LECTURE-18

VED V. DATAR*

Residue theorem

Let f be a holomorphic function in $D_\varepsilon(p) \setminus \{p\}$ with a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n.$$

The *residue* of $f(z)$ at $z = p$ is then defined by

$$\mathrm{Res}_{z=p} f(z) = a_{-1}.$$

Then by Theorem 2 in Lecture 16, for any $r < \varepsilon$,

$$\mathrm{Res}_{z=p} f(z) = \frac{1}{2\pi i} \int_{|z-p|=r} f(z)\, dz.$$

More generally, we have the following fundamental result.

Theorem 1 (Residue Theorem)**.** *Let Ω be open, $\{p_k\} \in \Omega$ a sequence of isolated points, and $f \in \mathcal{O}(\Omega^*)$, where $\Omega^* := \Omega \setminus \{p_1, \cdots\}$. Then for any cycle $\gamma \sim_\Omega 0$ in Ω such that no p_k lies on Supp(γ), we have*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_k n(\gamma, p_k) \mathrm{Res}_{p_k} f(z).$$

Moreover, for any given γ as above, $n(\gamma, p_k) = 0$ for all but finitely many k, and hence the summation above has only a finite number of non-zero terms.

Proof. First let us assume that there are only a finite number of singular points. Let C_k be a small circle around p_k enclosing a disc D_k, such that $\overline{D_k} \subset \Omega$ and such that C_k does not intersect *Supp*(γ). We now claim that

$$\gamma \sim_{\Omega^*} \sum_k n(\gamma, p_k) C_k. \tag{0.1}$$

Assuming this, we are then done by the generalised Cauchy theorem, since

$$\begin{aligned}\int_\gamma f(z)\, dz &= \sum_k n(\gamma, p_k) \int_{C_k} f(z)\, dz \\ &= 2\pi i \sum_k n(\gamma, p_k) \mathrm{Res}_{z=p_k} f(z).\end{aligned}$$

To prove (0.1) let $a \notin \Omega^*$. We need to show that

$$n(\gamma, a) = \sum_k n(\gamma, p_k) n(C_k, a). \tag{0.2}$$

If $a \notin \Omega$, then by construction $a \notin \overline{D_k}$, and since a disc is simply connected, $n(C_k, a) = 0$. On the other hand, since $\gamma \sim_\Omega 0$, we also have that $n(\gamma, a) = 0$, and hence (0.2) is trivially satisfied. If $a \in \Omega$, then $a = p_j$ for some j. Once again, as above, $n(C_k, a) = 0$ for all $k \neq j$. On the other hand, $n(C_j, a) = 1$. This verifies (0.2) and completes the proof of (0.1).

Finally, suppose the number of singularities is infinite in number. It is enough to prove that $n(\gamma, p_k) = 0$ for all but finitely many k. Note that

$$U_0 := \{a \in \mathbb{C} \mid n(\gamma, a) = 0\}$$

is an open set (since the index is locally constant). Moreover it contains the annulus $A_{R.\infty}(0)$ for R large enough. As a consequence, the set $U^c = \mathbb{C} \backslash U_0$ is a compact set and must contain only finitely many p_k (since the singularities are isolated). □

An important corollary is the following.

Corollary 0.1. *Let f be holomorphic in Ω except possibly at isolate points $\{p_1, p_2, \cdots, \}$ in Ω, and let γ be a positively oriented, simple, closed curve in Ω not passing through any of the singularities. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{p_k \in Int(\gamma)} \mathrm{Res}_{z=p_k} f(z).$$

Proof. This follows from the residue theorem and the fact that

$$n(\gamma, p_k) = \begin{cases} 1, & z \in Int(\gamma) \\ 0, & z \in Ext(\gamma). \end{cases}$$

□

Our next result helps in computing the residue at poles.

Proposition 0.1. *Let f have a pole of order m at p. Then*

$$\mathrm{Res}_{z=p} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\Big|_{z=p} (z-p)^m f(z).$$

Proof. If f has a pole of order m, then $(z-p)^m f(z)$ has a removable singularity at $z = p$. Moreover, if the Laurent series expansion for f at p is given by

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-p)^n,$$

then

$$(z-p)^m f(z) = \sum_{k=0}^{\infty} a_{k-m} (z-p)^k,$$

and hence

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\Big|_{z=p}(z-p)^m f(z) = a_{m-1-m} = a_{-1} = \mathrm{Res}_{z=p} f(z).$$

□

Example 0.1. *Let us evaluate*

$$\int_{S_R} \frac{e^{\pi/2z}}{1+z^2}\, dz,$$

where S_R is a square of side length $2R$ centred at the origin and oriented in an anti-clockwise direction. Let $f(z)$ be the integrand. Then it has three isolated singularities, namely an essential one at 0 and poles of order one at $\pm i$. Let us compute the residue at each of the singularities.

- **Residue at $z = 0$.** *The Laurent series expansion is given by*

$$\frac{e^{(\pi/2z)}}{z^2+1} = \Big(\sum_{n=0}^{\infty} \frac{\pi^n}{2^n n!} z^{-n}\Big)\Big(\sum_{m=0}^{\infty} (-1)^{2m} z^{2m}\Big),$$

hence the residue, which is the coefficient of z^{-1} is given by

$$\mathrm{Res}_{z=0} f(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{\pi}{2}\right)^{2m+1} = \sin\left(\frac{\pi}{2}\right) = 1.$$

- **Residue at $z = i$.** *By Proposition 0.1, the residue is given by*

$$\mathrm{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)\frac{e^{\pi/2z}}{z^2+i} = \frac{e^{\pi/2i}}{2i} = -\frac{1}{2}.$$

- **Residue at $z = -i$.** *Once again by Proposition 0.1, the residue is given by*

$$\mathrm{Res}_{z=-i} f(z) = \lim_{z\to -i}(z+i)\frac{e^{\pi/2z}}{z^2+i} = \frac{e^{-\pi/2i}}{-2i} = -\frac{1}{2}.$$

Then by the residue theorem, we have

$$\int_{S_R} \frac{e^{\pi/2z}}{1+z^2}\, dz = \begin{cases} 1, & R < 1, \\ 0, & R > 1. \end{cases}$$

The argument principle

Theorem 2 (The argument principle)**.** *Let Ω be a domain and $f \in \mathcal{M}(\Omega)$ zeroes at $\{a_j\}$ or orders $\{m_j\}$ and poles at $\{b_k\}$ of orders $\{n_k\}$. Then for every cycle $\gamma \sim_\Omega 0$ which does not pass through any zeroes or poles, we have that*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_j n(\gamma, a_j) m_j - \sum_k n(\gamma, b_k) n_k.$$

Furthermore the two summations are finite summations.

As a simple corollary we have the following.

Corollary 0.2. *Let Ω be a simply connected domain and $f \in \mathcal{O}(\Omega)$ with zeroes at $\{a_j\}$ or orders $\{m_j\}$. Then for any simple, closed, positively oriented curve γ no passing through any of the roots, we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{a_j \in Int(\gamma)} m_j.$$

Proof of the argument principle. Note that f'/f has poles precisely at the zeroes and poles of $f(z)$, and is holomorphic everywhere else. So the integral can be computed by using the residue theorem. To do so, we need to compute the residues of f'/f. There are two cases.

(1) **Residue of f'/f at $z = a_j$.** Near a_j, say on $D_{\varepsilon_j}(a_j)$, we can write $f(z) = (z - a_j)^{m_j} g_j(z)$, where $g_j(z)$ is holomorphic and zero free on $D_{\varepsilon_j}(a_j)$. Then

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z - a_j} + \frac{g_j'(z)}{g_j(z)}.$$

Since g_j'/g_j is holomorphic, we have that

$$\text{Res}_{z=a_j} \frac{f'(z)}{f(z)} = m_j.$$

(2) **Residue of f'/f at $z = b_k$.** Near b_k, say on $D_{\varepsilon_k}(b_k)$, we can write $f(z) = (z - b_k)^{-n_k} g(z)$, where $g_k(z)$ is holomorphic and zero free on $D_{\varepsilon_k}(b_k)$. Then

$$\frac{f'(z)}{f(z)} = -\frac{n_k}{z - a_j} + \frac{g_k'(z)}{g_k(z)}.$$

Once again, since g_k'/g_k is holomorphic, we have that

$$\text{Res}_{z=b_k} \frac{f'(z)}{f(z)} = -n_k.$$

The theorem then follows by an application of the residue theorem. $\square$

Remark 0.1. *More generally, if f is holomorphic, and we take $f(z) - w$, then for any simple closed curve γ such that $w \notin f(\text{Supp}(\gamma))$,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - w}\,dz = \sum_{a_j \in Int(\gamma), f(a_j) = w} m_j,$$

where m_j is the order of the zero of $f(z) - w$ at a_j.

Argument principle as an index calculation. If γ is a short curve such that $|\gamma(0)| = |\gamma(1)|$, and not passing through the origin, then the index $n(\gamma, 0)$ computes the change in the argument of $\gamma(t)$ (upto a factor of 2π). To see this, note that is γ is short enough, then one can define a holomorphic branch of the logarithm $\log z$ in a neighborhood of $\text{Supp}(\gamma)$. Then

$$n(\gamma, 0) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = \frac{\log\gamma(1) - \log\gamma(0)}{2\pi i} = \frac{\arg\gamma(1) - \arg(0)}{2\pi}.$$

Now, let $\gamma : [0,1] \to \Omega$ be a curve and $f \in \mathcal{O}(\Omega)$. Then $\Gamma(t) = f(\gamma(t))$ defines a curve in $\mathbb{C}$ with $\Gamma'(t) = f'(\gamma(t))\gamma'(t)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt = \frac{1}{2\pi i}\int_0^1 \frac{\Gamma'(t)}{\Gamma(t)}\,dt = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w},$$

and hence we conclude that

$$n(\Gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

So the integral of f'/f along γ essentially measures the change in argument of $f(z)$. More generally, together with the remark above, we see that for any $w \notin f(\mathrm{Supp}(\gamma))$,

$$n(\Gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - w}\,dz.$$

* Department of Mathematics, Indian Institute of Science
Email address: vvdatar@iisc.ac.in