LECTURE-15

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A *Laurent series* centered at $z = a$ is an infinite series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} c_n (z-a)^n \tag{0.1}$$

We can combine this into one infinite sum

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n = \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots,$$

by setting

$$a_n = \begin{cases} b_{-n}, & n \leq -1 \\ c_n, & n \geq 0. \end{cases} \tag{0.2}$$

We say that the Laurent series in (0.1) is *convergent* at z if both the infinite series are convergent. The first term above is an infinite series of the form

$$b_1(z-a)^{-1} + \cdots. \tag{0.3}$$

Changing the variable to $w = (z-a)^{-1}$, we can re-write this as a usual power series -

$$b_1 w + b_2 w^2 + \cdots.$$

Then by the fundamental theorem for power series, there exists an R_1 such that the series converges on the disc $|w| < R_1^{-1}$ (or equivalently the annulus $|z| > R_1$), where

$$R_1 = \limsup_{n\to\infty} |b_n|^{1/n}$$

Or equivalently, the series (0.3) converges for $|z-a| > R_1$. On the other hand the second series in (0.1) is a regular power series, and hence setting

$$R_2 = (\limsup_{n\to\infty} |c_n|^{1/n})^{-1},$$

the second series is convergent for $|z-a| < R_2$. Combining this, we have the following theorem.

Theorem 1. *If R_1, R_2 given by the formulae above satisfy $R_1 < R_2$, then the Laurent series 0.1 converges for all $z \in \mathbb{C}$ such that $R_1 < |z-a| < R_2$. Moreover, the convergence is uniform and absolute in the region $r_1 \leq |z-a| \leq r_2$ for any r_1, r_2 satisfying $R_1 < r_1 < r_2 < R_2$. As a consequence, the limiting function is holomorphic in the annulus $R_1 < |z-a| < R_2$.*

Henceforth if $R_1 < R_2$ we will denote the annulus of inner radius R_1 and outer radius R_2 by

$$A_{R_1,R_2}(a) = \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\}.$$

Our main result in this chapter is a converse.

Theorem 2. *Let $R_1 < R_2$, and f be holomorphic on a domain containing the closure of the annulus $A_{R_1,R_2}(a)$. Then for all $z \in A_{R_1,R_2}(a)$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta,$$

for any $r \in [R_1, R_2]$. Moreover, the series converges uniformly and absolutely on any compact subset of $A_{R_1,R_2}(a)$.

First we need the following elementary observations.

Lemma 1. *Let F be holomorphic on any domain containing the closure of the annulus $A_{R_1,R_2}(a)$. Then*

$$\int_{C_r(a)} F(z) \, dz$$

is independent of $r \in [R_1, R_2]$.

Proof. Let $R_1 < r_1 < r_2 < R_2$. For simplicity let us denote $C_{r_i}(a) = C_i$. We claim that $C_1 \sim_{A_{R_1,R_2}(a)} C_2$. The lemma then follows from the generalized Cauchy theorem. To prove the claim, we need to compute indices. Let $w \notin A_{R_1,R_2}(a)$. Then either $|w| > R_2$ or $|w| < R_1$. If it is the former, then $w \notin Int(C_1)$ and $w \notin Int(C_2)$. Hence $n(C_2, w) = n(C_1, w) = 0$. On the other hand, if it is the latter, then $w \in Int(C_1) \subset Int(C_2)$, and so $n(C_1, w) = n(C_2, w) = 1$. In either case, for all $w \notin A_{R_1,R_2}(a)$, $n(C_1, w) = n(C_2, w)$, and hence by definition $C_1 \sim_{A_{R_1,R_2}(a)} C_2$. □

Lemma 2 (CIF for annuli)**.** *Let f be holomorphic on a domain containing the closure of the annulus $A_{R_1,R_2}(a)$. Then for all $z \in \mathbb{C}$ such that $R_1 < |z-a| < R_2$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof. For convenience, we use the notation $A = A_{R_1,R_2}(a)$. Fix $z \in A$, and consider the function

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z}, \ \zeta \neq z \\ f'(z), \ \zeta = z. \end{cases}$$

Clearly $g(\zeta)$ is holomorphic on the punctured annulus $A \setminus \{z\}$. But it is continuous on the whole of the annulus since f is holomorphic at z. Hence by the theorem on removable singularities, $g(\zeta)$ is holomorphic on all of A. Then by the above lemma

$$\int_{C_{R_2}} g(\zeta)\, d\zeta = \int_{C_{R_1}} g(\zeta)\, d\zeta.$$

Since $z \notin C_{R_2}$ or C_{R_1}, the above is equivalent to

$$\begin{aligned}
\int_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \int_{C_{R_2}} \frac{f(z)}{\zeta - z}\, d\zeta - \int_{C_{R_1}} \frac{f(z)}{\zeta - z}\, d\zeta \\
&= f(z) \int_{C_{R_2}} \frac{d\zeta}{\zeta - z} - f(z) \int_{C_{R_1}} \frac{d\zeta}{\zeta - z} \\
&= 2\pi i f(z)(n(C_{R_2}, z) + n(C_{R_1}, z))
\end{aligned}$$

Since $z \in Int(C_{R_2})$ but lies in $Ext(C_{R_1})$, $n(C_{R_2}, z) = 1$ and $n(C_{R_1}, z) = 0$, and this completes the proof of the Lemma. □

Proof of theorem 2. This is similar to the proof of analyticity, and the key tool as before is the geometric series expansion

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n,$$

which is valid in the region $|w| < 1$. By the Lemma above

$$f(z) = \frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta := I_2 - I_1.$$

To evaluate I_2, we write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a - (z-a)} = \frac{1}{\zeta - a}\Big(\frac{1}{1 - (z-a)/(\zeta - a)}\Big)$$

Since 'a' is the center of the annulus, if $\zeta \in C_{R_2}$, and $z \in Int(C_{R_2})$, then

$$\frac{|z-a|}{|\zeta - a|} = \frac{|z-a|}{R_2} < 1.$$

Applying the geometric series expansion with $w = (z-a)/(\zeta - a)$ we see that

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(\zeta)}{\zeta - a} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^n}\, d\zeta \\
&= \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta\Big) \cdot (z-a)^n \\
&= \sum_{n=0}^{\infty} a_n (z-a)^n,
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta.$$

To analyze I_1, we write

$$\frac{1}{\zeta - z} = -\frac{1}{z-a}\Big(\frac{1}{1-(\zeta-a)/(z-a)}\Big).$$

But now, if $\zeta \in C_{R_1}$, then for $z \in A_{R_1,R_2}(a)$ we have that

$$\frac{|\zeta - a|}{|z-a|} = \frac{R_1}{|z-a|} < 1,$$

and so again from the geometric series expansion it follows that

$$\begin{aligned} I_1 &= -\frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\zeta)}{z-a}\sum_{k=0}^{\infty}\frac{(\zeta-a)^k}{(z-a)^k}\,d\zeta \\ &= -\sum_{k=0}^{\infty}\Big(\frac{1}{2\pi i}\int_{C_{R_1}} f(\zeta)(\zeta-a)^k\,d\zeta\Big)(z-a)^{-k-1}. \end{aligned}$$

Putting $k+1=-n$, we can write

$$I_1 = -\sum_{n=-1}^{-\infty} a_n(z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta.$$

This completes the proof of the theorem.

Application to study of isolated singularities

Corollary 1. *Let $f : \Omega_p^* \to \mathbb{C}$ holomorphic. The for any disc $D_R(p)$ such that $\overline{D_R(p)} \subset \Omega$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-p)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\partial D_R(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,d\zeta.$$

Proof. Apply the theorem to the annulus $A_{r,R}(p)$ and let $r \to 0$. □

We then have the following characterization of isolated singularities based on the Laurent series expansion.

Theorem 3. *Let $f : \Omega_p^*$ holomorphic with Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-p)^n$$

around p. Then

(1) *p is a removable singularity if and only if $a_n = 0$ for all $n < 0$.*
(2) *p is a pole of order m if and only if $a_n = 0$ for all $n < -m$.*

(3) *p is an essential singularity if and only if for any $N > 0$, there exists an $n < -N$ such that $a_n \neq 0$. That is, there are infinitely many non-zero negative exponent terms in the Laurent series expansion.*

Proof. Note that if $\overline{D_R(p)} \subset \Omega$, then the coefficients are given by the formula

$$a_n = \frac{1}{2\pi i} \int_{\partial D_R(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}}\, d\zeta.$$

So if p is a removable singularity, then for integers $n < 0$, $(\zeta - p)^{-n-1} f(\zeta)$ is holomorphic on the entire disc $D_R(p)$, and hence by Cauchy's theorem for discs, $a_n = 0$ for all $n < 0$. Conversely, if $a_n = 0$ for $n < 0$, the Laurent series reduces to a power series, and we know that power series are holomorphic on the entire disc of convergence.

To prove the characterization of poles, apply the same argument to the function $(\zeta - p)^m f(\zeta)$. The characterization of essential singularities then follows from the definition and the first two parts. □

Remark 1. *Note that in the event the function has only poles, the Laurent series of the function centered at some other points might have infinitely many negative exponent terms. The theorem only states that the Laurent series centered at the isolated singularity can have only finitely many negative exponent terms. For an illustration of this, see Example 2 below.*

Example 1. *Consider the function*

$$\frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}.$$

It has two singularities at $z = 1$ and $z = 2$ which are clearly poles. We can expand the function as a Laurent series centered at either of the poles. To illustrate this, let us find the Laurent series expansion centered at $z = 1$. One approach is to use the formula for the coefficients in Theorem 2 and compute out all the integrals. An easier approach is to use the geometric series expansion, namely that

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$$

whenever $|w| < 1$. Note that the function is holomorphic on the annulus $0 < |z-1| < 1$, and so we can hope to have a Laurent series expansion on that domain. Writing

$$\frac{1}{z-2} = \frac{1}{z-1-1} = -\frac{1}{1-(z-1)}.$$

Since $|z-1| < 1$, using the geometric series expansion (with $w = z-1$) we see that

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} (z-1)^n,$$

and so

$$\frac{1}{z^2-3z+2} = -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n.$$

Example 2. *Sticking with the function from the previous example, one can also try to find a Laurent series expansion on other annuli. For instance the function is holomorphic on the annulus $A_{1,2}(0) = 1 < |z| < 2$. We consider each of the terms in the partial fraction decomposition separately. For $z \in A_{1,2}(0)$, $|z| > 1$ and so applying the geometric series expansion above to $w = 1/z$, we see that*

$$\frac{1}{z-1} = \frac{1}{z}\Big(\frac{1}{1-1/z}\Big) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n}.$$

On the other hand, for $z \in A_{1,2}(0)$, $|z| < 2$ and hence once again applying the geometric series expansion to $w = z/2$,

$$\frac{1}{z-2} = -\frac{1}{2}\Big(\frac{1}{1-z/2}\Big) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

Putting it all together, we see that on $1 < |z| < 2$,

$$\frac{1}{z^2-3z+2} = -\sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

So even though the function only has poles, it Laurent series centred around $z = 0$ has infinitely many negative exponent terms.

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