LECTURE-17

VED V. DATAR*

Proof of the Mittag-Leffler theorem.

Recall that the Mittag-Leffler theorem was as follows.

Theorem 0.1 (Mittag-Leffler)**.** *Let $\{p_k\}$ be a discrete set of points in Ω, and for each k, let $Q_k(z)$ be a polynomial without a constant term. There there exists a $f \in \mathcal{M}(\Omega)$ with poles at p_k and holomorphic everywhere else, with principle part at p_k given by $Q_k(1/(z-p_k))$. Moreover, all such meromorphic functions are of the form*

$$f(z) = \sum_k \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big) + H(z),$$

where each $q_k(z)$ and $H(z)$ are holomorphic functions on Ω, and q_k depends only on Q_k. Furthermore:

1. *If $\{p_k\}$ is a finite sequence, then one could take $q_k \equiv 0$.*
2. *If $\Omega = \mathbb{C}$, and $|p_k| \to \infty$, then one could each q_k to be a polynomial.*

Proof. The proof of the theorem in case (1) is trivial, and we leave it as an exercise. We prove the theorem only in the case (2) above. For a general Ω with possibly infinite sequence $\{p_k\}$, the proof relies on Runge's theorem and is out of the scope of the present course.

So from now, suppose $\Omega = \mathbb{C}$. Without loss of generality, we can assume that no p_k is equal to 0. Suppose we order them such that $0 < |p_1| \leq \cdots$. By the first part, we can assume that the number of poles is infinite, and hence that $\lim_{k\to\infty} |p_k| = \infty$. Since each $Q_k(1/(z-p_k))$ is holomorphic on $|z| < |p_k|$, it can be expanded as a Taylor series around $z = 0$. Let q_k be the partial sum of degree d_k of this Taylor expansion. Let

$$M_k = \sup_{|z|\leq|p_k|/2} \Big|Q_k\Big(\frac{1}{z-p_k}\Big)\Big|.$$

Claim. For all z such that $|z| \leq |p_k|/4$, we have the estimate

$$\Big|Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big| \leq 2M_k\Big(\frac{2|z|}{|p_k|}\Big)^{d_k+1} \leq M_k 2^{-d_k}.$$

Date: 24 August 2016.

We assume the claim for the moment. Now pick $d_k >> 1$ such that $2^{d_k} \geq M_k 2^k$, and consider the series

$$f(z) = \sum_k \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big).$$

For any compact set $K \subset\subset \mathbb{C} \setminus \{p_1, \cdots, \}$, there exists a N such that for $k > N$, $K \subset D_{|p_k|/4}(0)$. By the claim and our choice of d_k, for all $k > N$, each term of the infinite series

$$\sum_{k=N+1}^{\infty} \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big)$$

is dominated by 2^{-k}, and hence by Weierstarss test the tail, represents a holomorphic function. On the other hand

$$\sum_{k=1}^{N} \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big)$$

is a meromorphic function on K with poles at $p_1, \cdots, p_N$ with prescribed principal parts. In particular, $f(z)$ is meromorphic on $\mathbb{C}$ with poles at p_k with principal part $Q_k((z-p_k)^{-1})$. To finish the proof, if $\tilde{f}$ is another such function, then clearly $\tilde{f} - f$ extends to an entire function.

Proof of the claim. Suppose

$$Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z) = \sum_{j=d_k+1}^{\infty} a_j z^j$$

in the disc $|z| < |p_k|/2$. By the Cauchy estimate, we have that $|a_j| \leq \frac{2^j M_k}{|p_k|^j}$, and so

$$\begin{aligned}
\Big|Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big| &\leq M_k \sum_{j=d_k+1}^{\infty} \frac{2^j |z|^j}{|p_k|^j} \\
&\leq M_k \Big(\frac{2|z|}{|p_k|}\Big)^{d_k+1} \sum_{j=0}^{\infty} \frac{1}{2^j} \\
&\leq 2M_k \Big(\frac{2|z|}{|p_k|}\Big)^{d_k+1},
\end{aligned}$$

where we used the fact that $|z| \leq |p_k|/4$ in the penultimate line. $\square$

Example 0.1. *In the previous lecture, we illustrated the theorem by obtaining an expansion for $\pi^2/\sin^2 \pi z$. We now obtain an expansion for $\pi \cot \pi z$ which is meromorphic on $\mathbb{C}$ with only simple poles at $z = n \in \mathbb{Z}$. In fact, the principal part is precisely $(z-n)^{-1}$. Unfortunately, the series $\sum (z-n)^{-1}$ is divergent, and hence one has to subtract off a polynomial, which in this* *case turns out to be a constant. Consider the series*

$$\sum_{n\neq 0}\frac{1}{z-n}+\frac{1}{n}=\sum_{n\neq 0}\frac{1}{(z-n)n},$$

which is compactly convergent on $\mathbb{C}\setminus\mathbb{Z}$ as can be seen by comparing with the series $\sum n^{-2}$. Hence the series represents a meromorphic function on $\mathbb{C}$ with simple poles at $z=n$. Then clearly

$$H(z)=\pi\cot\pi z-\frac{1}{z}-\sum_{n\neq 0}\frac{1}{z-n}+\frac{1}{n}$$

is an entire function. Moreover, by direct calculation, one can see that for $z\notin\mathbb{Z}$ (and hence everywhere),

$$H'(z)=\frac{\pi^2}{\sin^2\pi z}-\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}=0$$

by our expansion from previous lecture. Hence $H(z)$ is a constant. Now, rewriting the function as

$$H(z)=\pi\cot\pi z-\lim_{m\to\infty}\Big(\frac{1}{z}+\sum_{n=-m}^{m}\frac{1}{z-n}+\frac{1}{n}\Big)=\pi\cot\pi z-\frac{1}{z}-\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

The right hand side is an odd function, and hence $H(z)$ must be zero. Thus we obtain the identity

$$\pi\cot\pi z=\frac{1}{z}+\sum_{n\neq 0}\Big(\frac{1}{z-n}+\frac{1}{n}\Big). \tag{0.1}$$

Infinite products

A infinite product of non-zero complex numbers $\Pi_{n=1}^{\infty}p_n$ is said to *converge* if

$$P:=\lim_{n\to\infty}\Pi_{k=1}^{n}p_k$$

exists. If some of the terms are allowed to be zero, then we say that the infinite product converges if the following two conditions hold

(1) At most a finite number of terms are zero.
(2) If $N>0$ such that $p_n\neq 0$ for all $k>N$, then $\Pi_{k=N+1}^{\infty}p_n$ converges in the above sense.

. If we denote the n^{th} partial product by $P_n=\Pi_{k=1}^{n}p_k$, then it is clear that it any convergent product $p_n=P_n/P_{n-1}$ converges to 1. Denoting $p_n=1+b_n$, we say that the product converges absolutely if $\Pi(1+|b_n|)$ converges. We then have the following basic fact.

Proposition 0.1. *Let $\{b_n\}$ be a sequence of complex functions, none of which is zero. Then the infinite product $\Pi(1+b_n)$ (absolutely) converges if and only the series $\sum\log(1+b_n)$ (absolutely) converges, where* log *is the principal branch of the logarithm.*

One can similarly talk about uniform convergence and compact convergence of infinite products. We then have the following counterpart of the above theorem.

Proposition 0.2. *Let $\{f_n\}$ be a sequence of entire. Suppose that for every compact set K, all but finitely many f_ns are zero free in K. Then $\Pi_{n=1}^{\infty}(1+f_n(z))$ converges compactly (resp. absolutely) on $\mathbb{C}$ if and only if $\sum \log f_n(z)$ converges compactly (resp. absolutely) on $\mathbb{C}$. In such a case the infinite product converges to an entire function.*

Note that absolutely convergent products also satisfy the "rearrangement property".

Weierstrass factorisation theorem

Theorem 0.2. *Let $\{a_n\}$ be an arbitrary sequence of non-zero complex numbers ordered such that $\lim_{n\to\infty} |a_n| = \infty$, if the sequence is infinite. Then there exists an entire function with zeroes at precisely the points a_n. Moreover, every entire function with these and no other zeroes (except possibly at $z = 0$) is given by*

$$f(z) = z^m e^{g(z)} \Pi_{n=1}^{\infty}\Big(1 - \frac{z}{a_n}\Big) e^{q_n(z)}, \tag{0.2}$$

where q_n is a polynomial given by

$$q_n(z) = \frac{z}{a_n} + \frac{1}{2}\Big(\frac{z}{a_n}\Big)^2 + \cdots + \frac{1}{m_n}\Big(\frac{z}{a_n}\Big)^{m_n}.$$

The convergence here is absolute, and uniform on compact sets. Furthermore, if there exists some integer $h > 0$, such that

$$\sum_n \frac{1}{|a_n|^{1+h}} < \infty, \tag{0.3}$$

then we can take $m_n = h$ for all n.

The expression (0.2) above is called the *canonical product* associated with $\{a_n\}$, and the smallest integer h satisfying (0.3) (if it exists) is called the *genus* of the canonical product. Else the genus is said to be infinite. If $g(z)$ above reduces to a polynomial, then we say that $f(z)$ is of finite genus, and the genus of $f(z)$ is defined to be the maximum of the degree of $g(z)$ and h.

Proof. We only prove the second part, and leave the more general statement as an exercise. So from now on assume that our sequence satisfies (0.3). We need to prove the existence of polynomials $q_n(z)$ such that the infinite product

$$\Pi_{n=1}^{\infty}\Big(1 - \frac{z}{a_n}\Big) e^{q_n(z)}$$

converges to an entire function. By Proposition 0.2, this holds if and only if the series with the general term

$$r_n(z) = \log\Big(1 - \frac{z}{a_n}\Big) + q_n(z)$$

converges compactly. Let $R > 1$ be arbitrary. We only consider those a_n such that $|a_n| > 2R$. Then on $|z| < R$, since $|z/a_n| < 1$, the first term in $r_n(z)$ has a power series expansion:

$$\log\Big(1 - \frac{z}{a_n}\Big) = -\frac{z}{a_n} - \frac{1}{2}\Big(\frac{z}{a_n}\Big)^2 - \cdots .$$

For the h as in the hypothesis, we let

$$q_n(z) = \frac{z}{a_n} + \frac{1}{2}\Big(\frac{z}{a_n}\Big)^2 + \cdots + \frac{1}{h}\Big(\frac{z}{a_n}\Big)^h,$$

so that

$$r_n(z) = -\frac{1}{1+h}\Big(\frac{z}{a_n}\Big)^{h+1} - \frac{1}{2+h}\Big(\frac{z}{a_n}\Big)^{h+2} - \cdots .$$

By comparison with a geometric series, we obtain the estimate that

$$|r_n(z)| \le \frac{1}{1+h}\Big(\frac{R}{|a_n|}\Big)^{1+h}\Big(1 - \frac{R}{|a_n|}\Big)^{-1} < \frac{2}{1+h}\Big(\frac{R}{|a_n|}\Big)^{1+h}.$$

By hypothesis, $\sum r_n(z)$ converges absolutely and compactly on $|z| \le R$, and hence so must the product. Note that even though the proof required us to work with $|a_n| > 2R$, just multiplying in the terms on the product with $|a_n| \le 2R$ does not affect convergence of the product, since such a_n's are only finite in number. This shows that the product with our choice of q_n converges to an entire function with the required properties. Now suppose $f(z)$ is any other such entire function with a zero of order m at $z = 0$. Then consider the entire function

$$F(z) = \frac{f(z)}{z^m \Pi_{n=1}^{\infty}\Big(1 - \frac{z}{a_n}\Big)e^{q_n(z)}}.$$

Since the zeroes of the numerator and denominator match up, $F(z)$ clearly extends as an entire function which is no-where zero. But then by simple connectivity of $\mathbb{C}$, this implies that one can take a holomorphic branch $g(z) = \log F(z)$. Then clearly $f(z)$ has the expression above. $\square$

Example 0.2. *Consider the function* $\sin \pi z$. *This is an entire function with zeroes at* $z = \pm n$. *Since*

$$\sum_n \frac{1}{n^{1+h}}$$

converges for $h = 2$ *(and* 2 *is the smallest such integer), we can apply the above proposition to obtain*

$$\sin \pi z = z e^{g(z)} \Pi_{n \in \mathbb{Z} \setminus \{0\}}\Big(1 - \frac{z}{n}\Big)e^{z/n}.$$

Claim. $e^{g(z)} \equiv \pi$.

To see this, we take a logarithmic derivative. Then

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0}\Big(\frac{1}{z-n} - \frac{1}{n}\Big).$$

Comparing with the expansion of $\pi \cot \pi z$, we see that $g'(z)$ has to vanish, and hence $g(z)$ is a constant. On the other hand,

$$\pi = \lim_{z\to 0} \frac{\sin \pi z}{z} = e^{g(0)},$$

and so $e^{g(z)} \equiv \pi$. It follows that $f(z)$ is an entire function of genus 1. Because of absolute convergence, we can rewrite

$$\frac{\sin \pi z}{\pi z} = \Pi_{n=1}^{\infty}\Big(1 - \frac{z^2}{n^2}\Big).$$

Expanding, and comparing the coefficients of z^2, we once again see that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This was the original approach of Euler.

We finish with the following elementary corollary, whose proof we leave as an exercise.

Corollary 0.1. *Any meromorphic function in $\mathbb{C}$ is the quotient of two entire functions.*

$*$ Department of Mathematics, Indian Institute of Science
Email address: vvdatar@iisc.ac.in