LECTURE-19

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In this lecture, we'll see three important applications of the argument principle.

Local mapping properties of holomorphic functions

Theorem 1. *Let Ω be connected, and $f \in \mathcal{O}(\Omega)$ be a non-constant holomorphic function. Suppose $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order m at z_0. Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that whenever w with $0 < |w - w_0| < \delta$, the equation $f(z) = w$ has exactly m distinct solutions in $B_\varepsilon(z_0)$ each of multiplicity one.*

Proof. First we choose $\varepsilon_0 > 0$ such that

(1) $f(z) - w_0$ has no other root in $\overline{D_{\varepsilon_0}(z_0)}$, and
(2) For all $z \in D_{\varepsilon_0}(z_0)$, $f'(z) \neq 0$.

The first condition can be achieved since $f(z) - w_0$ is not constant, and zeroes are isolated. For the second condition, if $m = 1$, then $f'(z_0) \neq 0$, and hence an $\varepsilon_0 > 0$ as above can be picked by continuity of $f'(z)$. If $m > 1$, then $f'(z_0) = 0$. But $f'(z)$ is also holomorphic, and hence it's zeroes must also be isolated. Let γ be the circle $|z - z_0| = \varepsilon$ oriented in the positive sense, and let $\Gamma = f \circ \gamma$. Now $w_0 \notin \mathrm{Supp}(\Gamma)$ by propert (1) above, and hence there exists a $\delta > 0$ such that $\overline{D_\delta(w_0)} \subset \mathbb{C} \setminus \mathrm{Supp}(\Gamma)$. For any $w \in D_\delta(w_0)$, since the index is locally constant, $n(\Gamma, w) = n(\Gamma, w_0)$. By our discussion in the previous lecture, $n(\Gamma, w_0)$ counts the number of zeroes of $f(z) - w_0$ (with multiplicity) in the interior of γ, which in this case is m. Hence $n(\Gamma, w) = m$, and so $f(z) - w$ also has exactly m solutions in $D_\varepsilon(z_0)$ counted with multiplicity. Now, look at $g(z) = f(z) - w$. Since $g'(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$, none of the roots of $g(z)$ can have multiplicity more than one. Hence $f(z) = w$ has exactly m distinct solutions in $D_\varepsilon(z_0)$, each with multiplicity one. □

Remark 0.1. *The theorem essentially says that locally, holomorphic functions are "branched" or "ramified" covers. That is if $f(z_0) = w_0$ with multiplicity m, and with ε, δ as above, the map $f : D_\varepsilon(z_0) \setminus \{z_0\} \to D_\delta(w_0) \setminus \{w_0\}$ is $m : 1$ covering map, and the m branches come together at z_0. If $m > 1$, we say that z_0 is a branch point, and that m is the branching order. The prototypical example that one should keep in mind is $f(z) = z^m$. Then in any small neighbourhood of $z = 0$ (excluding at zero), then function is $m : 1$. Namely, for any $w \neq 0$, then if $w = re^{i\theta}$, $f(r^{1/m}\zeta_m^k e^{i\theta/m}) = w$ for $k = 0, 1, \cdots, m-1$, where $\zeta_m = e^{2\pi i/m}$ is the primitive m^{th} root of unity.*

Corollary 0.1 (Open mapping theorem)**.** *Let U be and open set, and $f : U \to \mathbb{C}$ be any non-constant holomorphic function. Then $f(U)$ is an open subset of $\mathbb{C}$.*

Proof. Let $w_0 \in f(U)$. Then there exists a $z_0 \in U$ such that $f(z_0) = w_0$. By the above theorem, there exists a $\varepsilon > 0$ and $\delta > 0$ such that $D_\varepsilon(z_0) \subset U$ and $f(z) = w$ has at least one solution in $D_\varepsilon(z)$ for each $w \in D_\delta(w_0)$. In particular, $D_\delta(w_0) \subset U$, and since w_0 was arbitrary, $f(U)$ is open. $\square$

Remark 0.2. *This is of course not true in the real setting, even for polynomials, much less real analytic functions. For instance, consider $f(x) = x^2$ on $(-1, 1)$. Then $f((-1, 1)) = [0, 1)$ which is not open.*

Given two open sets U and V, we say that $f : U \to V$ is a *biholomorphism* if f is bijective, holomorphic, and it's inverse $f^{-1} : V \to U$ is also holomorphic.

Corollary 0.2 (Inverse function theorem)**.** *Let $f \in \mathcal{O}(\Omega)$, and $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, and put $w_0 = f(z_0)$. Then there exist $\varepsilon, \delta > 0$ such that for every $w \in D_\delta(w_0)$ there exists a unique $z_w \in D_\varepsilon(z_0)$ such that $f(z_w) = w$. Moreover we have the following explicit formula for z_w:*

$$z_w = \frac{1}{2\pi i} \int_{|z-z_0|=r} z \frac{f'(z)}{f(z) - w} \, dz, \tag{0.1}$$

where $|z_w - z_0| < r < \varepsilon$. In particular, if we set $U = f^{-1}(D_\delta(w_0)) \cap D_\varepsilon(z_0)$, then $f : U \to D_\delta(w_0)$ is a biholomorphism with $f^{-1}(w) = z_w$ and $(f^{-1})'(w) = 1/f'(z_w)$.

Proof. Since $f'(z_0) \neq 0$, the multiplicity of $f(z) = w_0$ is exactly one at $z = z_0$. By Theorem 1, there exists $\varepsilon, \delta > 0$ such that for all $w \in D_\delta(w_0)$, there is a unique z_w such that $f(z_w) = w$ in the disc $D_\varepsilon(z_0)$. Also note that $f'(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$. locally To prove the formula for z_w, we use the residue theorem. Consider the function

$$H_w(z) = \frac{z f'(z)}{f(z) - w}.$$

Then since $f(z) = w$ has a unique solution in $|z - z_0| < \varepsilon$, $H_w(z)$ has a pole exactly order one at $z = z_w$, and is holomorphic everywhere else. Also note that $f'(z_w) \neq 0$. This follows since $f(z) - w$ has a zero of multiplicity one at z_w. We then compute the residue

$$\begin{aligned} \mathrm{Res}_{z=z_w} H_w(z) &= \lim_{z \to z_w} (z - z_w) \frac{z f'(z)}{f(z) - w} \\ &= z_w f'(z_w) \lim_{z \to z_w} \frac{z - z_w}{f(z) - f(z_w)} = z_w. \end{aligned}$$

Then (0.1) is proved by an application of the residue theorem. In particular, as in the statement of the theorem, if we set $U = f^{-1}(D_\delta(w_0)) \cap D_\varepsilon(z_0)$, then $f : U \to D_\delta(w_0)$ is an injective function with a well defined inverse function $f^{-1} : D_\delta(w_0) \to U$. By the open mapping theorem this inverse function is continuous. In fact since in the formula for f^{-1}, the integrand depends holomorphically on w, an argument similar to the proof of the CIF for derivative, shows that f^{-1} is holomorphic. By the chain rule then $(f^{-1})'(w) = 1/f'(z_w)$.

□

Remark 0.3. *Another proof can be obtained by using the inverse function theorem from multivariable calculus. Recall that if $J_f(z_0)$ is the Jacobian (determinant) of f when thought of as a map from subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, then $J_f(z_0) = |f'(z_0)|^2 \neq 0$. Hence from the inverse function theorem in calculus, there exists a local inverse f^{-1} on an open neighbourhood V of w_0 with continuous partial derivatives. Possibly by shrinking V one can assume that that $f'(z) \neq 0$ on U. All one needs to do is to show that $f^{-1} : V \to U$ is holomorphic. It is enough to prove that f^{-1} satisfies CR equations. By chain rule,*

$$0 = \frac{\partial}{\partial \bar{w}} f \circ f^{-1} = \frac{\partial f}{\partial z}\frac{\partial f^{-1}}{\partial \bar{w}} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \overline{f^{-1}}}{\partial \bar{w}} = f'(z)\frac{\partial f^{-1}}{\partial \bar{w}},$$

since f is holomorphic. But then since $f'(z) \neq 0$, we see that $\partial f^{-1}/\partial \bar{w} = 0$ at each point.

An elementary but important consequence of the proof is the following.

Corollary 0.3. *A holomorphic function is locally injective on an open set U if and only if for all $z \in U$, $f'(z) \neq 0$.*

Proof. Suppose $f'(z)$ is never zero, then the inverse function theorem implies that the function is locally injective. Conversely, suppose the function is injective on some $D_r(z_0)$, but $f'(z_0) = 0$. Then by Theorem 1 there exists a $\delta > 0$, and $w \in D_\delta(f(z_0))$ such that $f(z) = w$ has at least two distinct solutions in $D_r(z_0)$ contradicting injectivity. □

Once again, the counterpart in real variable theory is false, as can be seen by considering the function $f(x) = x^3$. This is globablly (and hence locally) injective, but $f'(0) = 0$.

The maximum modulus principle

The next theorem says that for non-constant holomorphic functions $f(z)$, $|f|$ cannot have local maximums.

Theorem 2 (Max modulus principle)**.** *Let Ω be connected and $f \in \mathcal{O}(\Omega)$. If there exists a $z_0 \in \Omega$ and a neighbourhood U such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$, then $f(z)$ is a constant.*

Proof. By assumption $|f(z_0)| = \sup_{z\in U} |f(z)|$. If $f(z)$ is non-constant on U, then by the open mapping theorem $f(U)$ is an open set. In particular, there exists a $\delta > 0$ such that for any $w \in D_\delta(f(z_0))$, there exists $z \in U$ such that $f(z) = w$. Now simply pick w_1 such that $|w_1| > |f(z_0)|$. Then there exists a $z_1 \in U$ such that $|f(z_0)| < |f(z_1)|$ which is a contradiction. Hence $f(z)$ must be a constant on U. But then by analytic continuation, $f(z)$ must be a constant on all of Ω. $\square$

As a consequence we have the following estimate.

Corollary 0.4. *Let Ω be a bounded set and $f \in \mathcal{O}(\Omega)$ such that f extends continuously to the boundary $\partial\Omega$. Then*

$$\sup_{z\in\Omega} |f(z)| \le \sup_{z\in\partial\Omega} |f(z)|.$$

Proof. It is enough to assume that Ω is connected (or else one could work with each connected component). Since $\overline{\Omega}$ is compact, there exists a $z_0 \in \overline{\Omega}$ such that $|f(z_0)| = \sup_{z\in\overline{\Omega}} |f(z)|$. If $z_0 \in \partial\Omega$, there is nothing to prove. If not, then by the above theorem, z_0 is an interior local maximum for $|f|$ and hence $f(z)$ must be a constant. But in that case the above inequality is trivial. $\square$

Note that a *minimum principle* does not hold, as can be seen easily by considering the function $f(z) = z$ on any neighbourhood of the origin. It turns out that this is only way in which a minimum principle can fail. Recall that $\mathcal{O}^*(\Omega)$ stands for holomorphic functions that are nowhere vanishing on Ω.

Corollary 0.5 (Minimum principle)**.** *Let Ω be connected and $f \in \mathcal{O}^*(\Omega)$. If there exists a $z_0 \in \Omega$ and a neighbourhood U such that $|f(z)| \ge |f(z_0)|$ for all $z \in U$, then $f(z)$ is a constant.*

Proof. Simply apply the maximum modulus principle to the holomorphic function $g(z) = 1/f(z)$. $\square$

Remark 0.4. *A function u is said to be* subharmonic *if $\Delta u \ge 0$ and* superharmonic *if $\Delta u \le 0$. It is a general fact that subharmonic functions satisfy a maximum principle while super harmonic functions satisfy a minimum principle. In particular, harmonic functions satisfy both a minimum and a maximum principle. If $f(z)$ is holomorphic, we can compute that*

$$\Delta|f|^2 = \frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}}|f(z)|^2 = \frac{1}{4}\frac{\partial}{\partial z} f(z)\overline{f'(z)} = |f'(z)|^2 \ge 0.$$

Hence $|f(z)|^2$ is subharmonic, and must satisfy a maximum principle. Hence $|f(z)|$ satisfies a maximum principle. On the other hand if $|f(z)|$ is nowhere vanishing, then $\log|f(z)|^2$ is smooth function, and in fact is harmonic as can be seen from the following computation

$$\Delta \log|f|^2 = \frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}} \log|f(z)|^2 = \frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}}(\log f(z) + \log\overline{f(z)}) = 0.$$

Note that since $f(z)$ is no-where vanishing at least locally near z one can define a holomorphic branch of log. *The upshot is that $\log|f(z)|^2$ must satisfy a minimum principle, and hence must $|f(z)|$ (since* log *is increasing).*

ROUCHE'S THEOREM

Theorem 3. *Let γ be a simple closed curve in Ω and $f, g \in \mathcal{O}(\Omega)$ such that for all $z \in Supp(\gamma)$,*

$$|f(z) - g(z)| < |g(z)|.$$

Then $f(z)$ and $g(z)$ have the same number of zeroes in $Int(\gamma)$.

Proof. Firstly, note that $f(z)$ and $g(z)$ have no zero on γ (the strictness of the inequality above is crucial precisely for this purpose). Moreover, for all $z \in Supp(\gamma)$, we have

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1.$$

Put $F(z) = f(z)/g(z)$. Then $F(z) \in \mathcal{M}(\Omega)$. Moreover, at the points where $f(z)$ and $g(z)$ are non-zero (in particular on $Supp(\gamma)$), one can easily see that

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}.$$

A quick way to see this is that in the nieghbourhood of such points, $\log F(z)$ is well defined and holomorphic, and moreover, $\log F(z) = \log f(z) - \log g(z)$. Now consider $\Gamma := F \circ \gamma$, then Γ is a close curve in $D_1(1)$. Since $D_1(1)$ is simply connected, and $0 \notin D_1(1)$, $n(\Gamma, 0) = 0$. By the argument principle,

$$0 = n(\Gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

Once again by argument principle, we see that $f(z)$ and $g(z)$ must have the same number of zeroes in $Int(\gamma)$. □

Typically, as can be seen in the example below, the theorem is applied to count the number of zeroes of $f(z)$. The heart of the matter is to come up with a suitable $g(z)$, whose zeroes can be counted easily, and such that the above (strict) inequality can be satisfied.

Example 0.1. *Consider the polynomial $p(z) = z^4 - 6z + 3$. We claim that all it's roots are contained in the disc $D_2(0)$, and three of them are contained in the annulus $A_{1,2}(0)$. We divide the proof into the following two cases.*

- **The disc $|z| < 2$.** *On the circle $|z| = 2$ we have the following estimate*

$$\begin{aligned} |p(z) - z^4| &= |6z - 3| \\ &\le 6|z| + 3 = 12 + 3 = 15 < |z|^4. \end{aligned}$$

  By Rouche's theorem, $p(z)$ has the same number of roots as z^4 in $|z| < 2$, and hence has four roots in that disc. But $p(z)$ is also a polynomial of degree four, and hence these four roots must include all the possible roots of $p(z)$.

- **The disc $|z| < 1+\varepsilon$ for $\varepsilon << 1$.** *Lets take $\varepsilon < 1/10$. On the circle $|z| = 1+\varepsilon$ we have the following estimate*

$$\begin{aligned} |p(z) - (-6z)| &= |z^4 + 3| \\ &\leq |z|^4 + 3 = (1+\varepsilon)^4 + 3 < 4.5 < 6(1+\varepsilon) = |-6z|. \end{aligned}$$

Once again by Rouche's theorem, $p(z)$ has exactly one root in $|z| < 1+\varepsilon$, and hence has exactly three roots in $1+\varepsilon \leq |z| < 2$. Since this is true for all $\varepsilon << 1$, in particular, it has exactly three roots in $A_{1,2}(0)$.

Remark 0.5. *Rouche's theorem can be used to give another proof of the fundamental theorem of algebra. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a general degree n polynomial (so $a_n \neq 0$). It is easy to see that for $R >> 1$, if $|z| = R$, then*

$$|p(z) - a_n z^n| < |a_n||z^n|.$$

This is essentially because $p(z) - a_n z^n$ is a polynomial of a strictly lower degree. Now by Rouche's theorem $p(z)$ and $a_n z^n$ have the same number of roots on $|z| < R$. In particular, $p(z)$ must have exactly n roots on $|z| < R$. In fact it can be shown easily (by induction for instance) that it cannot have any further zeroes.

Appendix : details left out in the proof of Corollary 0.2

To spell out the details on the holomorphicity of f^{-1} and that the derivative is $1/f'(z_w)$, we first note that

$$\begin{aligned} \frac{f^{-1}(w+h) - f^{-1}(w)}{h} &= \frac{1}{2\pi i h}\int_{|z-z_0|=r} zf'(z)\Big(\frac{1}{f(z)-w-h} - \frac{1}{f(z)-w}\Big)\,dz \\ &= \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{zf'(z)}{(f(z)-w-h)(f(z)-w)}\,dz \end{aligned}$$

Now the integrand is continuous and bounded for $|h| << 1$, and hence we can take compute the limit by swapping the integral and the limit. That is,

$$\lim_{h\to 0}\frac{f^{-1}(w+h) - f^{-1}(w)}{h} = \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{zf'(z)}{(f(z)-w)^2}\,dz.$$

Another application of the residue theorem shows that the second integral is precisely $1/f'(z_w)$. To see this, we observe that

$$\frac{zf'(z)}{(f(z)-w)^2} = \frac{(z-z_w)f'(z)}{(f(z)-w)^2} + z_w\frac{f'(z)}{(f(z)-w)^2}.$$

From the geometric series expansion, one can see that the second term is of the form

$$z_w\frac{f'(z)}{(f(z)-w)^2} = \frac{z_w}{f'(z_w)^2}(z-z_w)^{-2} + g(z),$$

where $g(z)$ is holomorphic near z_w. This relies on the fact that $f'(z_w) \neq 0$, and hence the numerator has a non-zero constant term in it's Taylor expansion. The upshot is that the second term does not contribute to the residue. The advantage now is that the first term has a simple pole at $z = z_w$, and hence we can compute the residue as

$$\mathrm{Res}_{z=z_w} \frac{zf'(z)}{(f(z)-w)^2} = \lim_{z \to z_w} \frac{(z-z_w)^2 f'(z)}{(f(z)-w)^2} = \frac{1}{f'(z_w)}.$$

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