LECTURE-20

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In this lecture, we'll systematically study real variables integrals that can be computed using contour integration.

Type-I: Integrals of rational functions

In this section we study integrals of the form

$$\int_{-\infty}^{\infty} R(x)\,dx,$$

where $R(x)$ is a rational function.

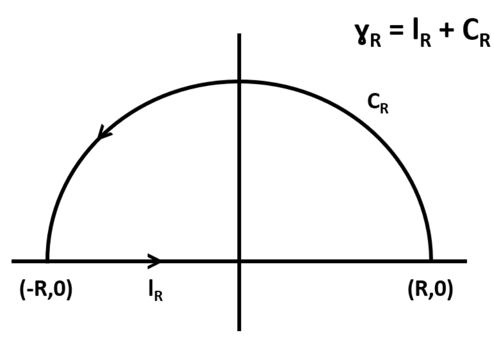
Assumption. $R(x) = P(x)/Q(x)$ with $\deg(Q) \geq \deg(P) + 2$ and $Q(x)$ has no real root. Recall that by definition,

$$\int_{-\infty}^{\infty} R(x)\,dx = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} R(x)\,dx.$$

By the hypothesis on degree and no real root of $Q(x)$, the integral is absolutely convergent and it follows that in fact,

$$\int_{-\infty}^{\infty} R(x)\,dx = \lim_{R \to \infty} \int_{-R}^{R} R(x)\,dx.$$

The method. We consider the contour γ_R consisting of a semi-circle of radius R centred at the origin and traversed in the anti-clockwise direction. We can decompose $\gamma_R = l_R + C_R$, where l_R is the interval $(-R, R)$ and C_R



is the semi-circular part, and hence

$$\int_{-R}^{R} R(x)\,dx = \int_{\gamma_R} R(z)\,dz - \int_{C_R} R(z)\,dz.$$

If $R >> 1$, then all roots of $Q(x)$ in the upper half plane lie in the interior of γ_R. By the residue theorem,

$$\int_{\gamma_R} R(z)\,dz = 2\pi i \sum_{\substack{Q(\alpha)=0\\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z).$$

On the other hand, since $\deg Q \geq \deg P + 2$, when $R >> 1$,

$$|R(z)| \leq \frac{M}{R^2}$$

for some $M > 0$. Hence,

$$\Big| \int_{C_R} R(z)\,dz \Big| \leq \frac{2\pi M}{R} \xrightarrow{R\to\infty} 0.$$

Putting it all together we have

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{\substack{Q(\alpha)=0\\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z).$$

Example 0.1. *Consider the integral*

$$\int_0^{\infty} \frac{x^2}{1+x^6}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\,dx = \pi i \sum_{\substack{1+\alpha^6=0\\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z),$$

where $R(z) = z^2/(1+z^6)$. The roots of the denominator are given by $\alpha_k = e^{\pi i/6 + 2\pi i k/6}$ for $k = 0, 1, \cdots, 5$. Of these only $\alpha_0 = e^{\pi i/6}, \alpha_1 = e^{\pi i/2} = i$ and $\alpha_3 = e^{5\pi i/6}$ are in the upper half region. Each of these is a simple pole, and hence we can compute that

$$\mathrm{Res}_{z=\alpha_k} R(z) = \lim_{z\to\alpha_k}(z-\alpha_k)\frac{z^2}{1+z^6} = \alpha_k^2 \lim_{z\to\alpha_k} \frac{z-\alpha_k}{1+z^6} = \frac{1}{6\alpha_k^3} = \begin{cases} -i/6, & k=0\\ i/6, & k=1\\ -i/6, & k=2. \end{cases}$$

Finally we get that

$$\int_0^{\infty} \frac{x^2}{1+x^6}\,dx = \pi i\Big(-\frac{i}{6}\Big) = \frac{\pi}{6}.$$

A variation. One also often encounters integrals (as in the above example) of the form

$$\int_0^{\infty} R(x)\,dx.$$

If $R(x)$ is an even function as in the example above, then one can simply convert it into an integral on all of $\mathbb{R}$ at the cost of an additional factor. But this trick will not work in general, so one might have to get more creative.

Example 0.2. *Consider the integral*

$$I = \int_0^\infty \frac{x}{1+x^4}\,dx = \lim_{R\to\infty}\int_0^R \frac{x}{1+x^4}\,dx.$$

As usual, we let I_R denote the integral on the right (inside the limit), and we let $f(z) = z/(1+z^4)$. We take the contour $\gamma_R = \gamma_{1,R} + C_R + \gamma_{2,R}$, where $\gamma_{1,R}$ consists of the straight line from (0.0) to $(R,0)$, C_R is the quadrant of the circle from $(R,0)$ to $(0,R)$, and $\gamma_{2,R}$ is the straight line from $(0,R)$ to $(0,0)$. By the residue theorem, since the only pole of $f(z)$ in the interior of γ_R is a simple pole at $z = e^{i\pi/4}$, we have

$$\begin{aligned}\int_{\gamma_R} f(z)\,dz &= 2\pi i \mathrm{Res}_{z=e^{i\pi/4}} f(z)\\ &= 2\pi i \lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})z}{1+z^4}\\ &= \frac{2\pi i e^{i\pi/4}}{4e^{3i\pi/4}} = \frac{\pi}{2}.\end{aligned}$$

Next, we observe that

$$\int_{\gamma_{1,R}} f(z)\,dz = I_R,$$

and by the discussion above,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

To compute the integral over $\gamma_{2,R}$ we parametrize $\gamma_{2,R}(t) = -it$, where $t \in (-R,0)$. Then $\gamma'_{2,R}(t) = -i$, and so

$$\int_{\gamma_{2,R}} f(z)\,dz = \int_{-R}^0 \frac{-it}{1+t^4}(-idt) = I_R.$$

Hence if $R > 1$, by the residue computation above,

$$2I_R + C_R = \frac{\pi}{2}.$$

Taking the limit as $R \to \infty$ we see that

$$I = \frac{\pi}{4}.$$

Remark 0.1. *This example was merely for illustration, since the integral can of course be computed in a more elementary way by a change of variables $x^2 = u$. The reader can challenge himself/herself with the following:*

$$\int_0^\infty \frac{x^3}{1+x^5}.$$

Type-II: Rational functions of sine and cosine

The type 2 integrals are of the following kind:

$$\int_0^{2\pi} R(\cos\theta, sin\theta)\, d\theta,$$

where $R(x)$ is again a rational function.
Assumption. No pole for $\theta \in [0, 2\pi)$.
The method. Consider $z = z(\theta) = e^{i\theta}$. Then

$$\begin{cases} \cos\theta = \frac{z+\bar{z}}{2} = \frac{z+1/z}{2} \\ \sin\theta = \frac{z-\bar{z}}{2i} = \frac{z-1/z}{2i}. \end{cases}$$

Moreover, $dz = z'(\theta)\theta = ie^{i\theta}\theta$, and hence

$$d\theta = -i\frac{dz}{z}.$$

One can then transform the integral to

$$\int_0^{2\pi} R(\cos\theta, sin\theta)\, d\theta = -i\int_{|z|=1} R\Big(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\Big)\frac{dz}{z}.$$

Example 0.3. *Consider the integral*

$$I := \int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta},\ a > 1.$$

Using the above transformations we can re-write the second integral as

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = -i\int_{|z|=1} \frac{1}{a+\frac{z+1/z}{2}}\frac{dz}{z} = -2i\int_{|z|=1} \frac{dz}{z^2+1+2az},$$

and so our required integral is

$$I = -i\int_{|z|=1} \frac{dz}{z^2+2az+1}.$$

Now the denominator has two real roots, $\alpha = -a + \sqrt{a^2-1}$ *and* $\beta = -a - \sqrt{a^2-1}$. *Since* $a > 1$, *it is easy to see that* $|\alpha| < 1$ *while* $|\beta| > 1$. *So*

$$I = (-i)(2\pi i)\text{Res}_{z=\alpha}\frac{1}{z^2+1+2az} = 2\pi\lim_{z\to\alpha}\frac{z-\alpha}{z^2+2az+1} = \frac{2\pi}{\alpha-\beta} = \frac{\pi}{\sqrt{a^2-1}}.$$

Type-III: Products of rational functions and triganometric functions

These are integrals of the kind

$$\int_{-\infty}^{\infty} R(x)\cos x\, dx,\ \int_{-\infty}^{\infty} R(x)\sin x,\ dx.$$

These can be combined into the analysis of one single integral

$$\int_{-\infty}^{\infty} R(x)e^{ix}\, dx.$$

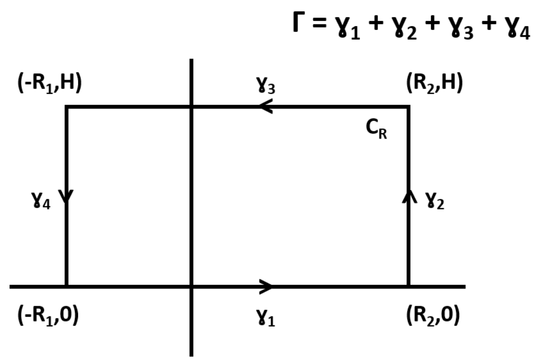
Assumptions. These come in two sub-types where the analysis substantially differs. As before, we let $R(x) = P(x)/Q(x)$.

- Type-III(a). We assume that $\deg Q \geq \deg P + 2$, and no root of Q is real.
- Type-III(b). Here we assume that $\deg Q = \deg P + 1$, and no root of Q is real.

The method. For Type-III(a), one proceeds exactly as in Type-I, and we do not spend additional time on this. For Type-III(b), it is not clear at all if the integral converges. It certainly will not absolutely converge, and hence we cannot use the semi-circle contour. But in many cases, the oscillating factor e^{ix} might make the integral converge conditionally. By definition, if the integral converges, then

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx = \lim_{R_1,R_2\to\infty} \int_{-R_1}^{R_2} R(x)e^{ix}\,dx.$$

Now let Γ be the rectangle with vertices $(-R_1, 0)$, $(R_2, 0)$, (R_2, H) and $(-R_1, H)$. with sides given by straight-line curves $\gamma_1, \gamma_2, \gamma_3$ and γ_4 (see figure below). If R_1, R_2 and H are big enough, the rectangle will contain all



the roots of Q with positive imaginary part. Hence

$$\sum_{i=1}^{4} \int_{\gamma_i} R(z)\,dz = 2\pi i \sum_{\substack{Q(\alpha)=0\\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z)e^{iz}.$$

The idea is to first fix R_1, R_2 and let $h \to \infty$. And then let $R_1, R_2 \to \infty$. We illustrate with an example.

Example 0.4. *Consider the integral*

$$I = \int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2} = \frac{1}{i}\int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2}\,dx,$$

since $x\cos x/(1+x^2)$ is an odd function and hence the integral will be zero. We let $f(z) = ze^{iz}/(1+z^2)$. If $H > 1$, then the rectangle will contain the *only pole of $f(z)$ with positive imaginary part, namely $z = i$. Hence*

$$\int_\Gamma f(z)\,dz = 2\pi i \mathrm{Res}_{z=i} f(z) = 2\pi i \lim_{z\to i}(z-i)\frac{ze^{iz}}{1+z^2} = 2\pi i\Big(\frac{ie^{i^2}}{i+i}\Big) = \frac{\pi i}{e}.$$

Now let us analyze each smooth curve in Γ.

- **The curve γ_1.** *This is the integral we are interested in*
$$\int_{\gamma_1} f(z)\,dz = \int_{-R_1}^{R_2} \frac{xe^{ix}}{1+x^2}\,dx.$$
- **The curve γ_2.** *On γ_2, $|z| > R_2$, and hence if $R_2 >> 1$, then*
$$\Big|\frac{z}{1+z^2}\Big| \le \frac{2}{R_2}.$$
Then parametrizing the curve by $\gamma_2(t) = R_2 + it$, and noting that $|e^{iz}| = e^{-t}$, we see that
$$\Big|\int_{\gamma_2} f(z)\,dz\Big| \le \frac{2}{R_2}\int_0^H e^{-t}dt \le \frac{2}{R_2}.$$
- **The curve γ_3.** *Again if $H >> 1$, we will have the estimate*
$$\Big|\frac{z}{1+z^2}\Big| \le \frac{2}{H}.$$
Also $|e^{iz}| = e^{-H}$ on γ_3. Hence $|f(z)| \le 2e^{-H}/H$, and we have
$$\Big|\int_{\gamma_2} f(z)\,dz\Big| \le \frac{2e^{-H}(R_2+R_1)}{H}.$$
- **The curve γ_4.** *As for the second curve, we once again have the bound*
$$\Big|\int_{\gamma_4} f(z)\,dz\Big| \le \frac{2}{R_1}.$$

Together, if we first let $H \to \infty$, and then let $R_1, R_2 \to \infty$ the integrals on the last three curves converge to 0, and hence

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2}\,dx = \int_\Gamma f(z)\,dz = \frac{\pi i}{e}.$$

Our original integral is then

$$\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2} = \frac{\pi}{e}.$$

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