LECTURE-12

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The logarithm

The purpose of this lecture is twofold - first, to characterize domains on which a holomorphic logarithm can be defined, and second, to show that the only obstruction to defining a holomorphic logarithm is in defining a continuous logarithm. From henceforth, we let $\mathcal{O}^*(\Omega)$ be the set of nowhere vanishing holomorphic functions on Ω. For instance, $e^z \in \mathcal{O}^*(\mathbb{C})$.

To set the stage, let us revisit the difficulties we had in defining a holomorphic logarithm. For $z = re^{i\theta}$, consider the function

$$\log z = \log|z| + i\theta, \tag{0.1}$$

where, for instance, we can let $\theta \in [\theta_0, \theta_0 + 2\pi)$. If $\arg p = \theta_0$, and we traverse a circle of radius $|p|$ centred at 0 and return to the point p, the argument goes from θ_0 to $\theta_0 + 2\pi$, and hence the $\log z$ does not return to the original value. In other words $\log z$ as defined is not continuous. On the other hand, if we return to p along a small circle not containing 0 in the interior, then the argument does return to θ_0, and $\log z$ does not jump in value. The difference between the situations if of course that the first curve goes around 0 while the second does not. Thus logarithm is an example of a *multivalued function*, and zero in this case is called a *branch* point.

In general, we can consider any holomorphic function $f : \Omega \to \mathbb{C}^*$. Then, a holomorphic function $g : \Omega \to \mathbb{C}$ (if it exists) is called a *branch* of the logarithm of f, and denoted by $\log f(z)$, if

$$e^{g(z)} = f(z)$$

for all $z \in \Omega$. A natural question to ask is the following.

Question 0.1. *Given a holomorphic function $f : \Omega \to \mathbb{C}^*$, when can we define a holomorphic branch of $\log f(z)$.*

From the point of view of the Cauchy theory, the multivalued behaviour of the logarithm function is esentially because $1/z$, which would be the derivative of a holomorphic logarithm function, does not integrate out to zero around curves that contain the origin in the interior. Keeping this in mind, we have the following basic theorem.

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Theorem 0.1. *Let Ω be a connected domain, and $f \in \mathcal{O}^*(\Omega)$ such that*

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for all closed loops $\gamma \subset \Omega$.

- *Then there exists a holomorphic function $g : \Omega \to \mathbb{C}$, denoted by $g(z) = \log f(z)$, such that*
$$e^{g(z)} = f(z).$$
- *$g' = f'/f$, and hence for any fixed $p \in \Omega$,*
$$g(z) = g(p) + \int_p^z \frac{f'(w)}{f(w)}\, dw.$$

Remark 0.1. *Note that different choices of $g(p)$ corresponding to the countable number of solutions to $e^z = f(p)$ give different formulae for $g(z)$, all of which differ by integral multiples of $2\pi i$. Conversely, if g_1 and g_2 are two logarithms, then they have to differ by a multiple of $2\pi i$. The various logarithm functions are called* branches.

Theorem 0.1 above combined with Cauchy's theorem for simply connected domains gives the following.

Corollary 0.1. *Let Ω be a simply connected domain.*

(1) *Then for any $f \in \mathcal{O}^*(\Omega)$, there exists a holomorphic $\log f(z)$ with $(\log f)' = f'/f$, and hence*
$$\log f(z) = \log f(p) + \int_p^z \frac{f'(w)}{f(w)}\, dw,$$
w here $\log f(p)$ is any *solution to $e^z = f(p)$.*

(2) *In particular, if $0 \notin \Omega$, then there is a holomorphic branch of $\log z$ on Ω with $(\log z)' = 1/z$. Moreover, for any $p \in \Omega$,*
$$\log z = \log p + \int_p^z \frac{1}{z}\, dz,$$
where we integrate along any path from p to z, and $\log p$ is any *solution to $e^z = p$.*

Proof of Theorem 0.1. Fix a point $p \in \Omega$, and let $g(p)$ be a solution to $e^{g(p)} = f(p)$. Since $f(p) \neq 0$, such a solution always exists. We then define $g(z)$ by

$$g(z) = g(p) + \int_p^z \frac{f'(w)}{f(w)}\, dw,$$

where we integrate over any curve joining p and z. By the hypothesis, this is independent of the path chosen. Then, by the same argument used before (as in the proof of Theorem), $g(z)$ is holomorphic with

$$g'(z) = \frac{f'(z)}{f(z)}.$$

Now, consider the function $F(z) = e^{-g(z)} f(z)$. Then

$$F'(z) = -e^{g(z)} g'(z) f(z) + e^{-g(z)} f'(z) = 0.$$

Since Ω is connected, this implies that $F(z)$ is a constant, and hence $F(z) = F(p) = 1$. This completes the proof. $\square$

Recall that the branch of $\log z$ defined by (0.1) is not even a continuous function over $\mathbb{C}^*$. This is not a coincidence. Our next theorem, says that continuity in fact, is the only obstruction to define a holomorphic logarithm.

Theorem 0.2. *Let $\Omega \subset \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous. If $e^{g(z)}$ is holomorphic, then so is $g(z)$.*

In other words if $f(z)$ is holomorphic, and we can define a continuous $\log f(z)$, then $\log f(z)$ is automatically holomorphic.

Proof. Let $f(z) = e^{g(z)}$, which by the hypothesis, is holomorphic, and fix a $p \in \Omega$. There is a $\delta > 0$ such that $\overline{D_\delta(p)} \subset \Omega$ and a holomorphic function $g_p(z)$ on $D_\delta(p)$ such that $e^{g_p(z)} = f(z)$. Then for all $z \in D_\delta(p)$,

$$\frac{g(z) - g_p(z)}{2\pi i} \in \mathbb{Z}.$$

Now, since $g(z)$ is continuous, $\frac{g(z)-g_p(z)}{2\pi i}$ is a continuous function on $D_\delta(p)$ which only takes integer values, and hence has to be a constant. That is,

$$g(z) = g_p(z) + 2\pi i n,$$

for some fixed $n \in \mathbb{Z}$. But then $g(z)$ has to be holomorphic in $D_\delta(p)$, since $g_p(z)$ is holomorphic, and hence is in complex differentiable at p. Since p was arbitrary, this completes the proof of the theorem.

$\square$

Some examples

Somewhat vaguely, for a (multi-valued) function $g(z)$, the point $z = a$ is defined to be a *branch point* if $g(z)$ is discontinuous while traversing an arbitrarily small circle around the point. We define infinity to be a branch point if $z = 0$ is the branch point of $g(1/z)$. An alternate way is to consider a curve “enclosing” infinity. This is a large curve that contains all the other branch points in it’s interior. Then infinity is a branch point if along this large curve, the function $g(z)$ is discontinuous. A *branch cut* is a union of curves such that $g(z)$ defines a single valued holomorphic function on the complement. Branch cuts should usually connect branch points to prevent the possibility of going around branch points and making the function value jump.

Principal branch of the logarithm. Since $\mathbb{C} \setminus \{\mathrm{Re}(z) \le 0\}$ is simply connected, this immediately implies that there is a holomorphic logarithm on that domain. The *principal branch* of the logarithm is defined to be the one with

$$\log 1 = 0.$$

By Theorem 0.1, for any $z \in \mathbb{C} \setminus \{\mathrm{Re}(z) \le 0\}$, we then have

$$\log z = \int_1^z \frac{1}{w}\, dw,$$

where we integrate over any piecewise smooth path from 1 to z. Suppose $z = re^{i\theta}$, then one such path is $C = C_1 + C_2$ where C_1 is parametrized by $z_1(t) : [0,1] \to \mathbb{C}$ with $z_1(t) = tr + (1-t)$, and C_2 is given by $z_2(t) : [0,\theta] \to \mathbb{C}$ where $z_2(t) = re^{e^{it}}$ So C is simply the path going first from 1 to r along the x-axis, and then the circular arc to z. Then

$$\int_{C_1} \frac{1}{w}\, dw = \int_0^1 \frac{r-1}{t(r-1)+1}\, dt = \log(t(r-1)+1)\Big|_{t=0}^{t=1} = \log r,$$

where the log is the usual logarithm defined on real numbers. On the other hand,

$$\int_{C_2} \frac{1}{w}\, dw = i\int_0^\theta dt = i\theta.$$

So the principal branch of the logarithm is given by

$$\log z = \log r + i\theta,$$

where $\theta \in (-\pi, \pi)$. We end with the following remark.

Remark 0.2. *Unlike the real logarithm, in the complex case, in general*

$$\log z_1 z_2 \neq \log z_1 + \log z_2.$$

For example, let $z_1 = e^{3\pi i/4}$,$z_2 = e^{\pi i/2}$ *and* $\log z$ *be the principal branch. Then* $\log z_1 = 3\pi i/4$ *and* $\log z_2 = \pi i/2$. *But* $z_1 z_2 = e^{5\pi i/4} = e^{-3\pi i/4}$ *(remember the range of* $\arg$ *is* $(-\pi, \pi]$, *and so* $\log z_1 z_2 = -3\pi i/4 \neq \log z_1 + \log z_2$. *Similarly, even though* $e^{\log z} = z$ *for all* z, $\log e^z \neq z$ *generally, again due to the periods of* e^z.

Branch cut for $\log(z^2 - 1)$**.** The points where $z^2 - 1 = 0$, namely $z = \pm 1$ are certainly branch points. Any branch cuts should include these two points. To see that infinity is also a branch point, note that the logarithm should be defined as a primitive of

$$\frac{2z}{z^2-1} = \frac{1}{z+1} + \frac{1}{z-1}.$$

It is clear that as we integrate along a curve of radius $R > 1$ around the origin, both terms will make a contribution with the same sign, and hence the integral will not be zero. In other words if we define log as a primitive of $\frac{2z}{z^2-1}$, it will have a jump if we traverse a large curve. Hence ∞ is a also branch point. Possible branch cuts, that will prevent going around $z = \pm 1$ or $z = \infty$ are

$$(-\infty, 1] \text{ or } (-\infty, 1] \cup [1, \infty) \text{ or } [-1, \infty).$$

Of course there are infintely many choices of branch cuts. Each of the above branch cuts renders the domain simply connected, and hence a holomorphic branch does exist.

A convenient way to write down a formula for the branch is by using "**double polar coordinates**". That is, we let

$$z = -1 + r_1 e^{i\theta_1} = 1 + r_2 e^{i\theta_2}.$$

If we restrict the "phases" θ_1, θ_2 to be in the usual range $(-\pi, \pi)$, then using the principal branch of the log we obtain

$$\log(z+1) = \log r_1 + i\theta_1$$
$$\log(z-1) = \log r_2 + i\theta_2.$$

Adding up these two we see that

$$g(z) = \log r_1 + \log r_2 + i(\theta_1 + \theta_2) \tag{0.2}$$

does define a branch of $\log(z^2 - 1)$ since it is easy to see that $e^{g(z)} = z^2 - 1$.

Claim. $g(z)$ defines a holomorphic branch of $\log(z^2 - 1)$ on $\mathbb{C} \setminus (-\infty, 1]$.

By Theorem 0.2 it is enough to check that it defines a continuous branch. But this is obvious since θ_1 is continuous everywhere except $z \leq -1$ and θ_2 is continuous everywhere except $z \leq 1$, and since these points are removed in the branch cut, $g(z)$ is continuous everywhere else.

If we instead, restrict $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (0, 2\pi)$, then formula (0.2) defines a holomorphic branch on the complement of the branch cut $(-\infty, 1] \cup [1, \infty)$. The reader should work these out carefully.

Branch cuts for $\log\left(\frac{z+1}{z-1}\right)$**.** A holomorphic definition would have primitive

$$\frac{d}{dz} \log\left(\frac{z+1}{z-1}\right) = \frac{1}{z+1} - \frac{1}{z-1}.$$

Clearly $z = \pm 1$ are branch points. To analyze branching at infinity, consider a large disc $D_R(0)$ with $R > 2$. Then both the terms contribute an integral of $2\pi i$ but with opposite signs, and hence the integral vanishes. In other words the argument does not change as we traverse this big circle. Hence infinity is NOT a branch point. Hence we can then choose the branch cut to be $[-1, 1]$, even though $\mathbb{C} \setminus [-1, 1]$ is not simply connected.

Again, lets analyze this using the double polar coordinates $z = -1 + r_1 e^{i\theta_1} = 1 + r_2 e^{i\theta_2}$, This time let $\theta_1, \theta_2 \in [-\pi, \pi)$, and define

$$g(z) = \log r_1 - \log r_2 + i(\theta_1 - \theta_2).$$

Clearly this defines a branch of $\log\left(\frac{z+1}{z-1}\right)$, and we need to check if it is a continuous branch. The function so defined is surely continuous (and hence holomorphic by Theorem 0.2) everywhere on $\mathbb{C} \setminus [-1,1]$ except possibly the real axis to the left of $z = -1$. As you approach this part of the real axis from the top, both $\theta_1, \theta_2 \to \pi$. On the other hand when you approach from the bottom, both $\theta_1, \theta_2 \to -\pi$, and so their difference cancels out. So the resulting function defines a continuous, and hence a holomorphic, $\log\left(\frac{z+1}{z-1}\right)$. Again if we change the domain for either θ_1 or θ_2 to $(0, 2\pi)$, we are forced ton consider other branch cuts. Once again, the reader should work both these cases out carefully.

n^{th}-ROOTS OF HOLOMORPHIC FUNCTIONS

Given a logarithm function, and a $w \in \mathbb{C}$, one can define a holomorphic complex power, by

$$z^w = e^{w \log z}. \tag{0.3}$$

When $w = 1/n$ is the reciprocal of a natural number, we call $z^{1/n}$ the n^{th} root of z.

Example 0.1. Roots of unity. *Consider the polynomial $z^n - 1$. Clearly $\zeta_n = e^{2\pi i/n}$, is a root. Moreover, ζ_n^k, for $k = 0, 1, \cdots, n-1$ is also a root, and since the degree of the polynomial is n, these are all the possible roots. We call ζ_n the primitive n^{th} root of unity.*

We then have the following analogs of Corollary 0.1 and Theorem 0.2.

Theorem 0.3.
(1) *If Ω is simply connected and $f(z)$ is holomorphic and zero free, then there exists a holomorphic function $g(z)$ such that $g(z)^n = f(z)$. Moreover, if $g_1(z)$ is any other such function, then $g_1(z) = \zeta_n^k g(z)$ where $\zeta_n = e^{2\pi i/n}$ is the primitive n^{th} root of unity and $k = 0, 1, \cdots, n-1$.*
(2) *If $g : \Omega \to \mathbb{C}$ is continuous such that $g(z)^n$ is holomorphic for some positive integer n, then $g(z)$ itself is holomorphic.*

Proof.
(1) For the first part, by Corollary 0.1, there exists a holomorphic $\log f(z)$. We then simply take
$$g(z) = e^{\frac{\log f(z)}{n}}.$$
It is also clear that if $g_1(z)$ is another such function, then $(g_1(z)/g(z))^n = 1$, and hence there exists
(2) We proceed as in the proof of Theorem 0.2. Let $f(z) = g(z)^n$. Then for any $p \in \Omega$ if $r > 0$ such that $\overline{D_r(p)} \subset \Omega$, by the first part, there exists a holomorphic function $g_p(z)$ on $D_r(p)$ such that $g_p(z)^n = f(z)$. But then on the disc, $(g/g_p)^n = 1$ and hence by continuity, there exists a fixed integer $0 \le k \le n-1$ (independent

of z) such that $g(z) = g_p(z)e^{2\pi ik/n}$, which in turn implies that $g(z)$ is holomorphic.

$\square$

Principal, and other branches of the square root. We can define the principal branch of the square root so that $\sqrt{1} = 1$. Doing a similar computation as above, we can then see that if $z = re^{i\theta}$ with $\theta \in (-\pi, \pi)$, then

$$\sqrt{z} = re^{i\theta/2}.$$

On the other hand, if we want $\sqrt{1} = -1$, then

$$\sqrt{z} = re^{i\pi + i\theta/2}.$$

Branch cuts for $\sqrt{z^2-1}$. Any of the branch cuts for $\log(z^2-1)$ will allow us to define $\sqrt{z^2-1}$ on their complement by equation (0.3). Each of those branch cuts extend out to infinity. But it turns out we can define a holomorphic branch of $\sqrt{z^2-1}$ on the complement of *finite* cut. This is possible because ∞ is not a branch point (even though it is a branch point of $\log(z^2-1)$).

To see this, we again make use of double polar coordinates. Let $z = -1 + r_1e^{i\theta_1} = 1 + r_2e^{i\theta_2}$ as before, where we let $\theta_j \in (-\pi, \pi)$, and we define

$$g(z) = \sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}.$$

Clearly this defines a branch of $\sqrt{z^2-1}$ since

$$g(z)^2 = r_1r_2e^{i(\theta_1+\theta_2)} = (z+1)(z-1) = z^2 - 1.$$

All we need to now do, is to find a branch cut such that $g(z)$ is continuous in the complement.

Clearly θ_1 is continuous everywhere except $(-\infty, -1]$ and θ_2 is continuous everywhere except $(-\infty, 1]$. So $g(z)$ is continuous everywhere except possibly for $(-\infty, 1]$. Let us analyze the two intervals $(-\infty, -1)$ and $[-1, 1]$. If z approaches $[-1, 1]$ from above, $\theta_1 \to 0$ but $\theta_2 \to \pi$, and so

$$g(z) \to \sqrt{r_1r_2}e^{i\pi/2} = i\sqrt{r_1r_2}.$$

But if it approaches from below, then $\theta_1 \to 0$ while $\theta_2 \to -\pi$. So

$$g(z) \to \sqrt{r_1r_2}e^{-\pi i/2} = -i\sqrt{r_1r_2},$$

and so $g(z)$ is discontinuous on $[-1, 1]$. Next we analyse continuity along $(-\infty, -1)$. As z approaches $(-\infty, -1)$ from above, $\theta_1 \to \pi$ and $\theta_2 \to \pi$, hence

$$g(z) \to e^{2i\pi/2}\sqrt{r_1r_2} = -\sqrt{r_1r_2}.$$

On the other hand when z approaches $(-\infty, -1)$ from below $\theta_1, \theta_2 \to -\pi$, and so

$$g(z) \to \sqrt{r_1r_2}e^{-i\pi} = -\sqrt{r_1r_2},$$

and hence $g(z)$ defines a continuous branch on $(-\infty, -1)$. The upshot is that $g(z)$ defines a continuous (and hence holomorphic) branch of $\sqrt{z^2-1}$ on $\mathbb{C} \setminus [-1, 1]$.

Note that with above formula, we can also compute the value of $\sqrt{z^2-1}$. For instance we demonstrate how to calculate the value of $\sqrt{i^2-1}$ for our particular branch. Of course the answer has to be either $\pm i\sqrt{2}$, but the question is which one of these values? We can write (draw a picture to see what is happening geometrically)

$$i = -1 + \sqrt{2}e^{i\pi/4} = 1 + \sqrt{2}e^{3\pi i/4},$$

so that $r_1 = r_2 = \sqrt{2}$ and $\theta_1 = \pi/4$ and $\theta_2 = 3\pi/4$. Since $\theta_1 + \theta_2 = \pi$ by the formula above

$$g(i) = \sqrt{\sqrt{2}\sqrt{2}}\, e^{i\pi/2} = i\sqrt{2}.$$

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