the volumes of all these metrics are fixed, it is well known, by a theorem of Gromov's, that $(M, \omega_k)$ converge in the Gromov-Hausdorff sense to a compact metric space $(Z, d)$. For sequences of Kähler-Einstein metrics, it was conjectured by Tian, and proved by Donaldson-Sun [4], that the limit is normal, projective variety. This had been a major stumbling block in proving the YTD conjecture for Fano manifolds. Indeed, in [2], Chen-Donaldson-Sun prove an analogous result for conical Kähler-Einstein metrics. For the smooth continuity method, Szekelyhidi [6] adapted the techniques of Chen-Donaldson-Sun and proved the following.

**Theorem 6.12.** *There exists a $r >> 1$ and embeddings $T_k : M \hookrightarrow \mathbb{P}^{N_r}$ by sections of $K_M^{-r}$ which are orthonormal with respect to hermitian metrics $h_k$, where $\omega_k = -\sqrt{-1}\partial\overline{\partial}\log h_k$. Moreover the flat limit $W$ of the family $T_k(M)$ is a normal projective variety homeomorphic to $(Z, d)$.*

The $W$ obtained above, is then a candidate for the central fibre of a destabilising special degeneration. A technical point is that $W$ is in the orbit closure of $PGL(N, \mathbb{C})$, the definition of $K$-stability requires the central fibre to be in the closure of a $\mathbb{C}^*$ subgroup. In [2], this is done by applying the Luna slice theorem from algebraic geometry to the pair $(W, \Delta)$, where $\Delta$ is a divisor in $W$ such that $W$ admits a KE metric with cone singularities along $\Delta$. In proving Theorem 6.11, one is forced to consider pairs $(W, \beta)$, where $\beta$ is a $(1,1)$ current on $W$. This space is of course infinite-dimensional and the Luna slice cannot be directly applied. This difficulty is overcome in [3] by approximating $\beta$ by currents that are concentrated along divisors, that is,

$$\beta \sim \sum_{i=1}^{K} [H_i \cap W],$$

and then applying Luna slice theorem to tuples $(W, H_1 \cap W, \cdots, H_K \cap W)$. $\square$

## Appendix A. Proof of Lemma 4.5

Though the original computation is due to Calabi himself, our proof follows the simplified computations due to Phong-Sturm-Sessum. We begin with a simple observation that if $H^k_{ij} := \Gamma^k_{ij} - \hat{\Gamma}^k_{ij} = g^{k\bar{l}}\hat{\nabla}_i g_{j\bar{l}}$, then

$$S := |\hat{\nabla}\omega|^2_\omega = g^{i\bar{j}}g^{k\bar{l}}g^{a\bar{b}}\hat{\nabla}_i g_{k\bar{b}}\overline{\hat{\nabla}_j g_{l\bar{a}}} = g^{i\bar{j}}g^{k\bar{l}}H^a_{ik}\overline{H^q_{jl}g_{q\bar{a}}} = g^{i\bar{j}}g^{k\bar{l}}g_{a\bar{q}}H^a_{ik}\overline{H^q_{jl}} = |H|^2,$$

where note that the norm is with respect to $\omega$. We compute using normal coordinates with respect to $\omega$. It is not difficult to see that

$$\Delta|H|^2 = |\nabla H|^2 + |\bar{\nabla}H|^2 + \mathrm{Re}\langle\nabla_a\nabla_{\bar{a}}H, H\rangle + \mathrm{Re}\langle\nabla_{\bar{a}}\nabla_a H, H\rangle,$$

where $\langle\nabla T, T\rangle = g_{p\bar{q}}g^{i\bar{j}}g^{k\bar{l}}T^p_{ik}\overline{T^q_{jl}}$ for any section $T$ of $T^{*(1,0)}M\otimes T^{*(1,0)}M\otimes T^{(1,0)}M$. Commuting the covariant derivatives, using the so-called Ricci identity (see Week-2 assignment for the corresponding formulae for one-forms and vector fields),

$$\nabla_a\nabla_{\bar{b}}H - \nabla_{\bar{b}}\nabla_a H = R_{a\bar{b}p}{}^kH^p_{ij} - R_{a\bar{b}i}{}^pH^k_{pj} - R_{a\bar{b}j}{}^pH^k_{ip},$$

and so

$$\begin{aligned}\nabla_{\bar{a}}\nabla_a H &= \nabla_a\nabla_{\bar{a}}H - R_p{}^kH^p_{ij} + R_i{}^pH^k_{pj} + R_j{}^pH^k_{ip}\\ &= \nabla_a\nabla_{\bar{a}}H - g^{k\bar{q}}R_{p\bar{q}}H^p_{ij} + g^{p\bar{q}}R_{i\bar{q}}H^k_{pj} + g^{p\bar{q}}R_{j\bar{q}}H^k_{ip}\end{aligned}$$

But from the equation $R_{p\bar{q}} = \lambda g_{p\bar{q}} + \rho_{p\bar{q}}$, and the assumption that $\Lambda^{-1}\hat{\omega} \le \omega \le \lambda\hat{\omega}$, we see that $\Lambda^{-1}\omega \le Ric(\omega) \le C\omega$, and hence

$$\mathrm{Re}\langle\nabla_{\bar{a}}\nabla_a H, H\rangle \ge \mathrm{Re}\langle\nabla_a\nabla_{\bar{a}}H, H\rangle - C|H|^2.$$

And so,

$$\Delta|H|^2 \ge 2\mathrm{Re}\langle\nabla_a\nabla_{\bar{a}}H, H\rangle - C|H|^2. \tag{10}$$

The advantage of having an barred covariant derivative first, is that since $H$ has only unbarred entries, covariant differentiation is the same as ordinary differentiation. To estimate the first term, we recall that $R_{i\bar{a}j}{}^k = -\Gamma^k_{ij;\bar{a}}$ (and a similar formula for $\hat{R}$) and hence we compute

$$\begin{aligned}\nabla_a\nabla_{\bar{a}}H^k_{ij} &= \nabla_a[\Gamma^k_{ij;\bar{a}} - \hat{\Gamma}^k_{ij;\bar{a}}]\\ &= -\nabla_a R_{i\bar{a}j}{}^k - \hat{\nabla}_a\hat{R}_{i\bar{a}j}{}^k + (\nabla_a - \hat{\nabla}_a)\hat{R}_{i\bar{a}j}{}^k\\ &= -\nabla_i R_{a\bar{a}j}{}^k - \hat{\nabla}_i\hat{R}_{a\bar{a}j}{}^k + (\nabla_a - \hat{\nabla}_a)\hat{R}_{i\bar{a}j}{}^k\\ &= -\nabla_i R_j{}^k - \hat{\nabla}_i\hat{R}_j{}^k + (\nabla_a - \hat{\nabla}_a)\hat{R}_{i\bar{a}j}{}^k,\end{aligned}$$

where we used the second Bianchi identity $\nabla_a R_{i\bar{a}j}{}^k = \nabla_i R_{a\bar{a}j}{}^k$ in the third line. Now the difference in the connections is precisely the quantity $H$ and hence

$$|\hat{\nabla}_i\hat{R}_j{}^k| + |(\nabla_a - \hat{\nabla}_a)\hat{R}_{i\bar{a}j}{}^k| \le C|H| + C.$$

For the first term,

$$\nabla_i R_j{}^k = (\nabla_i - \hat{\nabla}_i)R_j{}^k + \hat{\nabla}R_j{}^k.$$

Now, from the equation, and the fact that $\omega$ and $\hat{\omega}$ are equivalent, this term can be controlled by $C|H| + C$, where $C$ might depend on $|\hat{\nabla}\rho|$. Putting all of this together,

$$|\mathrm{Re}\langle\nabla_a\nabla_{\bar{a}}H^k_{ij}\rangle| \le C|H|^2 + C|H|,$$

and hence

$$\Delta|H|^2 \ge -C|H|^2 - C|H| - C,$$

from which the result follows since $|H|$ can be estimated by $C + C|H|^2$.

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