LECTURE-2

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1. Topology of the complex plane

A consequence of the triangle inequality discussed in the previous lecture is that the function

$$d(z,w) := |z-w|$$

defines a distance function on $\mathbb{C}$, and for simplicity, we denote the corresponding metric space by $(\mathbb{C}, |\cdot|)$. Given a $z_0 \in \mathbb{C}$, the *open* disc of radius R around z_0 is given by

$$D_R(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < R\}.$$

We now review a few standard definitions from topology. The *complement* of a set S, denoted by S^c is the set of all complex numbers NOT in S. Given any set $S \subset \mathbb{C}$, a point $p \in \mathbb{C}$ is a *limit* or an *accumulation* point if for any $r > 0$, the disc $D_r(p)$ has at least one point in common with S other than possibly p itself. A point $p \in S$ is said to be *isolated* if p is not a limit point of S. The *closure* of a set S, denoted by $\bar{S}$ is the union of S with all it's accumulation points. The *interior* of S, denoted by $\mathring{S}$, is the set of all points $p \in S$ such that $D_r(p) \subset S$ for some $r > 0$. The *boundary* of a set S is the set of points $p \in \mathbb{C}$ such that for all $r > 0$, the disc $D_r(p)$ contains at least one point from S and S^c. For instance the boundary of the open disc $D_r(p)$ is the circle of radius r centered at p.

A set S is called *open* is for any point $p \in S$, there exists a disc $D_r(p) \subset S$. That is each point has a neighborhood that is completely contained in the set. A set is called *closed* is it's complement is open. An equivalent definition (why are they equivalent?) is that a set is closed if and only if it completely contains it's boundary. So for any set S, the interior $\mathring{S}$ is the largest open set contained in S and the closure $\bar{S}$ is the smallest closed set containing S. A basic property of open and closed sets is the following.

Proposition 1.1.
- *Arbitrary union (possibly infinite) of open sets is again open. Finite intersection of open sets is open.*
- *Arbitrary intersection of closed sets is close. Finite union of closed sets is closed.*

Given a sequence $\{z_n\}$ we say that it *converges* to $p \in \mathbb{C}$ if for all $\varepsilon > 0$, there exists an N such that

$$|z_n - p| < \varepsilon.$$

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Proposition 1.2. *$z_n \to p$ if and only if $Re(z_n) \to Re(p)$ and $Im(z_n) \to Im(p)$.*

This is a consequence of the fact that for any $z \in \mathbb{C}$,

$$\max(|Re(z)|, |Im(z)|) \leq |z| \leq \sqrt{2}\max(|Re(z)|, |Im(z)|).$$

A disadvantage of the above definition of convergence is that one needs to know the limit a priori, to even decide if a sequence is converging. A convenient alternative is of course the notion of a *Cauchy sequence*. Recall that a sequence z_n is said to be Cauchy if for all $\varepsilon > 0$, there exists an $N > 0$ such that for all $n, m > N$ we have

$$|z_n - z_m| < \varepsilon.$$

It is easy to see (prove it!) that every convergent sequence is Cauchy. Conversely, we have the following fundamental fact.

Theorem 1.1. *Every Cauchy sequence in $\mathbb{C}$ converges. That is, $(\mathbb{C}, |\cdot|)$ is a complete metric space.*

. The theorem follows from the proposition above and the fact that real numbers from a complete metric space. Recall that a set is called *compact* if every open cover has a finite sub-cover. A consequence of completeness is the following useful characterization of compact sets in $\mathbb{C}$.

Theorem 1.2. *The following are equivalent for a subset $K \subset \mathbb{C}$.*

(1) *K is compact.*
(2) *K is closed and bounded.*
(3) *K is* sequentially compact. *That is, any infinite sequence $\{z_n\} \subset K$ has an accumulation point $p \in K$.*

The last notion we need is that of a connected set. A subset $S \subset \mathbb{C}$ is called connected, if

$$S = (U \cap S) \cup (V \cap S)$$

for some disjoint open sets U and V. If S itself is open, this reduces to saying that S cannot be written as the union of two disjoint open sets. An open, connected subset is called a *region*. We have the following elementary characterization of regions.

2. Functions on the complex plane

Let $S \subset \mathbb{C}$ be a subset. A function $f : S \to \mathbb{C}$ is a rule that assigns *unique* complex number, denoted by $f(z)$ to every number $z \in S$. The set S is called the *domain* of the function, and

$$f(S) := \{f(z) \mid z \in S\},$$

is called the *range*. The *pre-image* of a set $T \subset \mathbb{C}$, denoted by $f^{-1}(T)$ is the subset of S defined by

$$f^{-1}(T) = \{z \in S \mid f(z) \in T\}.$$

A function is called *injective* or *one-one* if the pre-image of every point in the range consists of exactly one point, i.e

$$f(z) = f(w) \implies z = w.$$

It is said to *surjective* or *onto* if the range is all of $\mathbb{C}$.

We say that the limit of $f(z)$ as z tends towards p is L, and denote it by

$$\lim_{z \to p} f(z) = L,$$

if the following holds - For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|z - p| < \delta, \ z \in S, \implies |f(z) - L| < \varepsilon.$$

We say that f is *continuous* at $p \in S$ if

$$\lim_{z \to p} f(z) = f(p).$$

f is simply called continuous if it is continuous at all points in its domain. We then have the basic fact.

Theorem 2.1. *$f : S \to \mathbb{C}$ is continuous if and only if $Re(f)$ and $Im(f)$ a continuous as real valued functions of two variables.*

So as far as topology, which is the study of continuous functions, is concerned, there is no difference between $\mathbb{C}$ and $\mathbb{R}^2$. With this remark, the following properties follow easily from what is already known about multi-variable functions.

Theorem 2.2. *Consider a function $f : \Omega \to \mathbb{C}$, where Ω is open.*

(1) *It is continuous if and only if $f^{-1}(U)$ is open for any open set $U \subset \mathbb{C}$.*
(2) *It is continuous if and only if $f^{-1}(K)$ is closed for every closed set $K \subset \mathbb{C}$.*
(3) *It is continuous at $p \in \Omega$ if and only if for any sequence $\{z_n\}$ such that $z_n \to p$, we have*

$$\lim_{z_n \to p} f(z_n) = f(p).$$

(4) *If f is continuous, then for any compact subset $K \subset \Omega$, $f(K)$ is compact.*
(5) *If f and $g : \Omega \to \mathbb{C}$ are continuous at p then so are $f \pm g$ and fg. If $g(p) \neq 0$, then f/g is also continuous at p.*
(6) *if f is continuous at p, and $g : f(\Omega) \to \mathbb{C}$ is continuous at $f(p)$, then the composition $g \circ f$ is also continuous at p.*

Example 2.1. *The function $f(z) = z^n$, where n is an integer, is continuous. To see this, note that*

$$z^n - p^n = (z - p)(z^{n-1} + z^{n-2}p \cdots + p^{n-1})$$

A polynomial is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_k \in \mathbb{C}$ *for* $k = 0, 1, \cdots, n$. *Then by the fact that sums of contiuous functions are continuous, it follows that polynomials are continuous at all points. A rational function is a quotient of two polynomials*

$$R(z) = \frac{p(z)}{q(z)},$$

wherever q *is non-zero. At all such points, by the quotient rule above, a rational function is also continuous.*

Example 2.2. *The function* $f(z) = \bar{z}$ *is continuous. Similarly, the function* $g(z) = |z|$ *is also continuous.*

Example 2.3. Arg(z) in not continuous. *Recall that if* $z = x + iy$*, then* $\arg(z)$ *is defined as the unique angle between* $(-\pi, \pi]$ *that the line joining the origin to* (x, y) *makes with the positive* x*-axis. Now consider any point on the the negative* x*-axis, say* $z = -1$.

2.1. **Curves in $\mathbb{C}$.** As a special case, we could take S to be an interval in $\mathbb{R}$ thought of as a subset of the x-axis in $\mathbb{C}$. A *path* is defined to be a continuous function $\gamma : I \to \mathbb{R}$, where I is an interval. We then have the following useful characterization of regions in $\mathbb{C}$.

Proposition 2.1. *Let* $\Omega \subset \mathbb{C}$ *be an open subset. Then* $\Omega \subset \mathbb{C}$ *is a region if and only if* Ω *is path connected, ie. for any* $z_0, z_1 \in \Omega$*, there exists a continuous map* $\gamma : [0, 1] \to \Omega$ *such that* $\gamma(0) = z_0$ *and* $\gamma(1) = z_1$.

Proof. Suppose Ω is a region. Fix $z_0 \in \Omega$, and let

$$A = \{z \in \Omega \mid \text{there exists a path in } \Omega \text{ connecting } z \text{ to } z_0\}.$$

Note that $z_0 \in A$, and hence A is non-empty. Since Ω is open, and clearly every disc is path connected, A is open. Now we claim that $A^c := \Omega \setminus A$ is also open. To see this, if A^c is non-empty we have some $w \in A^c$. Then since Ω is open, there is disc $D_r(w) \subset \Omega$. Clearly, $D_r(w) \cap A = \Phi$, for if, $z_1 \in D_r(w) \cap A$, then one could simply connect z to z_0, by connecting z to z_1, and z_1 to z_0, contradicting our assumption that $w \in A^c$. This shows that $D_r(w) \subset A^c$, and hence A^c is open. But since Ω is connected and A is non-empty, this forces $A^c = \Phi$. Conversely, suppose Ω is path connected, but is disconnected. Then we can write $\Omega = A \cup A^c$, where both A and A^c are open and non-empty. □

2.2. **Convergence of functions.** There are two notions of convergence, that of point-wise, and uniform convergence. We say that

- the sequence of functions $f_n : \Omega \to \mathbb{C}$ converges *point-wise* to $f : \Omega \to \mathbb{C}$, if for every $z \in \Omega$, the sequence $f_n(z) \to f(z)$. Or equivalently, given any $\varepsilon > 0$, and any $z \in \Omega$, there exists an $N > 0$, possibly depending both on ε and z, such that

$$n > N \implies |f_n(z) - f(z)| < \varepsilon.$$

- the sequence of functions is said to converge *uniformly* if given any $\varepsilon > 0$, there exists an $N > 0$ depending only on ε such that for all $z \in \Omega$ and $n > N$ we have that

$$|f_n(z) - f(z)| < \varepsilon.$$

Theorem 2.3. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of continuous functions which converge uniformly to $f : \Omega \to \mathbb{C}$, then f itself is continuous.*

3. Holomorphic functions

Let $\Omega \subset \mathbb{C}$ be an open subset. We say that a function $f : \Omega \to \mathbb{C}$ is *complex differentiable* at $z = p \in \Omega$, if the limit

$$f'(p) = \frac{d}{dz}\Big|_{z=p} f(z) = \lim_{z\to p} \frac{f(z) - f(p)}{z - p} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

exists and is finite. The limit, denoted by $f'(p)$, is then called the derivative (or the complex derivative if the context is not clear) of f at p. The function is called *holomorphic*, if it is complex differentiable at all points in the domain. Formally this definition is identical to the one for real valued functions of one variable. We have also seen that functions of a complex variable can be thought of as vector fields in two variables. In multivariable calculus, there is already a notion of derivatives of such functions. A natural question is to ask for the relation between these two notions. We will return to this question shortly. But first we collect some basic properties of holomorphic functions, the proofs of which are identical to those in for differentiable functions of one real variable.

Theorem 3.1. *Let $f, g : \Omega \to \mathbb{C}$ be differentiable at z, and $a, b \in \mathbb{C}$*

(1) *If f is constant in a neighbourhood of z, then $f'(z) = 0$.*
(2) *(Linearity) $af + bg$ is complex differentiable at z, and*

$$[af + bg]'(z) = af'(z) + bg'(z)$$

(3) *(Product rule) fg is complex differentiable at z and*

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

(4) *(Quotient rule) If $g'(z) \neq 0$, then f/g is complex differentiable at z and*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}.$$

(5) *(Chain rule) If h is complex differentiable at $f(z)$, then $h \circ f$ is complex differentiable at z and*

$$[h \circ f]'(z) = h'(f(z)) \cdot f'(z).$$

The main theme of the course is that holomorphicity imposes severe restrictions on the functions under consideration. Just to get our feet wet, we start with the following elementary observation.

Proposition 3.1. *If a function f is complex differentiable at $z = a$ then it is automatically continuous at $z = a$.*

Proof. We proceed by contradiction. So suppose f is not continuous at a. Then there exists an $\varepsilon > 0$ and a sequence $z_n \to a$ such that $|f(z_n) - f(a)| > \varepsilon$. By holomorphicity, there exists an N such that

$$\left|\frac{f(z_n) - f(a)}{z_n - a} - f'(a)\right| < 1,$$

whenever $n > N$. Or equivalently, that

$$|f(z_n) - f(a) - (z_n - a)f'(a)| < |z_n - a|.$$

By the triangle inequality

$$\begin{aligned} \varepsilon < |f(z_n) - f(a)| &= |f(z_n) - f(a) - (z_n - a)f'(a) + (z_n - a)f'(a)| \\ &< |f(z_n) - f(a) - (z_n - a)f'(a)| + |(z_n - a)f'(a)| \\ &< |z_n - a|(1 + |f'(a)|) \end{aligned}$$

Suppose now N is chosen large enough so that

$$|z_n - a| < \frac{\varepsilon}{2(1 + |f'(a)|)},$$

then the above chain of inequality yields

$$\varepsilon < |f(z_n) - f(a)| < \frac{\varepsilon}{2},$$

which is absurd. □

Example 3.1. *As a first example, we compute the derivative of $f(z) = z^n$, where n is an integer. We first assume that $n \geq 1$. Then it is easy to see (try to prove it!) that*

$$z^n - a^n = (z - a)(z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1}).$$

So then

$$f'(a) = \lim_{z\to a} \frac{z^n - a^n}{z - a} = \lim_{z\to a} z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1} = na^{n-1}.$$

For negative integers n we can apply quotient rule to again obtain the same formula. So for integers n we have that

$$\boxed{\left.\frac{d}{dz}\right|_{z=a} z^n = na^{n-1}}$$

Example 3.2. Polynomials and Rational functions. *Recall that polynomials are functions of the type*

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

Then by the above theorem, such functions are holomorphic. Moreover, by the above calculation the derivative of a degree n polynomial is again a polynomial, but of degree $n - 1$. Recall also that rational functions are quotients *of two polynomials. By the quotient rule, these are holomorphic at all points where the denominator does not vanish. That is, if*

$$R(z) = \frac{p(z)}{q(z)},$$

then $R(z)$ is holomorphic at $z = a$ if and only if $q(a) \neq 0$.

Example 3.3. *The function $f(z) = \bar{z}$ is not holomorphic. To see this, consider the difference quotient*

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h}.$$

Then if $h \to 0$ along the real axis i.e. $h \in \mathbb{R}$, this difference quotient is 1. On the other hand if $h \to 0$ along the imaginary axis, i.e. $h = ik$ where $k \in \mathbb{R}$, then this quotient is always -1. So the limit cannot exist.

Example 3.4. *The function $f(z) = |z|$ is not holomorphic. By the product rule, it is enough to show that $g(z) = |z|^2$ is not holomorphic. The difference quotient is*

$$\frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h} = \frac{z\bar{h} + \bar{z}h + |h|^2}{h} = z\frac{\bar{h}}{h} + z + \bar{h}.$$

The limit of the last two terms as $h \to 0$ is $\bar{z}$, but, as we saw in the previous example, the limit of the first term does not exist. So $|z|^2$, and hence $|z|$, is not differentiable.