LECTURE-16

VED V. DATAR*

Meromorphic functions

A function on a domain Ω is called meromorphic, if there exists a sequence of points $p_1, p_2, \cdots$ with no limit point in Ω such that if we denote $\Omega^* = \Omega \setminus \{p_1, \cdots\}$

- $f : \Omega^* \to \mathbb{C}$ is holomorphic.
- f has poles at $p_1, p_2 \cdots$.

We denote the collection of meromorphic functions on Ω by $\mathcal{M}(\Omega)$. We have the following observation, whose proof we leave as an exercise.

Proposition 0.1. *The class of meromorphic function forms a field over $\mathbb{C}$. That is, given any meromorphic functions $f, g, h \in \mathcal{M}(\Omega)$, we have that*

(1) $f \pm g \in \mathcal{M}(\Omega)$,
(2) $fg \in \mathcal{M}(\Omega)$,
(3) $f(g+h) = fg + fh$.
(4) $f \pm 0 = f,\ f \cdot 1 = f$,
(5) $1/f \in \mathcal{M}$.

Recall that if a holomorphic function has finitely many roots, then it can be "factored" as a product of a polynomial and a no-where vanishing holomorphic function. Something similar holds true for meromorphic functions.

Proposition 0.2. *Let $f \in \mathcal{M}(\Omega)$ such that f has only finitely many poles $\{p_1, \cdots, p_n\}$ with orders $\{m_1, \cdots, m_n\}$. Then there exist holomorphic functions $g, h \in \mathcal{O}(\Omega)$ such that for all $z \in \Omega \setminus \{p_1, \cdots, p_n\}$,*

$$f(z) = \frac{g(z)}{h(z)}.$$

Moreover, we can choose g and h such that $f(z)$ and $g(z)$ have the exact same roots with same multiplicities, while $h(z)$ has zeroes precisely at $p_1, \cdots, p_n$ with multiplicities exactly $m_1, \cdots, m_n$.

Proof. We define $g : \Omega \setminus \{p_1, \cdots, p_m\}$ by

$$g(z) = \Big(\Pi_{k=1}^{n}(z - p_k)^{m_k}\Big) f(z).$$

This is clearly a holomorphic function. Moreover, since $f(z)$ has a pole of order m_k at p_k, $g(z)$ is bounded in a neighbourhood of p_k. Thus, by the

Date: 24 August 2016.

theorem on removable singularities, $g(z)$ can be extended as a holomorphic function on Ω. The theorem is then proved with

$$h(z) = (z - p_1)^{m_1} \cdots (z - p_n)^{m_n}.$$

□

Remark 0.1. *The same is true even if the meromorphic function has infinite number of poles. This is a consequence of Weierstrass' factorization theorem. We will prove this theorem for the special case when $\Omega = \mathbb{C}$. For a general open set the proof requires the use of Runge's approximation theorem.*

Partial fraction decomposition of meromorphic functions on $\mathbb{C}$.

Recall that if f has a pole of order m at p, then the Laurent series expansion can be written as

$$f(z) = Q_p f\Big(\frac{1}{z-p}\Big) + H_p f(z),$$

where $H_p f$ is holomorphic near p, and $Q_p f(w)$ is a polynomial

$$Q_p f(w) = a_{-m} w^m + \cdots a_{-1} w,$$

where for each $n = 1, 2, \cdots, m$ and each $\varepsilon << 1$, we have

$$a_{-n} = \frac{1}{2\pi i} \int_{|z-p|=\varepsilon} f(z)(z-p)^{n-1}.$$

The difference $f(z) - H_p f(z)$ is called the *principal* part of $f(z)$ at p. We then have the following fundamental theorem.

Theorem 0.1 (Mittag-Leffler)**.** *Let $\{p_k\}$ be a discrete set of points in Ω, and for each k, let $Q_k(w)$ be a polynomial without a constant term. There there exists a $f \in \mathcal{M}(\Omega)$ with poles at p_n and holomorphic everywhere else, with principle part at p_k given by $Q_k(1/(z-p_k))$. Moreover, all such meromorphic functions are of the form*

$$f(z) = \sum_k \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big) + H(z),$$

where each $q_k(z)$ and $H(z)$ are holomorphic functions on Ω. Furthermore:

1. *If $\{p_k\}$ is a finite sequence, then one could take $q_k \equiv 0$.*
2. *If $\Omega = \mathbb{C}$, and $|p_k| \to \infty$, then one could take each q_k to be a polynomial.*

We will prove parts (1) and (2) in the next lecture. For a general Ω and infinitely many poles, the proof require's Runge's theorem, and an outline will be provided in the appendix to the next lecture.

Remark 0.2. *Note that if $\sum_k g_k$ is a compactly convergent series on Ω (for instance, take a power series), then $\tilde{q}_k = q_k + g_k$, and $\tilde{H} = H + \sum_k g_k$ will give another representation for the function $f(z)$, and hence q_k and H are by no means unique.*

Remark 0.3. *Note that given a meromorphic function, the theorem does not say whether that particular function has a partial fraction decomposition (unlike say for rational functions, as we will see in the next section, or more generally meromorphic functions with only a finite number of poles). In other words, we are not claiming that the converse holds (even when $\Omega = \mathbb{C}$). It does turn out that the converse holds under some additional conditions on the distribution of poles. But in particular examples, one can get away by a more hands-on approach. We will see a beautiful illustration of this below.*

Example 0.1. *Consider the meromorphic function $f(z) = \pi^2/\sin^2 \pi z$ which is a meromorphic function with poles at integers. Near zero,*

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{\Big(\pi z + O(z^3)\Big)^2} = \frac{1}{z^2\Big(1 - z^2/6 + \cdots\Big)^2} = \frac{1}{z^2}\Big(1 + \frac{z^2}{6} + \cdots\Big)^2,$$

and so the principal part of $f(z)$ is given by $1/z^2$. Using the identity $\sin^2(\pi(z-n)) = \sin^2 \pi z$, it is easy to see that the principal parts at $z = n$ are given by $(z-n)^{-2}$. Now consider the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

This converges uniformly on all compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, and hence represents a meromorphic function on $\mathbb{C}$ with poles of order two at all integers. Moreover the principle parts at each pole $z = n$ is given by $Q_n((z-n)^{-1})$, where $Q_p(w) = w^2$. It is then easy to see that the difference

$$H(z) := \frac{\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

extends to an entire function.

Claim. $H \equiv 0$.

Proof. Note that the series and the function on the left are both periodic with period 1, and hence so is $H(z)$. That is, $H(z+1) = H(z)$ for all $z \in \mathbb{C}$. Also by Euler's identity, if $z = x + iy$, then

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = \sin(\pi x)\cosh(\pi y) + \sinh(\pi y)\cos(\pi x),$$

and so

$$|\sin \pi z|^2 = \cosh^2(\pi y) - \cos^2(\pi x) \geq \cosh^2(\pi y) - 1 \to \infty$$

uniformly as $|y| \to \infty$. As a consequence $\pi^2/\sin^2 \pi z$ converges uniformly to zero as $|y| \to \infty$. But the infinite series also shares this property. Indeed, since the series converges uniformly on $|y| \geq 1$, we can take pointwise limit, and clearly each $(z-n)^{-2} \to 0$ uniformly as $|y| \to \infty$. The upshot is that $H(z)$ converges uniformly to zero as $|y| \to 0$. In particular, $H(z)$ is bounded on the strip $\{z \in \mathbb{C} \mid 0 \leq Re(z) \leq 1\}$. But then since H is periodic with period one, this means that H is a bounded entire function, and hence a constant by Liouville. But since $\lim_{y \to 0} H(iy) = 0$, we can conclude that $H \equiv 0$. $\square$

Assuming this, we get the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

for all $z \notin \mathbb{Z}$. Plugging in $z = 1/2$, we obtain the identity

$$\frac{\pi^2}{4} = \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Now let, $S = \sum_{m=0}^{\infty} m^{-2}$. Then we have

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \frac{\pi^2}{8} + \frac{S}{4}. \end{aligned}$$

Solving for S, we get the beautiful identity

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Proving this identity was the so-called Basel problem, first "solved" by Euler. But his "proof" would not pass our modern day standards of rigour. Euler used a "facotrization" for sine, but a rigorous development of the theory of infinite product factorizations of entire functions had to wait till Weierstrass came along many decades later. Nevertheless, Euler's insights were of course crucial in all subsequent developments.

Meromorphic functions on the extended complex plane

It is often useful to think of $z = \infty$ on the same footing as other points in the complex plane, and to define the *extended complex plane*

$$\hat{\mathbb{C}} = \mathbb{C} \cup \infty.$$

We can then think of meromorphic functions $f : \Omega \setminus \{p_1, \cdots, p_j, \cdots\} \to \mathbb{C}$, as functions $f : \Omega \to \hat{\mathbb{C}}$, by defining

$$f(p_j) = \infty$$

for all the poles p_js. Similarly, in studying meromorphic functions on $\mathbb{C}$, it is also useful to consider the extension of the function themselves to $\hat{\mathbb{C}}$. We say that $z = \infty$ is a pole of order m (resp. removable or essential singularity) if $z = 0$ is a pole of order m (resp. removable or essential singularity) for the function

$$\hat{f}(z) = f(1/z).$$

Similarly we can also define a zero of order m at infinity. We then say that a meromorphic function on $\mathbb{C}$ is meromorphic on the extended plane, if it does not have an essential singularity at $z = \infty$. It turns out that meromorphic functions on $\hat{\mathbb{C}}$ can be classified. Recall that a rational function on $\mathbb{C}$ is a function of the form

$$R(z) = \frac{P(z)}{Q(z)}$$

where both $P(z)$ and $Q(z)$ are polynomials.

Example 0.2.
(1) *A polynomial $P(z) = a_n z^n + \cdots + a_0$ with $a_n \neq 0$ has a pole of order n at infinity. In fact, conversely, ever entire function $p(z)$ with a pole of order n at infinity is a polynomial of degree n. This follows from the Cacuhy estimates.*
(2) *The function e^z has an essential singularity at infinity.*
(3) *A rational function has a pole or removable singularity at infinity. In fact a rational function $R(z) = P(z)/Q(z)$ as above has*
- *a pole of order $\deg P - \deg Q$ at infinity if $\deg(P) > \deg(Q)$,*
- *a removable singularity at infinity if $\deg(P) \leq \deg(Q)$,*
- *a zero of order $\deg(Q) - \deg(P)$ if $\deg(P) < \deg(Q)$.*

Theorem 0.2. *The only meromorphic functions on $\hat{\mathbb{C}}$ are rational functions.*

Proof. Let $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic function.

Claim-1. F has only finitely many poles $\{p_1, \cdots, p_n\}$ in the complex plane C.

To see this, note that $F(1/z)$ has either a pole or zero at $z = 0$. In either case there is a small neighborhood $|z| < \varepsilon$ which has no other pole. Which is the same as saying that F has no finite pole in $|z| > 1/\varepsilon$. But $|z| \leq 1/\varepsilon$ is compact, and since all poles are isolated, this shows that there are only finitely many poles. Now, corresponding to each of the poles $p_k \in \mathbb{C}$ there exists a polynomial Q_k (see Remark 0.2 in Lecture-20) such that

$$F(z) = Q_k\Big(\frac{1}{z - p_k}\Big) + H_k(z),$$

where G_k is holomorphic on a whole neighborhood around p_k (including at the point p_k). Similarly if $|z| > R$, we can write

$$F(z) = Q_\infty(z) + H_\infty\Big(\frac{1}{z}\Big),$$

where as before, $H_\infty(z)$ is holomorphic in a neighborhood of $z = 0$.

Claim-2. The function

$$G(z) = F(z) - Q_\infty(z) - \sum_{k=1}^{n} Q_k\Big(\frac{1}{z-p_k}\Big)$$

is an entire and bounded function.

Assuming the claim, by Liouville's theorem, $G(z)$ is a constant, and hence $F(z)$ must be rational, and the theorem is proved. To prove the claim, first note that clearly, $G(z)$ is holomorphic away from $\{p_1, \cdots, p_n\}$. At some $z = p_k$, $Q_j(1/z - p_j)$ is holomorphic for all $j \neq k$. On the other hand, near p_k,

$$F(z) - Q_k\Big(\frac{1}{z-p_k}\Big) = H_k(z)$$

which is holomorphic. This shows that $G(z)$ is entire. As a consequence, to show boundedness, we only need to show boundedness on $|z| > R$ for some large R. To see, first observe that since Q_k are polynomials,

$$\lim_{z\to\infty} Q_k\Big(\frac{1}{z-p_k}\Big) = 0.$$

Hence it is enough to show that $F(z) - Q_\infty(z)$ is bounded near infinity. But this follows immediately from noting that

$$H_\infty(z) = F\Big(\frac{1}{z}\Big) - Q_\infty\Big(\frac{1}{z}\Big)$$

is holomorphic near $z = 0$ and hence is bounded on $|z| < \varepsilon$ for some $\varepsilon > 0$. In particular $F(z) - Q_\infty(z)$ is bounded on $|z| > 1/\varepsilon$. This proves the claim, and hence completes the proof of the theorem.

$\square$

Remark 0.4. *A meromorphic function $f \in \mathcal{M}(\hat{\mathbb{C}})$ gives rise to a holomorphic map $F : \mathbb{P}^! \to \mathbb{P}^1$. Conversely, given any map $F : \mathbb{P}^1 \to \mathbb{P}^1$, one gets a meromorphic map from $\hat{\mathbb{C}} \to \mathbb{C}$ with poles at $F^{-1}([0,1])$. So the theorem can be reformulated in the following way - all holomorphic maps from $F : \mathbb{P}^1 \to \mathbb{P}^1$ are given by rational functions of two variables, where the numerator and denominator are homogenous polynomials.*

A simple consequence of the proof is the following theorem on partial fraction decomposition that we take for granted as an important tool in integration theory, but never see the proof of.

Corollary 0.1. *For any rational function $R(z) = P(z)/Q(z)$ has a partial fraction decomposition of the form*

$$R(z) = Q_\infty(z) + \sum_{k=1}^{n} Q_k\Big(\frac{1}{z-p_k}\Big),$$

where p_k is a root of $Q(z)$ of order m_k, Q_k is a polynomial of degree m_k, and $\deg Q_\infty = \deg P - \deg Q$ if this number is non-negative. Else we have that $Q_\infty \equiv 0$.

∗ Department of Mathematics, Indian Institute of Science
Email address: vvdatar@iisc.ac.in