LECTURE-10

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Index of a curve

For a piecewise smooth (not necessarily close) curve, we defined the *index* or *winding number* around a point $p \notin \gamma$ by

$$n(\gamma, p) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-p}\,dz.$$

We have already seen that if γ is a circle traversed n number of times, then

$$n(\gamma, p) = \begin{cases} n, & p \text{ is inside the disc bounded by } \gamma \\ 0, & \text{otherwise} \end{cases}$$

We now argue that this number is a measure of the change in the argument along the curve. For simplicity, lets suppose that $p = 0$, and that the curve γ joins $u = re^{i\theta}$ to $v = re^{i\varphi}$ (note that γ need not be a circular arc), where $\theta, \varphi \in (-\pi, \pi)$. In particular, the curve lies in $\mathbb{C} \setminus \{z < 0\}$. On this domain, $1/z$ has a primitive, which we take to be the principal branch of the logarithm. Then by the fundamental theorem

$$n(\gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz = \log v - \log u = \frac{\varphi - \theta}{2\pi},$$

and so up to a factor of 2π, the index measures the change in the argument. The following theorem contains the basic properties of the index.

Theorem 1. *Let γ be any closed curve. Then*

1. *$n(\gamma, p)$ is an integer for any $p \in \mathbb{C} \setminus \gamma$.*
2. *$n(\gamma, p)$ is a continuous function on $\mathbb{C} \setminus \gamma$, and hence is locally constant.*
3. *If γ is any curve lying in the interior of a disc D, then $n(\gamma, p) = 0$ for all $p \in \mathbb{C} \setminus \overline{D}$.*

Proof. (1) If we could take holomorphic logarithms freely, and argument as above would suffice. Instead we will give a computational proof. Let $\gamma : [0, 1] \to \mathbb{C}$ be parametrization. Then

$$n(\gamma, p) = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t) - p}\,dt.$$

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Consider a function

$$g(s) = \frac{1}{2\pi i} \int_0^s \frac{\gamma'(t)}{\gamma(t) - p}\, dt.$$

This is a continuous function on $[0, 1]$, and is differentiable wherever $\gamma'(t)$ is continuous (and hence at all but finitely many points) with derivative

$$g'(s) = \frac{1}{2\pi i} \frac{\gamma'(t)}{\gamma(t) - p}.$$

Then, letting

$$G(s) = \exp(-2\pi i g(s))(\gamma(s) - p),$$

at all but finitely many points, $G'(s) = 0$. This shows that $G(s)$ is locally constant, which together with continuity, forces it to be a constant. In particular, $G(1) = G(0)$, from which it follows (since $g(0) = 1$ and $\gamma(0) = \gamma(1)$) that

$$e^{2\pi i g(1)} = \frac{\gamma(0) - p}{\gamma(1) - p} = 1.$$

So $g(1) = n(\gamma, p)$ has to be an integer.

(2) Continuity is easy to check since $p \notin \gamma$. Since the index is integer valued it has to be then locally constant.
(3) If $|p| >> 1$, then clearly $n(\gamma, p)$ can be made really small. But then since the index is locally constant, and $\mathbb{C} \setminus \overline{D}$ is connected, it ought to be zero for all $p \in \mathbb{C} \setminus \overline{D}$.

□

Remark 1. *There is a deeper reason that the index is always an integer, and a full explanation requires some knowledge of covering space theory. If $a \in \mathbb{C}$ does not lie on $\gamma : [0, 1] \to \mathbb{C}$, we can think of γ as a curve in $\mathbb{C}_a^* := \mathbb{C} \setminus \{a\}$. Then it follows from standard covering space theory that γ has a "lift" to a curve $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ such that $e^{2\pi i \tilde{\gamma}(t)} = \gamma(t) - a$. The relevant jargon is that* $\exp : \mathbb{C} \to \mathbb{C}^*$ *is a covering space map. Now additionally if γ is closed, then $\gamma(1) = \gamma(0)$, and hence $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer. On the other hand, by chain rule, $\gamma'(t) = 2\pi i e^{2\pi i \tilde{\gamma}(t)} \tilde{\gamma}'(t)$, and hence*

$$\tilde{\gamma}'(t) = \frac{1}{2\pi i} \cdot \frac{\gamma'(t)}{\gamma(t) - a}.$$

Integrating both sides we see that

$$\mathbb{Z} \ni \tilde{\gamma}(1) - \tilde{\gamma}(0) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t) - a}\, dt = \int_\gamma \frac{dz}{z - a} := n(\gamma, a),$$

and hence $n(\gamma, a)$ is an integer.

THE GENERALISED CAUCHY THEOREM

A *chain* is a formal linear combination of curves $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$, where $a_i \in \mathbb{Z}$ and each γ_i is a regular curve. We interpret $a\gamma$ as γ traversed a times if $a > 0$ and γ traversed in the the reverse direction $-a$ times if $a < 0$. Chains can be then added in an obvious way. The union of the set theoretic images of γ_j is called the *support* of γ and denoted by $\mathrm{Supp}(\gamma)$. A chain is called a *cycle* if each of it's components γ_j is a closed curve. We can extend the definition of the index $n(\gamma, p)$ for a cycle γ and a point p that does not lie on any of the components of γ. Then it has the same properties as in Theorem 1. Additionally, we have the linearity property that

$$n(\gamma_1 + \gamma_2, p) = n(\gamma_1, p) + n(\gamma_2, p).$$

We say that a chain γ is *homologous* to zero in Ω, and write $\gamma \sim_\Omega 0$ (or $\gamma \sim 0(mod\Omega)$), if for any point $a \in \Omega^c$, $n(\gamma, p) = 0$. We also say that γ_1 is homologous to γ_2 in Ω and write $\gamma_1 \sim_\Omega \gamma_2$ if $\gamma_1 - \gamma_2 \sim_\Omega 0$, or equivalently if $n(\gamma_1, p) = n(\gamma_2, p)$ for all $p \in \Omega^c$. Note that if $\Omega \subset \Omega'$ then $\gamma \sim_\Omega 0$ implies that $\gamma \sim_{\Omega'} 0$, but the converse need not be true as can be seen in the example below.

Example 1. *Consider the disc $D_3(0)$ and the annulus $A_{1,2}(0) := \{z \in \mathbb{C} \mid 1 < |z| < 3\}$. By Cauchy's theorem for discs, any curve γ in $D_3(0)$ is homologous to zero. On the other hand the curve $\gamma(t) = 2e^{2\pi i t},\ t \in [0, 1]$ is NOT homologous to zero in $A_{1,2}(0)$. This is because $n(\gamma, 0) = 1 \neq 0$.*

Now we are ready to state the most general form of Cauchy's theorem.

Theorem 2 (Generalised Cauchy's theorem)**.** *If $f : \Omega \to \mathbb{C}$ is holomorphic and $\gamma \sim_\Omega 0$, then*

$$\int_\gamma f(z)\,dz = 0.$$

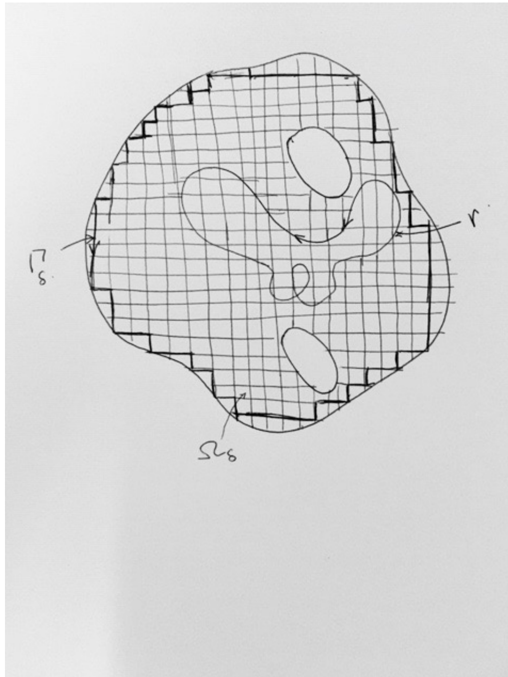
More generally, if γ_1, γ_2 are curves in Ω such that $\gamma_1 \sim_\Omega \gamma_2$, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

In other words, Cauchy's theorem says that if the integrals of the holomorphic functions $1/(z - a)$ is zero on a closed curve in Ω, then the integral of *any* holomorphic function on γ is zero.

Proof. We first assume that Ω is bounded. For a small $\delta > 0$, we cover the plane with a net consisting of squares with sides parallel to the axes, and length δ. Since Ω is bounded, and if $\delta > 0$ is chosen sufficiently small, there exists a finite set of number of cubes $Q_1, \cdots, Q_N$ such that

(1) The collection of squares $\{Q_1, \cdots, Q_j\}$ are all the squares in the net which lie completely inside Ω. That is, if Q is a square in the net, then $Q = Q_j$ for some j if and only if $Q \subset \Omega$.
(2) $\gamma \subset \Omega_\delta := \left(\cup_{j=1}^N Q_j \right)^\circ$.



Consider the cycle

$$\Gamma_\delta = \sum_j \partial Q_j,$$

where the boundary of each Q_j is oriented in the anti-clockwise direction. Then Γ_δ is *equivalent* to $\partial\Omega_\delta$ in the sense that for any function

$$\int_{\Gamma_\delta} f(z)\, dz = \int_{\partial\Omega_\delta} f(z)\, dz,$$

since the integrals over the common boundaries cancel.

Now, let γ be a cycle homologous to zero in Ω. Let $\zeta \in \Omega \setminus \Omega_\delta$, and let Q be a square in the net such that $\zeta \in Q$. By definition of Ω_δ, $Q \neq Q_j$ for any j. Again, by our choice of the squares that make up Ω_δ, there exists a point $\zeta_0 \in Q \cap \Omega^c$. Since $\zeta_0 \notin \Omega$ and $\gamma \sim_\Omega 0$, we have that $n(\gamma, \zeta_0) = 0$. But then by continuity, since ζ_0 and ζ can be joined by a straight in Q and hence not intersecting γ, we have $n(\gamma, \zeta) = 0$. In particular, $n(\gamma, \zeta) = 0$ for all $\zeta \in \partial\Omega_\delta$.

Suppose now that $f(z)$ is holomorphic on Ω. For any $z \in \mathring{Q}_{j_0}$, we have

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} f(z), \ j = j_0, \\ 0, \ \text{otherwise}, \end{cases}$$

and hence

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{0.1}$$

Since j_0 was arbitrary, (0.1) holds for all $z \in \cup_j Q_j^\circ$. But since both sides are continuous in z, clearly (0.1) must hold on all of Ω_δ. As a consequence,

$$\int_\gamma f(z)\,dz = \int_\gamma \Big(\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z}\,d\zeta\Big)\,dz.$$

By Fubini's theorem (which can be applied since the integrand is continuous in both z and ζ),

$$\int_\gamma f(z)\,dz = \int_{\Gamma_\delta} f(\zeta)\Big(\frac{1}{2\pi i}\int_\gamma \frac{dz}{\zeta - z}\Big)\,d\zeta = \int_{\Gamma_\delta} f(\zeta) n(\gamma,\zeta)\,d\zeta = 0.$$

Finally, if Ω is not bounded, consider a large disc $D_R(0)$ which contains $\mathrm{supp}(\gamma)$ in the interior, and let $\Omega' = \Omega \cap D_R(0)$. Then since $\gamma \sim_{D_R(0)} 0$, one can easily see that $\gamma \sim_{\Omega'} 0$, and the previous argument then applies to Ω' completing the proof. □

Using the generalised Cauchy theorem, we can prove the following generalisation of the CIF.

Theorem 3 (Generalised Cauchy integral formula (GCIF))**.** *Let $f \in \mathcal{O}(\Omega)$, and $\gamma \subset \Omega$ a cycle. If $\gamma \sim_\Omega 0$, then for any $z \in \Omega \setminus \mathrm{Supp}(\gamma)$,*

$$n(\gamma, z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof. Fix a $z \in \Omega\setminus\mathrm{Supp}(\gamma)$, and let $\varepsilon_0 > 0$ such that $D_{\varepsilon_0}(z)\cap\mathrm{Supp}(\gamma) = \phi$. For $\varepsilon \in (0, \varepsilon_0)$, let C_ε be the circle centred at z with radius ε.
Claim. For every $\varepsilon \in (0, \varepsilon_0)$, $\gamma \sim_{\Omega_z^*} n(\gamma, z)C_\varepsilon$, where as usual $\Omega_z^* = \Omega \setminus \{z\}$.

Proof of the Claim. We need to prove that for any $\zeta \in \mathbb{C} \setminus \Omega_z^*$,

$$n(\gamma,\zeta) = n(\gamma, z) n(C_\varepsilon, \zeta). \tag{0.2}$$

First suppose $\zeta \in \mathbb{C}\setminus\Omega$. Then in particular, ζ lies outside $D_\varepsilon(z)$ and hence $n(C_\varepsilon,\zeta) = 0$. On the other hand since $\gamma \sim_\Omega 0$, we also have $n(\gamma,\zeta) = 0$, and hence (0.2) is verified. The only other possibility is that $\zeta = z$. But then $n(C_\varepsilon, \zeta) = 1$, and hence again (0.2) is verified.

□

Now, applying Cauchy's theorem to the holomorphic function $f(\zeta)/(\zeta - z)$ on Ω_z^*, we see that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{n(\gamma, z)}{2\pi i}\int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{0.3}$$

The integral on the right is $n(\gamma, z)f(z)$ by the CIF for discs, and we are done. But it is in fact possible to avoid the CIF altogether, and in the process provide a second proof for the CIF on discs. The argument is as follows. Given any $\eta > 0$, by choosing $\varepsilon << 1$, we can make sure that

$$|f(\zeta) - f(z)| < \eta$$

for all $\zeta \in C_\varepsilon$. Then

$$\Big|\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z)\Big| = \Big|\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\Big| < \eta.$$

Hence from (0.3),

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = n(\gamma, z)\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f(\zeta)}{\zeta - z}\,d\zeta = n(\gamma, z)f(z).$$

□

Simply connected domains

There are many equivalent ways of defining simply connected domains. Following Ahlfors, we take a slightly non-standard route. Consider the sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

We denote it's north pole by $N = (0, 0, 1)$, and south pole by $S = (0, 0, -1)$. Then consider the stereographic projection $\Phi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ defined by

$$\Phi(x, y, z) = \frac{x + iy}{1 - z}.$$

Then Φ_N is a bijection, and is in fact a homeomorphism. We can thus identify $\mathbb{S}^2$ as a one-point compactification of $\mathbb{C}$, and call it the extended complex plane, and think of N as the "point at infinity".

We then say that a domain $\Omega \subset \mathbb{C}$ is *simply connected* if it is connected, *and* $\mathbb{S}^2 \setminus \Omega$ is also connected. We note that this is not a commonly used definition since it does not work in higher dimensions. In the next lecture, we will provide equivalent characterisations, one of which will be what is the modern "textbook" definition.

Example 2. *A disc, $\mathbb{C}$ itself, and half planes are simply connected. A parallel strip, say $\{z \mid a \le Im(\zeta) \le b\}$ is also simply connected. This shows the importance of taking the complement of Ω in the extended plane, rather than $\mathbb{C}$ itself. Similarly $\mathbb{C} \setminus \{\zeta \le 0\}$ is also simply connected. In this case the complement of the set in the extended plane is the complement of half a great circle in $\mathbb{S}^2$. On the other hand the complement of a line L passing through the origin in $\mathbb{C}$ in the extended plane is an entire great circle, and hence $\mathbb{C} \setminus L$ is not simply connected. Similarly, $\mathbb{C}^*$ is not simply connected, since $\mathbb{S}^2 \setminus \mathbb{C}^* = \{N, S\}$*

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