# LECTURE-11 

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## Characterisations of simply connected sets

Recall that a connected subset $\Omega \subset \mathbb{C}$ is called simply connected if the complement in the extended complex plane is also connected. We now provide several equivalent characterisations of being simply connected. First we need to introduce two important notions.

Recall that a curve $\gamma:[a, b] \rightarrow \Omega$ is a simple, closed curve if $\gamma$ is injective on $(a, b)$ and $\gamma(a)=\gamma(b)$. Such curves are called Jordan curves, and their name stems from the following historically significant theorem.

Theorem 1 (Jordan curve theorem). Let $\gamma$ be a Jordan curve and $C$ be it's image. Then it's complement $\mathbb{C} \backslash C$ consists of exactly two open connected subsets. One of these components is bounded while the other is unbounded.

The bounded component is called the interior and the unbounded component is called the exterior, denoted respectively by $\operatorname{int}(\gamma)$ and $\operatorname{ext}(\gamma)$. While intuitively obvious, the proof is extremely non-trivial. So much so that the the theorem is notorious for numerous incorrect proofs from well known mathematicians. In fact it will not be an exaggeration to say that attempts to prove this theorem led to the modern development of algebraic topology.

We also need to introduce the notion of homotopy. We say that a closed piecewise regular curve $\gamma:[0,1] \rightarrow \Omega$ is contractible, or null homotopic in $\Omega$ if there exists a point $p \in \Omega$ and a continuous function $H:[0,1] \times[0,1] \rightarrow \Omega$ such that

$$
\left\{\begin{array}{l}
H(0, t)=\gamma(t), H(1, t)=p, \forall t \in[0,1] \\
H(s, 0)=H(s, 1), \forall s \in[0,1]
\end{array}\right.
$$

We use the notation $\gamma_{s}(t):=H(s, t)$.
Example 1. (1) Let $D$ be a disc centred at $p$. Then for every curve $\gamma$, consider the homotopy

$$
H(s, t)=(1-s)(\gamma(t)-p)+p .
$$

Then $H(1, t)=p$ and hence $\gamma$ is null homotopic. More generally any convex domain has the property that every closed curve is null homotopic. To see this, let $\gamma$ be an aribitrary closed curve. Then

[^0](2) On the other hand, consider the curve $\gamma(t)=e^{i t}$ in the annulus $A_{0,2}(0):=\{z|0<|z|<2\}$. Then $\gamma$ is not null-homotopic, as can for instance be seen using the theorem below.

We then have the following fundamental result.
Theorem 2. Let $\Omega \subset \mathbb{C}$ be a connected set. Then the following are equivalent.
(1) $\Omega$ is simply connected.
(2) For every Jordan curve $\gamma$ in $\Omega$, int $(\gamma) \subset \Omega$.
(3) For all cycles $\gamma$ in $\Omega, \gamma \sim_{\Omega} 0$.
(4) (Cauchy theorem for simply connected domains) For all holomorphic functions $f \in \mathcal{O}$ and all cycles $\gamma$ in $\Omega$,

$$
\int_{\gamma} f(z) d z=0 .
$$

(5) Every closed piecewise regular curve is null homotopic.

We need to use the following crucial lemma which we state without proof.
Lemma 1. Let $\gamma$ be a piecewise regular curve which is null homotopic. Then one can choose the homotopy $H$ such that each $\gamma_{s}$ is piecewise regular.

Proof. The implication $(2) \Longleftrightarrow(3)$ is a consequence of the generalised Cauchy theorem. W

- (1) $\Longrightarrow(2)$. We can assume that $\Omega \neq \mathbb{C}$ for else the implication is trivial. Suppose $a \in \operatorname{int}(\gamma) \cap \Omega^{c}$. Then since $\Omega$ is simply connected, $\mathbb{S}^{2} \backslash \Omega$ is connected, and hence there exists a $p \in \Omega^{c} \cap \operatorname{ext}(\gamma)$, and a path $\sigma$ lying in $\Omega^{c}$ and connecting $a$ to $p$. But since $\operatorname{int}(\gamma)$ is connected this is a contradiction.
- (1) $\Longrightarrow(3)$. Let $\gamma$ be a cycle in $\Omega$ and $p \notin \Omega$. Since $\mathbb{S}^{2} \backslash \Omega$ is connected, there is a sequence of points $p_{n}$ such that $\left|p_{n}\right| \rightarrow \infty$ and there is a path $\sigma_{n}$ from $p$ to $p_{n}$. Since $\lim _{n \rightarrow \infty} n\left(\gamma, p_{n}\right)=0$ and index is locally constant, this implies that $n(\gamma, p)=0$.
- (3) $\Longrightarrow$ (1). Suppose $\Omega$ is not simply connected. Then $\mathbb{S}^{2} \backslash \Omega=$ $A \cup B$, where $B$ is the component at infinity, and $A$ is a compact (possibly disconnected) set. Let

$$
\delta:=\inf \{|z-w| \mid z, \in A, w \in B\} .
$$

Then $\delta>0$. Now we cover the entire plane with a net $\mathcal{N}$ of squares of a fixed side length $\delta / 4$ (any side length strictly smaller than $\delta / \sqrt{2}$ will do). We choose the net so that a certain square, say $Q_{1}$ has the point $a \in A$ at it's centre. Let $Q_{1}, \cdots, Q_{N}$ be the squares whose interiors have a non-empty intersection with $A$, and let

$$
\Gamma=\partial\left(\underset{2}{\left.\cup_{j=1}^{N} Q_{j}\right)}\right.
$$

oriented in an anticlockwise direction. Note that

$$
n\left(\partial Q_{j}, a\right)=\left\{\begin{array}{l}
1, j=1 \\
0, j>1
\end{array}\right.
$$

and hence $n(\partial \Gamma, a)=1$, since the integrals over the common boundaries vanish. But then since $\Gamma$ clearly does not meet $B$, we have found a cycle in $\Omega$ such that $n(\Gamma, a) \neq 0$ but $a \in \Omega^{c}$. This is a contradiction.

- $(2) \Longrightarrow(1)$. Suppose $\Omega$ is not simply connected, then the $\Gamma$ produced above gives a Jordan curve whose interior is not completely contained inside $\Omega$.
- $(4) \Longrightarrow(2)$. It is enough to prove that every closed smooth curve that index zero. Let $\gamma$ be one such curve, and let $p \in \Omega^{c}$. There exists a homotopy $H:[0,1] \times[0,1] \rightarrow \Omega$ contracting $\gamma$ to a point $a \in \Omega$. For $s \in[0,1]$, consider the function

$$
f(s):=\frac{1}{2 \pi i} \int_{0}^{1} \frac{\gamma_{s}^{\prime}(t)}{\gamma_{s}(t)-p} d t .
$$

Note that the definition makes sense because of the Lemma above. Now, since $p \in \Omega^{c}$, clearly $f(s)$ is a continuous function. Moreover, $f(1)=0$, and hence $f(0)=n(\gamma, p)=0$.

- $(2) \Longrightarrow(4)$. Since this implication will not play any further role in the course, we simply direct the reader to the argument on page 252 of Complex Analysis by Theodore Gamelin.

An important consequence of this is the following.
Theorem 3. Let $\Omega$ be a simply connected domain and $f \in \mathcal{O}$. Then $f$ has a primitive on $\Omega$.

Proof. The proof is along the lines of the proof for existence of primitives on disc that was used in the proof of Cauchy's theorem. So we fix a $p \in \Omega$, and define

$$
F(z)=\int_{\gamma_{z}} f(w) d w
$$

where the integral is along some path $\gamma_{z}$ joining $p$ to $z$. If we choose another path $\tilde{\gamma}_{z}$ joining the two points, then $\gamma_{z}-\tilde{\gamma}_{z}$ will form a cycle. Since the domain is simply connected, $\gamma_{z} \sim_{\Omega} \tilde{\gamma}_{z}$, and hence the integral of $f$ along both would be the same. Hence our definition is actually independent of the path. By openness of $\Omega$, for any $h$ small, the straight line joining $z$ to $z+h$ will lie entirely in $\Omega$, and we call this path as $l$. Then $\gamma_{z+h}-l$ and $\gamma_{z}$ are both piecewise smooth paths joining $p$ to $z$, so once again by simple connectedness of $\Omega$ and Theorem 2

$$
\int_{\gamma_{z+h}} f(w) d w-\int_{l} f(w) d w=\int_{\gamma_{z}} f(w) d w,
$$

or equivalently

$$
F(z+h)-F(z)=\int_{l} f(w) d w
$$

Then the same argument as in the proof of Theorem 2.1 in Lecture- 7 implies that $F(z)$ is holomorphic with $F^{\prime}(z)=f(z)$.

## Cauchy's theorem for multiply connected domain

A connected domain $\Omega \subset \mathbb{C}$ is said to be $n$-connected if it's complement in the extended complex plane has $n$-connected components. So for instance, a simply connected set is 1 -connected, and an $n$-connected set has $n-1$ number of "holes". Our convention will be to label the components as $A_{1}, \cdots, A_{n}$, where $A_{n}$ is component containing the north pole (or the point at "infinity"). Using the argument in the proof of the implication (2) $\Longrightarrow$ (1) in Theorem 2 we obtain the following.

Theorem 4. For every $i=1, \cdots, n-1$, there exists a cycle $\gamma_{i}$ such that

$$
n\left(\gamma_{i}, p\right)=\left\{\begin{array}{l}
1, p \in A_{i}  \tag{0.1}\\
0, p \in \Omega^{c} \backslash A_{i} .
\end{array}\right.
$$

Moreover we have the following observations.
(1) The set $\left\{\gamma_{1}, \cdots, \gamma_{n}\right\}$ is a linearly independent set, in the sense that $\sum_{i=1}^{n} c_{i} \gamma_{i} \sim_{\Omega} 0$ if and only if $c_{i}=0$ for all $i$.
(2) The set $\left\{\gamma_{1}, \cdots, \gamma_{n}\right\}$ is a spanning set, in the sense that if $\gamma$ is any other cycle in $\Omega$, then

$$
\gamma \sim_{\Omega} c_{1} \gamma_{1}+\cdots c_{n-1} \gamma_{n-1}
$$

where $c_{i}=n\left(\gamma, p_{i}\right)$ for any $p_{i} \in A_{i}$.
(3) For any $f \in \mathcal{O}(\Omega)$, we have

$$
\int_{\gamma} f(z) d z=\sum_{i=1}^{n-1} n\left(\gamma, p_{i}\right) \int_{\gamma_{i}} f(z) d z
$$

where $\left(p_{1}, p_{2}, \cdots, p_{n-1}\right)$ is any collection of points in $A_{1} \times \cdots A_{n-1}$.
(4) If $\left\{\sigma_{1}, \cdots, \sigma_{m}\right\}$ is a linearly independent spanning set, then $m=n$.

We call the collection $\left\{\gamma_{1} \cdots, \gamma_{n}\right\}$ the homology basis for $\Omega$. In general there will be multiple choices of homology bases, but by an elementary theorem in linear algebra, all of these must be $n$ in number. In fact, if

Example 2. Consider the domain $\Omega:=D_{2}(0) \backslash\{-1, i, 1\}$. Then clearly the domain is 4 -connected. In fact if we label the points by $p_{1}=-1, p_{2}=i, p_{3}=$ 1 , then $A_{i}=\left\{p_{i}\right\}$ for $i=1,2,3$ and $A_{4}=\mathbb{S}^{2} \backslash D_{2}(0)$. Let $\gamma_{i}$ be given by

$$
\gamma_{i}(t)=p_{i}+\frac{1}{2} e^{i t} .
$$

Clearly $n\left(\gamma_{i}, p_{i}\right)=1$. On the other hand if $p \in \Omega^{c} \backslash\left\{p_{i}\right\}$, then $p$ lies outside $D_{1 / 2}\left(p_{i}\right)$ and hence $n\left(\gamma_{i}, p\right)=0$. Hence by the above theorem, $\left\{\gamma_{1}, \gamma_{2}, \gamma_{3}\right\}$ forms a homology basis for $\Omega$.

As a consequence of the above result, we have the following. We say that a Jordan curve $\gamma$ is positively oriented if while the curve is traversed, $\operatorname{int}(\gamma)$ remains to the left.

Corollary 1. Let $\Gamma, \Gamma_{1}, \cdots, \Gamma_{n}$ be positively oriented, pairwise non intersecting, piecewise smooth Jordan curves such that $\Gamma_{j} \subset \operatorname{int}(\Gamma)$ for all $j=$ $1, \cdots, n$. Let $\Omega=\operatorname{int}(\Gamma) \cap\left(\cap_{j=1}^{n} \operatorname{ext}\left(\Gamma_{j}\right)\right)$. Let $f$ be a function that is holomorphic in a neighbourhood of $\Omega$. Then

$$
\int_{\Gamma} f(z) d z=\sum_{j} \int_{\Gamma_{j}} f(z) d z .
$$

Proof. Firstly, one can construct Jordan curves $C, C_{1}, \cdots, C_{n}$ such that $\operatorname{int}(\Gamma) \subset \operatorname{int}(C)$ and $\operatorname{int}(C) \subset \operatorname{int}(\Gamma)$ for all $i$, and such that $f$ is holomorphic in $\Omega^{\prime}:=\operatorname{int}(C) \cap\left(\cap_{j=1}^{n} \operatorname{ext}\left(C_{j}\right)\right)$ which of course contains $\Omega$. Let $p_{i} \in \operatorname{int}\left(C_{i}\right)$. It is then easy to check that $\Gamma_{1}, \cdots, \Gamma_{n}$ forms a homology basis for $\Omega^{\prime}$ and the result then follows from Theorem 4. Note that constructing the Jordan curves $C_{i}$ is non-trivial. If $\Gamma_{i}$ is smooth, then one can construct $C_{i}$ by perturbing slightly in the direction of the inner normal (and similarly $C$ be perturbing $\Gamma$ a little bit along the outer normal). But since our curves are only piecewise smooth, extra care must be take to "round off" the "corners".

A real variable integral. We will now apply Cauchy's theorem to compute a real variable integral. Later in the course, once we prove a further generalization of Cauchy's theorem, namely the residue theorem, we will conduct a more systematic study of the applications of complex integration to real variable integration. For now, let us compute

$$
\int_{-\infty}^{\infty} \frac{\cos x}{1+x^{2}} d x
$$

This is an improper integral which is convergent, so by definition

$$
\int_{-\infty}^{\infty} \frac{\cos x}{x^{2}} d x=\lim _{R \rightarrow \infty} \int_{-R}^{R} \frac{1-\cos x}{x^{2}} d x
$$

Consider now the function

$$
f(z)=\frac{e^{i z}}{1+z^{2}}
$$

This is a holomorphic function on $\mathbb{C} \backslash\{i\}$. Moreover, on the real line the real part of this function is precisely the function that we are looking to integrate. Now consider a contour $\Gamma_{R}:=\{z \in \mathbb{C}| | z \mid=R, \operatorname{Im}(z)>0\} \cup\{(x, 0) \mid x \in$ $(-R, R)\}$ oriented in the anti-clockwise direction. Let $C_{\varepsilon} ;=\{|z-i|=\varepsilon\}$ be
a small circle around $i$. Then for $R \gg \varepsilon$, we clearly have that $\Gamma_{R} \sim_{\mathbb{C}_{i}^{*}} C_{\varepsilon}$, and hence by Theorem 4,

$$
\int_{\Gamma_{R}} f(z) d z=\int_{C_{\varepsilon}} f(z) d z
$$

Now note that

$$
f(z)=\frac{g(z)}{z-i}
$$

where $g(z)=e^{i z} /(z+i)$ is holomorphic in a neighbourhood of the disc $D_{\varepsilon}(i)$, and hence by the Cauchy integral formula (applied to $g(z)$ ),

$$
\int_{C_{\varepsilon}} f(z) d z=\int_{C_{\varepsilon}} \frac{g(z)}{z-i} d z=2 \pi i g(i)=\frac{\pi}{e} .
$$

Letting $S_{R}$ be the semi-circle $\{z \in \mathbb{C}||z|=R, \operatorname{Im}(z)>0\}$, we see that

$$
\int_{\Gamma_{R}} f(z) d z=\int_{\gamma_{R}} f(z) d z+\int_{-R}^{R} \frac{e^{i x}}{1+x^{2}}=\int_{\gamma_{R}} f(z) d z+\int_{-R}^{R} \frac{\cos x}{1+x^{2}}
$$

where we have used the fact that $\sin x$ is an odd function. On $\gamma_{R}$ we claim that

$$
\left|\frac{1-e^{i z}}{z^{2}}\right| \leq \frac{2}{R^{2}}
$$

To see this, for $z \in \gamma_{R}$, we can write $z=x+i y$ with $y>0$. So $\left|e^{i z}\right|=e^{-y}<1$, and hence by triangle inequality $\left|1-e^{i z}\right|<2$ which proves the claim since $|z|=R$ on $\gamma_{R}$. Using this we can estimate that

$$
\left|\int_{\gamma_{R}} \frac{1-e^{i z}}{z^{2}} d z\right| \leq \frac{2}{R} \operatorname{len}\left(\gamma_{R}\right)=\frac{2 \pi}{R} \rightarrow 0
$$

as $R \rightarrow \infty$. So the contribution on $\gamma_{R}$ goes to zero as $R$ goes to infinity, and hence

$$
\int_{-\infty}^{\infty} \frac{\cos x}{1+x^{2}}=\frac{\pi}{e}
$$

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