LECTURE-21

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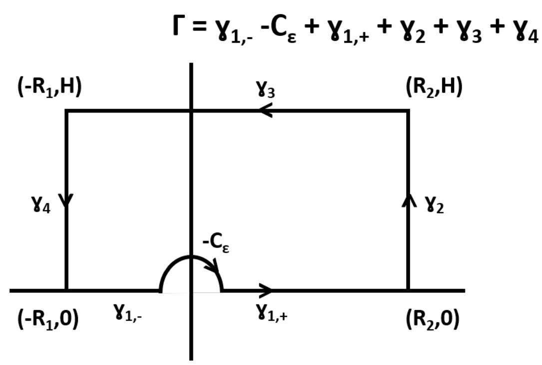
This is a continuation of our previous lecture on computing real variable integrals using the residue theorem. In the present lecture, we'll focus on integrations involving branch cuts. But first we refine a method introduced in the previous lecture.

Type-III(b): Principal values

Now suppose that $Q(x)$ has a simple zero on the real line. In this case even though the integral of $R(x)e^{ix}$ will not converge, the integrals of $R(x)\sin x$ or $R(x)\cos x$ might converge if the simple zero happens to coincide with a zero of $\sin x$ and $\cos x$ respectively. To illustrate the work-around in this case, suppose we wish to evaluate the integral

$$\int_{-\infty}^{\infty} R(x)\sin x.$$

Assumption. $\deg Q(x) = \deg P(x)+1$ and $Q(x)$ has a simple zero at $x = 0$.
The method. Consider the contour Γ in the figure below. Using the



analysis from the final section of the previous lecture, we can see that

$$\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\,dx+\int_{\varepsilon}^{\infty} R(x)e^{ix}\,dx-\int_{C_\varepsilon} R(z)e^{iz}\,dz = 2\pi i \sum_{\substack{Q(\alpha)=0\\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha}R(z)e^{iz}.$$

To compute the third integral, note that near $z = 0$,

$$R(z)e^{iz} = \frac{A}{z} + R_0(z),$$

where $R_0(z)$ is holomorphic near $z = 0$ (and hence bounded) and $A = \mathrm{Res}_{z=0} R(z)e^{iz}$. Then by an explicit calculation it is easy to see that

$$\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon} R(z)e^{iz}\, dz = \pi i A,$$

and hence

(0.1)

$$\lim_{\varepsilon \to 0^+} \Big(\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\, dx + \int_{\varepsilon}^{\infty} R(x)e^{ix}\, dx \Big) = 2\pi i \Big(\frac{1}{2} \mathrm{Res}_{z=0} R(z)e^{iz} + \sum_{\substack{Q(\alpha)=0 \\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z)e^{iz} \Big).$$

The *principal value* of the integral of $R(x)e^{ix}$ on $\mathbb{R}$ is defined to be

$$\text{p.v.}\Big(\int_{-\infty}^{\infty} R(x)e^{ix}\, dx \Big) := \lim_{\varepsilon \to 0^+} \Big(\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\, dx + \int_{\varepsilon}^{\infty} R(x)e^{ix}\, dx \Big),$$

if the limit exists. Note that if $Q(x)$ has a simple pole at $x = 0$, then the above limit will exist, as the analysis above shows. Moreover, under these assumptions, clearly $R(z)\sin z$ is integrable near zero and infinity, and so

$$\int_{-\infty}^{\infty} R(x) \sin x\, dx = Im\Big(\text{p.v.}\Big(\int_{-\infty}^{\infty} R(x)e^{ix}\, dx \Big) \Big),$$

and the latter principal value can be computed by the analysis above. Let us illustrate this via a famous integral.

Example 0.1. *Consider the integral of*

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

Then by the above analysis,

$$I = Im\Big(\pi i \mathrm{Res}_{z=0} \frac{e^{iz}}{z} \Big),$$

and so

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Remark 0.1. *This is the standard example of a function whose improper integral is finite, and yet the Lebesgue integral does not converge. I cannot resist the temptation to include another method of computing the integral, using the so called "Feynman technique". This is essentially just differentiation under the integral sign, but was popularised by Feynman as his way of handling integrals that others needed residue calculus for! Let I be the integral above. We introduce a parameter '$a > 0$' and consider the the family of integrals*

$$I(a) = \int_0^{\infty} \frac{\sin(x)e^{-ax}}{x}\, dx.$$

One can check that $I(a)$ is a differentiable function of a, and that differentiation under the integral works. Then,

$$I'(a) = -\int_0^\infty e^{-ax} \sin x \, dx.$$

We can compute the integral on the right, by applying integration by parts twice. Indeed

$$\begin{aligned} J(a) := \int_0^\infty e^{-ax} \sin x \, dx &= -\frac{1}{a}\int_0^\infty \sin x \, de^{-ax} \\ &= \frac{1}{a}\int_0^\infty e^{-ax} d \sin x \\ &= \frac{1}{a}\int_0^\infty e^{-ax} \cos x \, dx \\ &= -\frac{1}{a^2}\int_0^\infty \cos x \, de^{-ax} \\ &= \frac{1}{a^2} + \int_0^\infty e^{-ax} \, d\cos x \\ &= \frac{1}{a^2} - \frac{J(a)}{a^2}. \end{aligned}$$

Solving for $J(a)$ we get that

$$I'(a) = -\frac{1}{1+a^2},$$

and hence

$$I(a) = -\arctan(a) + C.$$

Since $\lim_{a\to\infty} I(a) = 0$, clearly $C = \pi/2$. But then taking $a \to 0^+$ we see that

$$I = \lim_{a\to 0^+} I(a) = C = \frac{\pi}{2}.$$

Type-IV: Products of rational functions and powers of x.

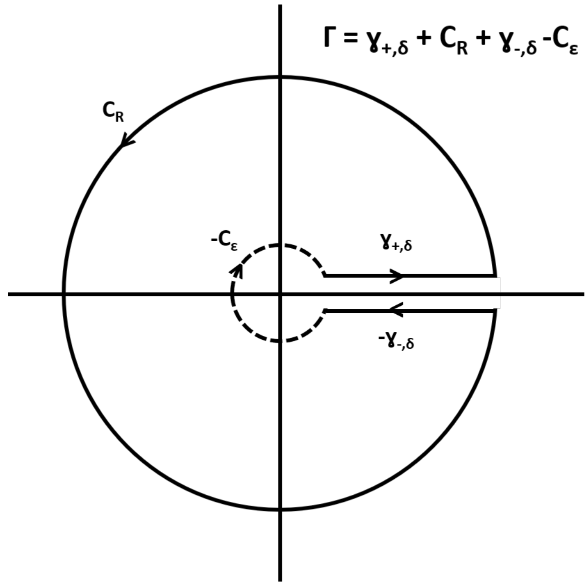
In this section we study integrals of the form

$$\int_{-\infty}^{\infty} x^\alpha R(x) \, dx,$$

where $R(x) = P(x)/Q(x)$ is a rational function and $\alpha \in (0, 1)$.
Assumption. $\deg Q \geq \deg P + 2$, and $Q(x)$ has a simple zero at $x = 0$ and no other real zero.
The method. Note that the assumption implies that the integral is absolutely convergent, and so

$$\int_{-\infty}^{\infty} x^\alpha R(x) \, dx = \lim_{R\to\infty} \int_{-R}^{R} x^\alpha R(x) \, dx.$$

Consider the contour in the figure below. We integrate the function $z^\alpha R(z)$ on the contour. Since we are dealing with fractional powers, we have to make



a choice of a branch cut and a a corresponding branch of the power. Given the geometry of the contour, it is clear that we have to use the branch cut $(0, \infty)$. Recall that $z^\alpha = e^{\alpha \log z}$, where $\log z = \log |z| + i \arg(z)$, and $\arg(z) \in (0, 2\pi)$.

We first have the following observations:

Lemma 0.1. *With the orientations as in the figure,*

$$\lim_{\delta \to 0} \int_{\gamma_{+,\delta}} z^\alpha R(z)\, dz = \int_\varepsilon^R x^\alpha R(x)\, dx,$$

$$\lim_{\delta \to 0} \int_{\gamma_{-,\delta}} z^\alpha R(z)\, dz = e^{2\pi i \alpha} \int_\varepsilon^R x^\alpha R(x)\, dx.$$

The proof relies on the fact that continuous functions on compact sets are uniformly continuous. For a fixed ε and R, and for small $\delta > 0$, the function $z^\alpha R(z)$ is continuous, and hence uniformly continuous, on the rectangle with vertices $(-\delta, \varepsilon)$, $(-\delta, R)$, (δ, R) and (δ, ε). The $e^{2\pi i \alpha}$ factor in the second integral is due to the fact that there is a jump of $e^{2\pi i \alpha}$ in the value of z^α across the branch cut $z > 0$. We leave the details to the reader.

By the residue theorem,

$$\int_{\gamma_{+,\delta}} z^\alpha R(z)\, dz + \int_{C_R} z^\alpha R(z)\, dz - \int_{\gamma_{-,\delta}} z^\alpha R(z)\, dz - \int_{C_\varepsilon} z^\alpha R(z)\, dz = 2\pi i \sum_{\substack{Q(\beta)=0 \\ \beta \neq 0}} \mathrm{Res}_{z=\beta} z^\alpha R(z).$$

Letting $\delta \to 0$, by the Lemma above,

$$(1 - e^{2\pi i \alpha}) \int_\varepsilon^R x^\alpha R(x)\, dx = 2\pi i \sum_{\substack{Q(\beta)=0 \\ \beta \neq 0}} \mathrm{Res}_{z=\beta} z^\alpha R(z) - \int_{C_R} z^\alpha R(z)\, dz + \int_{C_\varepsilon} z^\alpha R(z)\, dz.$$

Finally, letting $\varepsilon \to 0^+$ and $R \to \infty$ one proves that the integrals on the right converge to zero under the given assumptions. We illustrate with an example.

Example 0.2. *Consider the integral*

$$I := \int_0^\infty \frac{x^{-a}}{1+x}\,dx, \;\; a \in (0,1).$$

To get it into the above form, we re-write this as

$$I = \int_0^\infty \frac{x^{1-a}}{x(1+x)}\,dx,$$

and let $I_{\varepsilon,R}$ be the corresponding integral from ε to R. Let $f(z) = z^{1-a}/(z(1+z))$. Recall that we are using the branch $z^{1-a} = e^{(1-a)\log z}$, where $\log(re^{i\theta}) = \log r + i\theta$ and $\theta \in (0, 2\pi)$. Now $f(z)$ has simple poles at $z = 0$ and $z = -1$. By the above discussion,

(0.2)

$$(1 - e^{2\pi i(1-a)})I_{\varepsilon,R} = 2\pi i \mathrm{Res}_{z=-1}\frac{z^{-a}}{1+z} - \int_{C_R} \frac{z^{1-a}}{z(1+z)}\,dz + \int_{C_\varepsilon} \frac{z^{1-a}}{z(1+z)}\,dz$$

We first estimate the two remaining integrals on the right.

- **The integral on C_R.** *When $|z| = R >> 1$, we have $|1+z| \geq |z| - 1 > R/2$, and so*

$$\Big|\frac{z^{-a}}{1+z}\Big| = \frac{R^{-a}}{|1+z|} \leq \frac{2R^{-a}}{R} = \frac{2}{R^{1+a}}.$$

  The integral then satisfies

$$\Big|\int_{C_R} \frac{z^{1-a}}{z(1+z)}\,dz\Big| \leq \frac{4\pi R}{R^{1+a}} = \frac{4\pi}{R^a} \xrightarrow{R\to\infty} 0.$$

- **The integral on C_ε.** *When $|z| = \varepsilon << 1$, we have $|1+z| \geq 1 - |z| > 1/2$, and so*

$$\Big|\frac{z^{-a}}{1+z}\Big| = \frac{\varepsilon^{-a}}{|1+z|} \leq 2\varepsilon^{-a}.$$

  The integral then satisfies

$$\Big|\int_{C_\varepsilon} \frac{z^{1-a}}{z(1+z)}\,dz\Big| \leq 4\pi\varepsilon^{1-a} \xrightarrow{\varepsilon\to 0^+} 0.$$

Finally we compute the residue at $z = -1$,

$$\mathrm{Res}_{z=-1} f(z) = \lim_{z\to -1} z^{-a} = e^{-a\log(-1)} = e^{-i\pi a}.$$

Putting all of this together, taking $\varepsilon \to 0^+$ and $R \to \infty$ in (0.2) we see that

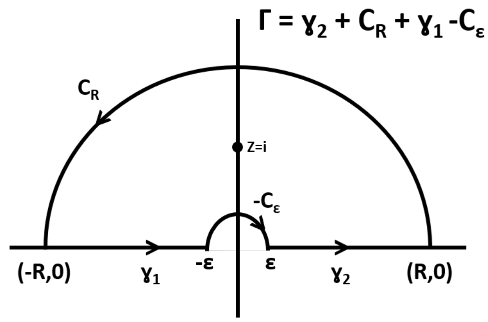
$$I = \frac{2\pi i e^{-i\pi a}}{(1 - e^{2\pi i(1-a)})} = \frac{2\pi i e^{-i\pi a}}{(1 - e^{-2\pi i a})} = \frac{2\pi i}{(e^{\pi i a} - e^{-\pi i a})} = \frac{\pi}{\sin \pi a}.$$

A BONUS INTEGRAL

As a final integral, let us compute

$$I := \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx.$$

We denote the corresponding integral over (ε, R) by $I_{\varepsilon,R}$. As an exercise, the reader should attempt to use the contour above too evaluate this integral. We will instead use a semicircular contour Γ in the figure below. We use the



branch cut $\{iy \mid y \in (-\infty, 0)\}$, and the branch of the logarithm defined by $\log(re^{i\theta}) = \log r + i\theta$, with $\theta \in (-\pi/2, 3\pi/2)$. Let

$$f(z) = \frac{(\log z)^2}{1+z^2}.$$

Then clearly,

$$\int_{\gamma_1} \frac{(\log z)^2}{1+z^2} = \int_\varepsilon^R \frac{(\log x)^2}{1+x^2}\,dx = I_{\varepsilon,R},$$
$$\int_{\gamma_2} \frac{(\log z)^2}{1+z^2} = \int_{-R}^{-\varepsilon} \frac{(\log|t| + i\pi)^2}{1+t^2}\,dt.$$

The limits follow from the fact that the integral is absolutely convergent. For the second integral above we parametrize $\gamma_2(x) = t = |t|e^{i\pi}$ with $t \in (-R, -\varepsilon)$. Putting $x = -t$ int he second integral, we see that

$$\begin{aligned}\int_{\gamma_2} \frac{(\log z)^2}{1+z^2} &= \int_{-R}^{-\varepsilon} \frac{(\log|t| + i\pi)^2}{1+t^2}\,dx \\ &= \int_\varepsilon^R \frac{(\log x + i\pi)^2}{1+x^2}\,dx \\ &= I_{\varepsilon,R} + 2\pi i \int_\varepsilon^R \frac{\log x}{1+x^2}\,dx - \pi^2 \int_\varepsilon^R \frac{dx}{1+x^2}.\end{aligned}$$

Notice that the third integral is $\arctan(R/\varepsilon)$, and so letting $\varepsilon \to 0^+$ and $R \to \infty$,

$$\lim_{\substack{\varepsilon\to 0^+\\ R\to\infty}} \Big(\int_{\gamma_1} \frac{(\log z)^2}{1+z^2} + \int_{\gamma_1} \frac{(\log z)^2}{1+z^2} \Big) = 2I - \frac{\pi^3}{2} + 2\pi i \int_\varepsilon^R \frac{\log x}{1+x^2}\,dx.$$

By the exact analysis as the previous section, one can prove that

$$\lim_{R\to\infty} \int_{C_R} \frac{(\log z)^2}{1+z^2}\,dz = \lim_{\varepsilon\to 0^+} \int_{C_\varepsilon} \frac{(\log z)^2}{1+z^2}\,dz = 0,$$

and so by the residue theorem,

$$\begin{aligned} 2I - \frac{\pi^3}{2} + 2\pi i \int_\varepsilon^R \frac{\log x}{1+x^2}\,dx &= 2\pi i \mathrm{Res}_{z=i} \frac{(\log z)^2}{1+z^2} \\ &= 2\pi i \lim_{z\to i} (z-i) \frac{(\log z)^2}{1+z^2} \\ &= 2\pi i \frac{(\log i)^2}{2i} \\ &= -\frac{\pi^3}{4}. \end{aligned} \tag{0.3}$$

Note that in the penultimate line, we used the fact that for our chosen branch of the logarithm, we have $\log i = i\pi/2$. Equating the real parts, and solving for I, we get that

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

Remark 0.2. *Notice that since $2\pi i$ times the residue above was completely real, the imaginary part in equation (0.3) above has to be zero. This yields a curious integral identity*

$$\int_0^\infty \frac{\log x}{1+x^2} = 0.$$

A simple change of variables yields an explanation. Namely, denoting the integral by J, if we let $t = 1/x$, we have

$$J = \int_\infty^0 \frac{-t^2 \log t}{1+t^2} \Big(\frac{-dt}{t^2} \Big) = -J,$$

and hence $J = 0$.

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