LECTURE-1 : THE COMPLEX NUMBERS

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1. Roadmap for the course

Complex analysis is one of the most beautiful branches of mathematics; a subject that lies at the heart of several other subjects, such as topology, algebraic geometry, differential geometry, harmonic analysis, and number theory.

The main objects in calculus are real valued functions defined on intervals. The starting point in complex analysis is to extend the notion of functions to include *complex valued* functions

$$f : \Omega \to \mathbb{C}$$

defined on subsets $\Omega \subset \mathbb{C}$ of complex numbers. Recall that complex numbers can be added, subtracted, multiplied and divided (if non-zero) just like real numbers. Every complex number can be written in the form

$$z = x + iy$$

where x and y are real numbers. So complex numbers can be identified as a set with Euclidean plane $\mathbb{R}^2$. The addition of complex numbers is also equivalent to addition of vectors in $\mathbb{R}^2$. So it might appear as if we are not adding much, and that nothing is lost by simply treating the complex valued function as a two variable vector field. In fact this is true, as we will see later, when talking about limits and continuity.

But there is one key difference between $\mathbb{R}^2$ and $\mathbb{C}$, that of multiplication and division. Indeed things change dramatically when we restrict our attention to *complex differentiable* or *holomorphic* functions, that is, functions for which

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. The important point being that h could be a complex number. Formally this definition is identical to that of a differentiable function in one-variable calculus. But quite surprisingly the mere change of perspective, the fact that h is allowed to take complex values as it goes to zero, produces beautiful new phenomenon that have no counterparts in one-variable calculus, or indeed even multivariable calculus. We now summarize some of these remarkable consequences of holomorphicity.

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- **Analyticity.** As we remarked above, complex valued functions can be thought as mapping between sets in $\mathbb{R}^2$. We will prove later in the course that for a holomorphic function, partial derivatives of all orders exist. And moreover, one also has that the Taylor series at ever point converges to the function value, that is holomorphic functions are *analytic*. Recall that this is not true for one-variable functions. For instance if

$$f(x) = \begin{cases} e^{-1/x^2}, \ x > 0 \\ 0, \ x \leq 0, \end{cases}$$

then it is easy to see that at $x = 0$, derivative of any order is zero. So the Taylor series of the function at $x = 0$ is zero, but the function is clearly not zero.
- **Analytic continuation.** Two holomorphic functions defined on an open connected domain are equal in a small open neighbourhood of a point, no matter how small the neighbourhood is, have to be identically equal.
- **Good convergence properties.** If a sequence of holomorphic functions converges uniformly, the limit function is again holomorphic. This is not true for differentiable one-variable functions. For instance, if $f_n : [-1, 1] \to \mathbb{R}$ is defined by

$$f_n(x) = \sqrt{\frac{1}{n} + x^2},$$

then one can show that $f_n \to |x|$ uniformly, but $|x|$ is not differentiable.
- **Liouville property.** A bounded holomorphic function defined on all of $\mathbb{C}$ is forced to be a constant. As a consequence, one can prove the fundamental theorem of algebra.

Part of the richness of the theory of holomorphic functions comes from the variety in the methods used to study the subject. We next summarize the approaches that we will touch upon in this course.

- **Partial differential equations.** It turns out that real and imaginary parts of holomorphic functions, thought of as real valued two-variable functions, satisfy a system of first-order partial differential functions, called the *Cauchy-Riemann* equations. As a consequence of this, the real and imaginary parts are *harmonic functions*. The theory of harmonic functions is rather well developed, and could be potentially exploited to study holomorphic functions. We will only touch upon the Cauchy-Riemann equations, but will not pursue this approach further. We will instead focus on integral methods.
- **Integral methods.** The viewpoint that we will adopt is centered on a remarkable formula called the Cauchy's integral formula. We will develop a notion of integration of complex valued functions

along curves, a generalization of the notion of line integrals in multi-variable calculus. The fundamental fact, which will be the theoretical basis for the rest of the course, is that the complex integral of a holomorphic function around a closed curve is zero. If the real and imaginary parts of the holomorphic function are assumed to have continuous partial derivatives, this result follows from Green's theorem. We will give an independent proof, not because we wish to be clever, but because remarkably this theorem will *imply* that the real and imaginary parts of the holomorphic function indeed have not only continuous partial derivatives but have partial derivatives of all orders, and are in fact *analytic*.

- **Power series methods.** As remarked above, every holomorphic function is represented by a power series. Since power series are algebraic objects, for the most part they can also be manipulated as if they were polynomials. Thus algebraic methods can be used to study holomorphic functions.
- **Geometric mathods.** An elementary but beautiful fact is that holomorphic functions, thought of as mappings (or transformations) between sets in $\mathbb{R}^2$ are *conformal maps*. That is, holomorphic mappings preserve angles between curves, and stretch the distances. We will study some standard examples of conformal maps. Towards the end of the course we will prove the following deep fact, first discovered by Riemann - Any domain in the complex plain which does not have a 'hole' and which is not the entire complex plane, can be mapped conformally to a disc centered at the origin of radius one. Our proof will be due to Koebe.

2. Complex numbers

It is known that certain polynomial equations with real coefficients need not have real roots. Complex numbers are obtained from the reals by formally adjoining a number i that solves the equation

$$i^2 = -1.$$

More formally, we define the set of complex numbers by

$$\mathbb{C} := \mathbb{R}[i] = \mathbb{R}[x]/(1 + x^2).$$

So a general complex number takes the form $z = x + iy$, where x and y are real numbers, and are called the real and imaginary part of z respectively. We use the notations $x = Re(z)$ and $y = Im(z)$. Clearly the real numbers can be identified as a subset of the complex numbers in a natural way as numbers with $Im(z) = 0$. We define addition and subtraction to be component-wise i.e. if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then we define

$$z_1 \pm z_2 = (x_1 + x_2) \pm i(y_1 + y_2).$$

Using this, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ as vector spaces. With this interpretation, a complex number represents a point in the xy-plane; with the x-coordinate given by $Re(z)$ and the y-coordinate given by $Im(z)$. This more geometric interpretation will be very useful to us.

But the complex numbers are much more than just 2-dimensional vectors. They also have a multiplicative structure, induced from the multiplicative structure of $\mathbb{R}[x]$. That is we can multiply two complex numbers to obtain another complex number. Indeed, if z_1 and z_2 are as above, we define

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

More simply, we define $i^2 = -1$, and then extend the product to satisfy the distributive property. It is not hard to verify that addition and multiplication satisfy the following properties:

P1 (Additive and Multiplicative identity.) For any complex number z,

$$z + 0 = z,\ \ z \cdot 1 = z.$$

P2 (Commutativity.) For any $z_1, z_2 \in \mathbb{C}$,

$$z_1 + z_2 = z_2 + z_1,\ \ z_1 \cdot z_2 = z_2 \cdot z_1.$$

P3 (Associativity.) For any complex numbers z_1, z_2, z_3,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),\ \ (z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3).$$

P4 (Distribution) For any $z_1, z_2, z_3 \in \mathbb{C}$,

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3.$$

P5 (Additive inverse.) For any $z \in \mathbb{C}$, $-z = (-1) \cdot z$ satisfies

$$z + (-z) = 0.$$

For notational convenience, we sometimes drop the *dot* when multiplying complex numbers. As remarked above, geometrically, addition of complex numbers corresponds to addition of vectors. What is the interpretation for multiplication? This is clearer if we use *polar coordinates*. Recall that any point (x, y) in the plane that is not the origin, can be represented uniquely by a pair (r, θ), where $r > 0$ and $\theta \in (-\pi, \pi]$ via the following transformation law:

$$x = r\cos\theta,\ \ y = r\sin\theta.$$

Then r is the geometric distance from the origin, and θ is the angle made by the line joining (x, y) to the origin with the positive x-axis. For instance the complex number i corresponds to $(1, \pi/2)$ in polar coordinates. So, any complex number can be represented as

$$z = r(\cos\theta + i\sin\theta).$$

If $w = \rho(\cos\alpha + i\sin\alpha)$ is another complex number, then it follows from the definition of the multiplication formula that

$$\begin{aligned} zw &= r\rho[(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + i(\cos\theta\sin\alpha + \sin\theta\cos\alpha)] \\ &= r\rho(\cos(\theta+\alpha) + i\sin(\theta+\alpha)), \end{aligned}$$

where we used the sum-angle formulas in the last equation. So geometrically multiplication simply corresponds to a dilation (i.e. scaling) and a rotation. For instance multiplication by i corresponds to rotating the vector representing the complex number by $\pi/2$. To form a good number system we will also need to be be able to divide by complex numbers. For any $z_2 \neq 0$, we say that $w = z_1/z_2$ if $z_1 = wz_2$. We call w, the *quotient* obtained by dividing z_1 by z_2. Clearly, if $z_2 = 0$, then by property P1, the quotient cannot be well defined. We next see that the quotient *is* in fact well defined when dividing by non-zero complex numbers.

Conjugate, Absolute value, Argument. To prove that a quotient always exists on dividing by a non-zero complex number, it is enough to have a formula for $1/z$, where $z \neq 0$. Let $z = x + iy$. Multiplying the numerator and denominator of $1/z$ by $x - iy$ (this is similar to rationalizing irrational denominators) we obtain

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}.$$

Note that in the last equation, the denominator is now a real number, and we already know how to divide by real non-zero numbers. The numerator, is called the *conjugate* of z and is denoted by

$$\bar{z} = x - iy.$$

Geometrically this amounts to reflection of the point representing z about the x-axis. Readers will notice that the denominator is the square of the distance of the point (x, y) from the origin. So we define the absolute value or the *length* of the complex number, denote by $|z|$ as

$$|z| = \sqrt{x^2+y^2}.$$

This is of course the 'r' in the polar coordinate representation. Some basic properties of these operations are the following.

- $\bar{\bar{z}} = z$.
- $|\bar{z}| = |z|$.
- $|z| = 0 \implies z = 0$.
- $\overline{zw} = \bar{z}\bar{w}$, $|zw| = |z||w|$.
- $\overline{z+w} = \bar{z} + \bar{w}$.

Note that contrary to the conjugate function, the modulus function is not additive. Instead we have an inequality; see Theorem 2.1. With these notations in place, we can re-write the above statement as

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2},$$

for any $z \neq 0$. So we in summary we shown that multiplication has another property, that every no-zero number has *multiplicative inverse.* That is we have

P6 (multiplicative inverse.) For every $z \in \mathbb{C}$, $z \neq 0$ there exists a complex number $1/z = \bar{z}/|z|^2$ such that

$$z \cdot \frac{1}{z} = 1.$$

With the two operations of addition and multiplication satisfying these six axioms, the set of complex numbers become what is called as *field* by algebraists. In fact, the complex numbers form an *algebraically closed* field, which means that any polynomial with complex coefficients can be completely factorized using complex roots. Later in the course, somewhat remarkably, we will prove this statement in algebra using our complex analysis techniques. We will in fact give multiple proofs, not just one!

The 'θ' in the polar coordinates also has a name, and is called the *argument* of z, and denote by $\arg(z)$. Using the new notation, it is also easy to see that

$$Re(z) = \frac{z + \bar{z}}{2}, \ Im(z) = \frac{z - \bar{z}}{2i}.$$

We can now define division by

$$\frac{z}{w} = \frac{z\bar{w}}{|w|^2}.$$

Integer Powers. Given any natural number $n \in \mathbb{N}$, we define z^n to be z multiplied to itself 'n' times. We also define $z^0 := 1$. For negative integers $-n$, we then define z^{-n} to be $1/z^n$ or the multiplicative inverse to z^n.

We end with an important inequality that will be crucial in most of the estimates.

Theorem 2.1 (Triangle inequality)**.** *Let $z, w \in \mathbb{C}$. Then we have the following inequalities*

(1)

$$|z + w| \leq |z| + |w|,$$

with equality if and only if $z = aw$ where $a \in \mathbb{R}$ i.e. z and w lie on the same line through the origin.

(2)

$$|z - w| \geq ||w| - |z||,$$

with equality again if z and w lie on the same line through the origin.

Proof. To see the first inequality, consider a triangle whose vertices are the origin O, the point represented by z, which we call A and the point represented by $z + w$ which we label B. Then by triangle inequality from elementary geometry, every side is smaller that the sum of the other two sides with equality only if all three points lie on the same line. So if they dont lie on the same line, we have

$$|OB| < |OA| + |AB|.$$

The proof is completed by noticing that $|OA| = |z + w|, |OA| = |z|$ and $|AB| = |w|$. For the second inequality, without loss of generality, suppose $|w| \geq |z|$. Then the right hand side is $|w| - |z|$. So applying the first inequality, since $w = (w - z) + z$,

$$|w| \leq |w - z| + |z|,$$

from which the desired inequality follows with equality if and only if $w - z$ and z lie on the same straight line. But then this happens if and only if z and w themselves lie on the same line through the origin. □

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