LECTURE-14

VED V. DATAR*

Isolated Singularities

A punctured domain is an open set with a point removed. For $p \in \Omega$, we use the notation

$$\Omega_p^* = \Omega \setminus \{p\},$$

or simply Ω^* for $\Omega \setminus \{0\}$, if $0 \in \Omega$ or when there is no confusion about the point removed. The aim of this lecture is to study functions that are holomorphic on punctured domains. The puncture, that is the point p in the above case, is called an isolated singularity. These come in three types -

- Removable singularities
- Poles
- Essential singularities

Removable singularities

A holomorphic function $f \in \mathcal{O}(\Omega_p^*)$ is said to have a *removable singularity* at p if there exists a holomorphic function $\tilde{f} \in \mathcal{O}(\Omega)$ such that

$$\tilde{f}\Big|_{\Omega_p^*} = f.$$

Theorem 0.1. *Let $f \in \mathcal{O}(\Omega_p^*)$. Then the following are equivalent.*

1. *f has a removable singularity at p.*
2. *f can be extended to a continuous function on Ω.*
3. *f is bounded in a neighborhood of p.*
4. $\lim_{z\to p}(z-p)f(z) = 0$.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are trivial. To complete the proof, we need to show that (4) $\implies$ (1).

For convenience, suppose $p = 0$. So suppose $f(z)$ satisfies

$$\lim_{z\to 0} zf(z) = 0,$$

and define a new function

$$g(z) = \begin{cases} z^2 f(z), & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Claim 1. *$g(z)$ is holomorphic on Ω and moreover, $g'(0) = 0$.*

Date: 24 August 2016.

Clearly $g(z)$ is holomorphic on Ω^*. So we only need to prove holomorphicity at $z = 0$. Let us compute the difference quotient. Since $g(0) = 0$, $g'(0)$ if it exists is equal to

$$\lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} hf(h) = 0$$

by hypothesis. This proves the claim. By analyticity, $g(z)$ has power series expansion in a neighborhood of $z = 0$. That is, there is a small disc $D_\varepsilon(0)$ such that for all $z \in D_\varepsilon(0)$,

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n.$$

Now, since $g(0) = g'(0) = 0$,

$$g(z) = z^2 \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-2}.$$

By comparing to the definition of $g(z)$ we see that for $z \in D_\varepsilon(0) \setminus \{0\}$,

$$f(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-2}.$$

and so we define

$$\tilde{f}(z) = \begin{cases} \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-2}, & z \in D_\varepsilon(0) \\ f(z), & z \in \Omega^*. \end{cases}$$

This is a well defined function, since on the intersection $\Omega^* \cap D_\varepsilon(0)$, $f(z)$ is equal to the infinite series. $\tilde{f}$ is clearly holomorphic on Ω^* since it equals $f(z)$ in this region. Moreover, since it is a power series in a neighborhood of $z = 0$, it is also holomorphic at $z = 0$. Hence $\tilde{f}$ satisfies all the properties in (1), and this completes the proof. □

Remark 0.1. *Recall that in lecture-7 we proved that Goursat's theorem was valid for functions that are holomorphic at all but one point in a domain, so long as they are bounded near that point. In view of the above theorem, such a result is not surprising, since the function does extend to a holomorphic function on the entire domain, to which Goursat's theorem applies.*

Example 0.1. *Consider the holomorphic function $Si : \mathbb{C}^* \to \mathbb{C}$ defined by*

$$Si(z) = \frac{\sin z}{z}.$$

Then clearly

$$\lim_{z\to 0} z \cdot \mathrm{Si}(z) = \lim_{z\to 0} \sin z = 0.$$

Hence by the theorem, $\mathrm{Si}(z)$ *has a removable singularity at $z = 0$ and hence can be extended to an entire function. It is instructive to look at the power* *series of* $\sin z$. *Recall that*

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...,$$

and so dividing by z, *we see that for* $z \neq 0$,

$$\mathrm{Si}(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}.$$

The power series on the right clearly defines an entire function (an is in particular also defined at $z = 0$*), and hence* $\mathrm{Si}(z)$ *defines an entire function.*

Poles

Let $f \in \mathcal{O}(\Omega_p^*)$. We say that p is a *pole* if

$$\lim_{z\to p} |f(z)| = \infty.$$

Theorem 0.2. *Let* $f \in \mathcal{O}(\Omega_p^*)$. *Then the following are equivalent.*

(1) f *has a pole at* p.

(2) *There exists a small disc* $D_\varepsilon(p)$ *and a holomorphic function* $h : D_\varepsilon(p) \to \mathbb{C}$ *such that* $h(p) = 0$ *and* $h(z) \neq 0$ *for any other* $z \in D_\varepsilon(p)$, *and*

$$f(z) = \frac{1}{h(z)}$$

for all $z \in D_\varepsilon(p) \setminus \{p\}$.

(3) *There exists a holomorphic function* $g : \Omega \to \mathbb{C}$ *such that* $g(p) \neq 0$, *and an integer* $m \geq 1$ *such that for all* $z \in \Omega_p^*$,

$$f(z) = \frac{g(z)}{(z-p)^m}.$$

(4) *There exists a* $M > 1$ *and integer* $m \geq 1$ *such that on some disc* $D_\varepsilon(p)$ *around* p, *we have the estimates*

$$\frac{1}{M|z-p|^m} \leq |f(z)| \leq \frac{M}{|z-p|^m}.$$

Note that the integer m in (3) and (4) above has to be the same, and is called the *order* of the pole at p, and written as $\nu_f(p)$.

Proof. Again for convenience, lets assume $p = 0$, and we denote $\Omega_p^* = \Omega^*$. Suppose $|f(z)| \to \infty$ as $z \to 0$. Then clearly there is a small disc $D_\varepsilon(0)$ on which f does not have a zero. Then $h(z) = 1/f(z)$ is holomorphic in the punctured disc $D_\varepsilon(0)^*$. Moreover,

$$\lim_{z\to 0} |h(z)| = \frac{1}{\lim_{z\to 0} |f(z)|} = 0,$$

and hence in particular is bounded near $z = 0$. By Theorem 0.1, $h(z)$ actually extends to a holomorphic function to the entire disc $D_\varepsilon(0)$, which we continue to call $h(z)$, and from the limit it is clear that $h(p) = 0$. So $h(z)$ atisfies all the conditions in (2), and this proves that (1) $\implies$ (2).

To show that (2) $\implies$ (3), note that by the theorem on zeroes, since h is not identically zero, there exists an integer m such that for all $z \in D_\varepsilon(0)$,

$$h(z) = z^m g_1(z),$$

where $g_1(p) \neq 0$. Moreover, since $h(p) = 0$, we must necessarily have $m \geq 1$. Now consider the function

$$g(z) = z^m f(z)$$

holomorphic on Ω^*. Then on $D_\varepsilon(0) \setminus \{0\}$, $g(z) = 1/g_1(z)$. Since $g_1(p) \neq 0$ and g_1 is holomorphic on $D_\varepsilon(0)$, we see that $1/g_1(z)$ is bounded on $D_\varepsilon(0)$. Hence by the removable singularity theorem, $g(z)$ extends to a holomorphic function on all of Ω, and satisfies all the conditions in (3).

To show that (3) $\implies$ (4), note that since g is holomorphic near $z = 0$, it will in particular be bounded in a neighborhood. So there exists $M > 0$ such that for all $z \in D_\varepsilon(0)$,

$$|g(z)| \leq M.$$

On the other hand, since $g(p) \neq 0$, by continuity, for the $\varepsilon > 0$ above, there exists a δ such that

$$|g(z)| \geq \delta$$

for all $z \in D_\varepsilon(0)$. Take M large enough so that $1/M < \delta$, then we see that on $D_\varepsilon(0)$,

$$\frac{1}{M} \leq |g(z)| \leq M,$$

and this proves (4).

(4) $\implies$ (1) also holds trivially, thus completing the proof of the Theorem. $\square$

Example 0.2. *The function*

$$\cot z = \frac{\cos z}{\sin z}.$$

has poles at all the zeroes of $\sin z$ *(since* $\cos z$ *and* $\sin z$ *do not share any zeroes, there is no "cancellation" of the poles). Let us find the order of the zero at* $z = 0$*. Near* $z = 0$, $\sin z \approx z$*. More precisely,*

$$z \cot z = \frac{z \cos z}{\sin z} = \frac{\cos z}{\mathrm{Si}(z)},$$

where $Si(z)$ *is the function from the last section. Then we saw from the power series expansion, that* $\mathrm{Si}(0) = 1$ *and hence* $\cos z/\mathrm{Si}(z) \to 1$ *as* $z \to 0$*. In particular, for a small* $\varepsilon > 0$, $1/2 < |\cos z/\mathrm{Si}(z)| < 2$*, and hence*

$$\frac{1}{2z} \leq \frac{\cos z}{\sin z} \leq \frac{2}{z},$$

and so $z = 0$ is a pole of order $m = 1$. It is once again instructive to look at an expansion near $z = 0$. For $z \neq 0$,

$$\begin{aligned}\frac{\cos z}{\sin z} &= \frac{1 - z^2/2 + \cdots}{z - z^3/6 + \cdots} \\ &= \frac{1}{z} \cdot \frac{1 - z^2/2 + \cdots}{1 - z^2/6 + \cdots} \\ &= \frac{1}{z}\Big(1 - \frac{z^2}{2} + \cdots\Big)\Big(1 + \frac{z^2}{6} + \cdots\Big) \\ &= \frac{1}{z} - \frac{z}{3} + \cdots\end{aligned}$$

From this it is clear that $\cot z$ has a pole of order $z = 0$.

Remark 0.2. *The idea of an expansion for a singular function near it's pole can be generalized. Let p be a pole for $f : \Omega_p^* \to \mathbb{C}$. Then from the theorem, we can write*

$$f(z) = \frac{g(z)}{(z-p)^m},$$

for some holomorphic $g : \Omega \to \mathbb{C}$ with $g(p) \neq 0$. By analyticity, in a neighborhood of p we can write

$$g(z) = \sum_{n=0}^{\infty} a_n (z-p)^n,$$

with $a_0 \neq 0$. Hence for $z \neq p$, we have the expansion

$$f(z) = \frac{a_0}{(z-p)^m} + \frac{a_1}{(z-p)^{m-1}} + \cdots + a_m + a_{m+1}(z-p) + a_{m+2}(z-p)^2 + \cdots .$$

Such an expansion is called a Laurent series expansion, which we will study in greater detail in the next lecture. The part with the negative powers is called the **principal part** *of f near p. In fact, if we denote by*

$$Q_p(w) := a_0 w^m + \cdots + a_{m-1} w,$$

then we can write

$$f(z) = Q_p\Big(\frac{1}{z-p}\Big) + h_p(z),$$

where h_p extends to a holomorphic function across p.

Essential singularities

If $f : \Omega_p^* \to \mathbb{C}$ is holomorphic, then p is called an *essential singularity* if it is neither a removable singularity nor a pole. Unlike in the case of removable singularities and poles, the function behaves rather erratically in any neighborhood around an essential singularity.

Theorem 0.3 (Casorati-Weierstrass)**.** *The following are equivalent.*

(1) *f has an essential singularity at p.*

(2) *For any disc* $D_\varepsilon(p)$*,* $f(D_\varepsilon(p))$ *is dense in* $\mathbb{C}$*, that is for any disc* $D_\varepsilon(p)$ *and any* $a \in \mathbb{C}$*, there exists a sequence* $\{z_n\} \in D_\varepsilon(p)$ *such that*

$$\lim_{n\to\infty} f(z_n) = a.$$

Proof. We first show that (2) $\implies$ (1). If p is a removable singularity, then for some disc $D_\varepsilon(p)$, $f(D_\varepsilon(0))$ is a bounded set in $\mathbb{C}$, and so cannot be dense. On the other hand if p is a pole, then $|f(z)| \to \infty$ as $z \to p$. In particular, there is a disc $D_\varepsilon(p)$ such that for all $z \in D_\varepsilon(p)$,

$$|f(z)| > 1,$$

and hence once again $D_\varepsilon(p)$ cannot be dense in $\mathbb{C}$. This forces p to be an essential singularity.

Conversely, suppose p is an essential singularity. We then have to show that (2) holds. If not, then there is a disc $D_{\varepsilon_0}(p)$ such that $f(D_{\varepsilon_0}(p) \setminus \{p\})$ is not dense in $\mathbb{C}$. Hence there exists an $a \in \mathbb{C}$ and an $r > 0$ such that

$$|f(z) - a| > r$$

for all $z \in D_\varepsilon(p) \setminus \{p\}$. Then define $g : D_\varepsilon(p) \setminus \{p\} \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z) - a}.$$

Since $f(z) \neq a$ on that punctured disc, $g(z)$ is holomorphic. Moreover $|g(z)| \leq 1/r$ in $D_\varepsilon(p) \setminus \{p\}$, and hence by the removable singularity Theorem 0.1, there exists an extension $\tilde{g}$ holomorphic on $D_\varepsilon(p)$. There are now two cases.

Case-1. $\tilde{g}(p) \neq 0$. Then by continuity, there is a smaller $r < \varepsilon$ and a $\delta > 0$ such that $|\tilde{g}(z)| > \delta$ on $D_r(p)$. But away from p,

$$f(z) = \frac{1}{g(z)} + a,$$

and so on $D_r(p) \setminus \{p\}$,

$$|f(z)| \leq \frac{1}{|\tilde{g}(z)|} + |a| < \frac{1}{\delta} + |a|,$$

and so $|f(z)|$ is bounded in a neighborhood of p. By the removable singularity theorem, f must have a removable singularity at $z = p$ which is a contradiction.

Case-2. $\tilde{g}(p) = 0$. Then for any $\varepsilon > 0$, there exists a $r > 0$ such that on D_r,

$$|\tilde{g}(z)| \leq \varepsilon.$$

So by triangle inequality, if ε small enough so that $|a| < 1/2\varepsilon$, then on $D_r(p) \setminus \{p\}$ we have

$$|f(z)| = \left|\frac{1}{g(z)} + a\right| \geq \left|\frac{1}{|g(z)|} - |a|\right| \geq \frac{1}{\varepsilon} - |a| > \frac{1}{2\varepsilon}$$

for all $z \in D_r(p)$. This shows that $\lim_{z\to p} |f(z)| = \infty$, which is a contradiction, completing the proof of the theorem. $\square$

Remark 0.3. *It is a theorem of Picard's that in any neighbourhood of an essential singularity, the image under f is not only dense in $\mathbb{C}$ but misses at most one point of $\mathbb{C}$!*

Example 0.3. *The function $f(z) = e^{1/z}$, which is holomorphic on $\mathbb{C}^*$, has an essential singularity at $z = 0$. To see this, we need to rule out the possibilities of f having a removable singularity or a pole at $z = 0$. Since*

$$f(1/n) = e^n \xrightarrow{n\to\infty} \infty,$$

$f(z)$ is not bounded in any neighborhood of $z = 0$, and hence cannot have a removable singularity. On the other hand,

$$f\Big(\frac{1}{2\pi ni}\Big) = e^{2\pi in} = 1.$$

Hence the limit $\lim_{z\to 0} f(z)$ cannot be infinity, and hence f cannot have a pole at $z = 0$. This shows that $f(z)$ has to have an essential singularity at $z = 0$. Again looking at an expansion, we see that for $z \neq 0$,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots .$$

So the expansion has infinitely many terms with negative powers of z. As we will see when we discuss Laurent series, this in fact characterizes essential singularities.

Remark 0.4. *We finally remark that non-isolated singularities can exist. For instance the function*

$$f(z) = \tan\Big(\frac{1}{z}\Big)$$

has singularities at 0 and points $p_n = 2/n\pi$ which converge to 0. The analysis in the present lecture does not apply to such singularities.

∗ Department of Mathematics, Indian Institute of Science
Email address: `vvdatar@iisc.ac.in`