LECTURE-22

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In this lecture, we will begin our study of some geometric properties of holomorphic maps.

Conformal maps

Loosely speaking, a conformal map is a map that preserves angles. We now try to make this concept more rigorous, and give a precise definition of conformal maps. Throughout this introductory part, we'll work with either $\mathbb{R}^2$ or $\mathbb{C}$, as is convenient, always remembering the natural identification. Let $f : \Omega \to \mathbb{R}^2$ be a C^1 map ie. the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ exist and are continuous. Given two curves $\gamma_1(t) = (x_1(t), y_1(t)) : (-\varepsilon, \varepsilon) \to \mathbb{C}$ and $\gamma_2(t) = (x_2(t), y_2(t)) : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = p$, we define the angle between them as

$$\measuredangle \gamma_1(0), \gamma_2(0)$$

Recall that if $\vec{v_1}$ and $\vec{v_2}$ are non-zero vectors in $\mathbb{R}^2$, then the angle between them is defined to be

$$\measuredangle \vec{v_1}, \vec{v_2} = \arccos\Big(\frac{\langle \vec{v_1}, \vec{v_2}\rangle}{|\vec{v_1}||\vec{v_2}|}\Big),$$

where $\langle \cdot, \cdot \rangle$ is the usual dot product on $\mathbb{R}^2$ and $|\cdot|$ is the usual norm (given by the square-root of the dot-product of the vector with itself). Motivate by this, we say that a <u>linear map</u> $T : \mathbb{R}^2 \to \mathbb{R}^2$ *preserves angles* or is *conformal* if $\det(T) > 0$ [1], and for any pair of non-zero vectors $\vec{v}, \vec{w} \in \mathbb{R}^2 \setminus \{\vec{0}\}$,

$$\frac{\langle \vec{v}, \vec{w}\rangle}{|\vec{v}||\vec{w}|} = \frac{\langle T\vec{v}, T\vec{w}\rangle}{|T\vec{v}||T\vec{w}|}.$$

More generally, we sat that a C^1 mapping $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is *conformal at* $(x_0, y_0) \in \Omega$ if the total derivative $\mathbf{D}_{(x_0,y_0)} f : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal. We say that f is *conformal* if it is conformal at all points in Ω. Recall that in the standard basis the matrix representing $\mathbf{D}_{(x_0,y_0)} f$ is given by the Jacobian matrix

$$\mathbf{D}_{(x_0,y_0)} f = \begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix},$$

where (u, v) are the components of f ie. in complex notation $f = u + iv$.

[1] This condition is equivalent to the map being orientation preserving. Some authors choose to not impose this extra condition on conformal maps

A slightly more geometric insight is obtained by looking at curves. Consider a pair of curves $\gamma_1(t), \gamma_2(t) : (-\varepsilon, \varepsilon) \to \mathbb{C}$ intersecting at $\gamma_1(0) = \gamma_2(0) = z_0$. We say that they intersect at an angle θ at z_0 if the angle between the tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ is θ. The a mapping is conformal at z_0 if and only if for any two curves γ_1 and γ_2 as above, the angle between them is equal to the angle between their images $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $f(z_0)$. To see this, it is enough to note that if $\gamma(t) : (-\varepsilon, \varepsilon) \to \Omega$ is a curve with $\gamma(0) = z_0 = (x_0, y_0)$ and $\gamma'(0) = \vec{v}$, then

$$\mathbf{D}_{(x_0.y_0)} f(\vec{v}) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)).$$

We are now ready to prove our main observation.

Proposition 1. *Let $f : \Omega \to \mathbb{C}$ be a C^1 map. Then f is conformal at z_0 if and only if f is complex differentiable at z_0 and $f'(z_0) \neq 0$. In particular, f is conformal if and only if it is holomorphic with nowhere vanishing complex derivative.* [2].

We first need two elementary lemmas from linear algebra.

Lemma 1. *A linear map $C : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal if and only if $C = \lambda Q$, where Q is an orientation preserving orthogonal transformation and $\lambda > 0$.*

Recall that Q is orthogonal if and only if $|Q\vec{v}| = |\vec{v}|$ for all $\vec{v}$ or equivalently $Q^T Q = I$, where Q^T is the transpose of Q, and I is the identity matrix.

Proof. Let $\vec{e_1}$ and $\vec{e_2}$ denote the standard basis vectors in the x and y directions, and let $C^T C = (a_{ij})$ be the matrix representation of $C^T C$ in this basis. By conformality, Then

$$0 = \langle \vec{e_1}, \vec{e_2} \rangle \implies 0 = \langle C\vec{e_1}, C\vec{e_2} \rangle = \vec{e_1}^T (C^T C) \vec{e_2} = a_{12}.$$

By the symmetry of $C^T C$, we also have that $a_{21} = 0$, and hence $C^T C$ is diagonal. On the other hand,

$$\begin{aligned} 0 = \langle \vec{e_1} - \vec{e_2}, \vec{e_1} + \vec{e_2} \rangle \implies 0 &= \langle C(\vec{e_1} - \vec{e_2}), C(\vec{e_1} + \vec{e_2}) \rangle \\ &= \langle C\vec{e_1}, C\vec{e_1} \rangle - \langle C\vec{e_2}, C\vec{e_2} \rangle \\ &= a_{11} - a_{22}. \end{aligned}$$

That is, the diagonal terms in $C^T C$ are equal, and hence $C^T C = \mu I$ for some $\mu \in \mathbb{R}$. Since $\det C > 0$ and the diagonal terms of $C^T C$ have to be non-negative, we have that $\mu > 0$ Now, let $Q = \mu^{-1/2} C$. Then

$$C^T C = \mu I \implies Q^T Q = I,$$

and this proves the claim with $\lambda = \sqrt{\mu}$. The converse, that a matrix $C = \lambda Q$ is conformal if Q is orthogonal is trivial. □

[2]If we drop orientation preserving from the definition of conformality, then a conformal map is either a holomorphic or anti-holomorphic map

Lemma 2. *Any orientation preserving orthogonal* 2×2 *matrix* Q *is a rotation matrix, that is, it is given by*

$$Q = R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some $\theta \in [0, 2\pi)$.

Proof. Since $Q^T Q = I$ and $\det Q > 0$, we have that $\det Q = 1$. Suppose

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the combination of $Q^T Q = I$ and $\det Q = 1$ gives us the three equations

$$\begin{cases} a^2 + c^2 = 1 \\ b^2 + d^2 = 1 \\ ad - bc = 1. \end{cases}$$

In particular, $(a-d)^2 + (b+c)^2 = 0$, and hence $a = d$ and $c = -b$. Since $a^2 + c^2 = 1$, we can set $a = \cos\theta$ and $c = \sin\theta$. The result then follows. □

Proof.

- Suppose f is conformal at z_0. Then $C := \mathbf{D}_{z_0} f$ is a conformal linear map, and by the above two lemmas, $C = \lambda R_\theta$ for some θ. In particular, the partial derivatives of f satisfy the Cauchy-Riemann equations, and since f is a C^1 map (in particular the total derivative exists), this implies that f is complex differentiable at z_0. Moreover, $0 < \det C = |f'(z_0)|^2$, and hence $f'(z_0) \neq 0$.
- Conversely, suppose f is complex differentiable at z_0 and $f'(z_0) \neq 0$. Then $C := \mathbf{D}_{z_0} f$ satisfies $\det C = |f'(z_0)|^2 > 0$. Moreover, it can be checked by direct calculation (using the fact that the Cauchy-Riemann equations are satisfied) that $Q = |f'(z_0)|^{-1} C$ is orthogonal. Then by Lemma 1, C is a conformal linear map, and hence by definition, f is a conformal map.

□

Corollary 1. *Any holomorphic, injective function is conformal.*

Proof. Let $f \in \mathcal{O}(\Omega)$ and injective, but not conformal. By Proposition 1 there exists a point $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. If $w_0 = f(z_0)$, then the equation $f(z) = w_0$ has a root of multiplicity $m > 1$ at $z = z_0$. By our fundamental theorem on the local mapping properties of holomorphic functions (Theorem 1 in Lecture-19), there exists $\varepsilon, \delta > 0$ such that for each w such that $0 < |w - w_0| < \delta$, there exists at least m distinct points in $B_\varepsilon(z_0)$ such that $f(z) = w$. In particular f is not injective, which is a contradiction. □

Remark 1. *Of course, there are plenty of conformal maps which are not injective. For example* $f(z) = z^n$ *on* $\mathbb{C}^*$ *is conformal, but not injective if* $n > 1$. *Other examples include* e^z, $\sin z$, $\cos z$ *etc.*

Mobius transformations

A *fractional linear transformation* or a *Mobius transformation* is a map of the form

$$w = T(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$. Clearly if each of a, b, c and d are scaled by the same complex number, then $T(z)$ remains invariant. Hence it is often convenient to normalize so that $ad - bc = 1$. The set of Mobius transformations is denoted by $\mathrm{Mob}(\mathbb{C})$. Note that the map is defined and holomorphic for all z except $z = -d/c$. Moreover,

$$\lim_{z\to\infty} T(z) = \frac{a}{c}.$$

In view of this, it is sometimes more convenient to define $T(-d/c) = \infty$ and $T(\infty) = a/c$ and think of $T(z)$ as a map from the extended complex plane $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$ to itself. We say that a map $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic, and write $T \in \mathcal{O}(\hat{\mathbb{C}})$ if the following three conditions hold:

(1) If $R\Big|_{\mathbb{C}}$ is a meromorphic function and
(2) $F(z) = T(1/z)$ is holomorphic in a neighbourhood of $z = 0$.

Lemma 3. *$\mathcal{O}(\hat{\mathbb{C}})$ can be identified with the set of rational functions on $\mathbb{C}$.*

This is essentially Theorem 0.2 in Lecture 16.

Theorem 1. (1) *A Mobius transformation T is a holomorphic, bijective map from $\hat{C}$ onto $\hat{C}$, and it's inverse is also a Mobius transformation.*
(2) *In particular if $T(z_0) = \infty$ for some $z_0 \in \mathbb{C}$, then $T\Big|_{\mathbb{C}\setminus\{z_0\}} \to \mathbb{C} \setminus \mathbb{C}$ is a conformal map.*
(3) *Moreover, if T and S are Mobius transformations, then so is $S \circ T$. In other words, $\mathrm{Mob}(\mathbb{C})$ forms a group under the law of composition with the identity element given by the transformation $I(z) = z$.*

Proof. Since T is a rational function, it is clearly in $\mathcal{O}(\hat{\mathbb{C}})$. The second point follows from the first easily. Let $T(z)$ be as above.

- **T is injective.** Suppose $T(z_1) = T(z_2)$ and neither of z_1 or z_2 is infinity or $-d/c$. Then rearranging, it is easy to see that $(ad - bc)z_1 = (ad - bc)z_2$, and since $ad - bc \neq 0$, we have $z_1 = z_2$. Now, suppose $z_1 = \infty$ then $T(z_1) = \frac{a}{c} = T(z_2)$. This forces z_2 to be infinity and hence $z_1 = z_2$. On the other hand, if $z_1 = -d/c$, then $T(z_1) = \infty = T(z_2)$. Again, this means that $z_2 = -d/c = z_1$.
- **T is surjective.** To prove this, we can simply solve the equation $w = T(z)$. That is, $w = T(z)$, if and only

$$z = \frac{dw-b}{a-wc}.$$

Combining this with the injectivity, we then have a well defined map $T^{-1} : \hat{C} \to \hat{C}$ defined by

$$T^{-1}(w) = \frac{dw - b}{a - wc}$$

which is again a Mobius transformation.

- **$S \circ T$ is a Mobius transformation.** Suppose T is as above, and S is another Mobius transformation

$$S(w) = \frac{pw + q}{rw + s}.$$

Then a simply computation gives

$$S \circ T(z) = \frac{(pa + cq)z + pb + qd}{(ra + cs)z + rb + ds}.$$

Moreover,

$$(pa + cq)(rb + ds) - (ra + cs)(pb + qd) = (ad - bc)(pr - qs) \neq 0,$$

and so $S \circ T$ is again a Mobius transformation.

□

The group $PSL(2, \mathbb{C})$. For a Mobius transformation, if we write the coefficients as a matrix, we get what we call the coefficient matrix

$$M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $ad - bc \neq 0$, the matrix $M(T)$ is an invertible matrix, that is $M(T) \in GL(2, \mathbb{C})$, the group of all invertible 2×2 complex valued matrices. If we normalize so that $ad - bc = 1$, then $M(T) \in SL(2, \mathbb{C})$, the so-called special linear group of 2×2 complex valued matrices. Even with this normalization, a particular Mobius transformation actually corresponds to 2 matrices, namely $M(T)$ and $-M(T)$, and hence a Mobius transformation actually corresponds to an *equivalence class* of matrices. To make this more precise we define the *projective special linear group* as

$$PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/A \sim \pm A.$$

We denote any element of $PSL(2, \mathbb{C})$ as $[A]$ where A is a matrix in $SL(2, \mathbb{C})$. One can prove that $PSL(2, \mathbb{C})$ forms a group with the multiplication

$$[A] \cdot [B] := [AB],$$

where AB is the usual matrix multiplication. One has to of course check that if you pick different representatives in $[A]$ (ie. -A instead of A) and/or $[B]$, then the multiplication gives the same element in $PSL(2, \mathbb{C})$. With this definition in place, we then obtain a map $\Phi : Mob(\mathbb{C}) \to PSL(2, \mathbb{C})$, given by

$$\Phi(T) = [M(T)].$$

By Theorem 1, part(3), it is clear that

$$[M(S \circ T)] = [M(S)M(T)].$$

where the multiplication on the right is simply the usual matrix multiplication. In fancier language, this says that the map Φ is a group homomorphism. In fact we have the following.

Theorem 2. *The map Φ is an isomorphism between the groups $Mob(\mathbb{C})$ and $PSL(2,\mathbb{C})$.*

Remark 2. *One can also define $PGL(2,\mathbb{C})$ as $GL(2,\mathbb{C})/\sim$ where $A \sim \lambda A$ where $\lambda \in \mathbb{C}^*$. It is then easy to see that $PGL(2,\mathbb{C})$ is isomorphic as a group to $PSL(2,\mathbb{C})$.*

Remark 3. *A convenient way to represent Mobius transformations is using homogenous coordinates, and this makes the role of $PSL(2,\mathbb{C})$ much more transparent. The extended complex plane $\hat{\mathbb{C}}$ can be identified with the set $\mathbb{P}^1$ of complex lines passing through the origin in $\mathbb{C}^2$. The identification is given by the complex slope. A line L in $\mathbb{C}^2$ passing through the origin is determined by a point $(\xi_1, \xi_2) \neq (0,0)$, and any other point on the line is given by $(t\xi_1, t\xi_2)$ for $t \in \mathbb{C}$. Hence we represent points in $\mathbb{P}^1$ as equivalence classes of these points $[\xi_1 : \xi_2]$. The complex slope is then given by $z = \xi_1/\xi_2$, and is a well defined number in $\hat{\mathbb{C}}$. For instance the points $[0:1]$ and $[1:0]$ correspond to the points 0 and ∞ respectively in $\hat{\mathbb{C}}$. We say that $[\xi_1 : \xi_2]$ are the homogenous coordinates of z. Note that homogenous coordinates are unique, only up to scaling ie. both (ξ_1, ξ_2) and $t\xi_1, t\xi_2)$ for $t \neq 0$, represent the same point z in $\hat{\mathbb{C}}$.*

With this identification, if $w = \zeta_1/\zeta_2$ and $z = \xi_1/\xi_2$, we can rewrite $w = Tz$ as

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

So the action of the Mobius transformation on $\hat{\mathbb{C}}$ is exactly the linear action of a 2×2 matrix on $\mathbb{C}^2 \setminus \{(0,0)\} = \{$set of homogenous coordinates$\}$.

The cross ratio. Given any four numbers z_1, z_2, z_3, z_4 in the extended complex plane $\hat{\mathbb{C}}$, the *cross ratio* is defined to be

$$(z_1, z_2, z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}.$$

Note that if one of the points is infinity, then the cross ratio is defined by taking a limit. For instance, if $z_1 = \infty$, then

$$(\infty, z_2, z_3, z_4) = \frac{z_4 - z_2}{z_3 - z_2}.$$

The importance of the cross ratio comes form the following theorem.

Theorem 3. (1) *Given any three points z_2, z_3, z_4, there exists a <u>unique</u> Mobius transformation mapping these points to $1, 0$ and ∞ respectively. In fact we can take*

$$S(z) = (z, z_2, z_3, z_4).$$

(2) *If T is any Mobius transformation, then*

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4).$$

Proof. (1) Clearly, $S(z) = (z, z_2, z_3, z_4)$ is a Mobius transformation that takes (z_2, z_3, z_4) to $(1, 0, \infty)$. Let T be another such Mobius transformation. Then $S \circ T^{-1}$ takes $(1, 0, \infty)$ to itself. Then it is not hard to prove that $S \circ T^{-1}(z) = z$, and hence $S(z) = T(z)$.

(2) Let $S(z) = (z, z_2, z_3, z_4)$. Then ST^{-1} carries (Tz_2, Tz_3, Tz_4) to $(1, 0, \infty)$. Then by part (1), $ST^{-1}(z) = (z, Tz_2, Tz_3, Tz_4)$. Applying this to Tz_1, we see that

$$(z_1, z_2, z_3, z_4) = Sz_1 = ST^{-1}Tz_1 = (Tz_1, Tz_2, Tz_3, Tz_4).$$

□

As a consequence we have the following.

Corollary 2. *Given any pair of three points (z_1, z_2, z_3) and (w_1, w_2, w_3), there exists a unique Mobius transformation that takes the first triple to the second.*

Proof. Let S and T be the Mobius transformations that take (z_1, z_2, z_3) and (w_1, w_2, w_3) to $(1, 0, \infty)$ respectively. Then $T \circ S^{-1}$ is the Mobius transformation that we need. Uniqueness follows from the fact that there is unique transofrmation (namely the identity) that takes $(1, 0, \infty)$ to itself. □

There is another, more geometric, application of the cross ratio. A *generalised circle* in $\mathbb{C}$ is either a circle (given by an equation $|z - z_0| = r$ or a straight line (given by the equation $\bar{a}z + a\bar{z} + b = 0$, where $b \in \mathbb{R}$). In principle, a straight line is being thought of as a circle with infinite radius. An additional justification for this terminology is that both circles and straight lines in $\mathbb{C}$ correspond to circles on the Riemann sphere via the stereographic projection (or rather via it's inverse). A key observation is that three points determine a unique generalised circle (in the case of a line, one of the points will be at infinity).

Theorem 4. *The cross ratio of of (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a generalised circle. Consequently, a Mobius transformation maps generalised circles to generalised circles.*

Proof. The second part follows from the first part, and the fact that the cross ratio is invariant under Mobius transformations. So we focus on proving the first part. The key is the following claim.

Claim. If T is a Mobius transformation, then T^{-1} maps the (extended) real axis $\hat{\mathbb{R}}$ to a generalised circle.

Assuming this we complex the proof. There are two directions.

- $\Longrightarrow$**.** Suppose the cross ratio (z_1, z_2, z_3, z_4) is real. Consider $Tz = (z, z_2, z_3, z_4)$. Then $Tz_1 \in \mathbb{R}$. Moreover, $(Tz_2, Tz_3, Tz_4) = (1, 0, \infty)$ and hence they already lie on the real line. Since T^{-1} maps $\mathbb{R}$ to a generalised circle, z_1, z_2, z_3 and z_4 lie on a generalised circle.

- $\Longleftarrow$ **.** Suppose z_1, z_2, z_3, z_4 lie on a generalised circle. Once again consider $Tz = (z, z_2, z_3, z_4)$. Once again, $(Tz_2, Tz_3, Tz_4) = (1, 0, \infty)$ and hence they already lie on the real line. So z_2, z_3, z_4 lie on the generalised circle $T^{-1}(\hat{\mathbb{R}})$. Hence z_1 also lies on $T^{-1}(\hat{\mathbb{R}})$, and so $Tz_1 \in \mathbb{R}$. In particular the cross ratio is real.

Proof of the claim: Let $z = T^{-1}(w)$. If w is real, then $Tz = \overline{Tz}$. If $T = \frac{az+b}{cz+d}$, then this condition translates to

$$\frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}.$$

Cross multiplying and simplifying, we get

$$(a\bar{c} - c\bar{a})|z|^2 + (a\bar{d} - c\bar{b})z + (b\bar{c} - d\bar{a})\bar{z} + b\bar{d} - d\bar{b} = 0.$$

There are two cases:

- **Case-1:** $a\bar{c} - c\bar{a} = 0$. Then since $ad - bc \neq 0$ we have that $a\bar{d} - c\bar{b} \neq 0$. Hence the equation above represents a straight line.
- **Case-2:** $a\bar{c} - c\bar{a} = \neq 0$. Completing the square, we can rewrite the equation as
$$\left|z + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a}\right| = \left|\frac{ad - bc}{\bar{a}c - \bar{c}a}\right|,$$
which clearly defines a circle.

□

The Cayley transform. Perhaps the most important Mobius transformation is the the *Cayley transform*:

$$\beta(z) = \frac{z-i}{z+i}.$$

To compute it's inverse, we set $w = \beta(z)$. and solve for z. It is easy to see that

$$\beta^{-1}(w) = i\frac{1+w}{1-w}.$$

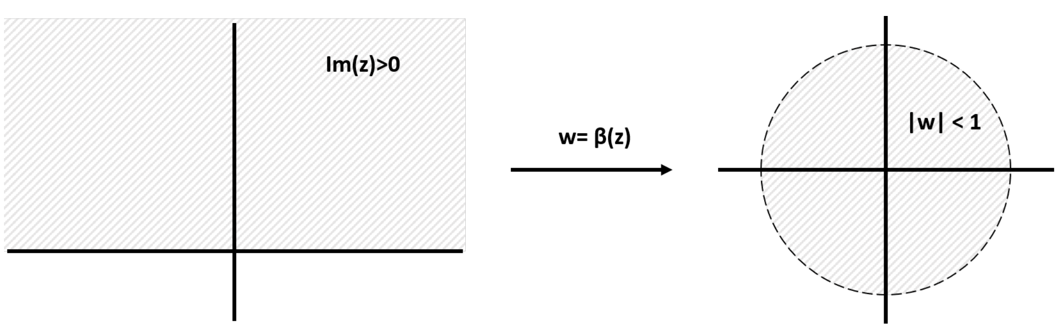

FIGURE 1. The Cayley transform

Lemma 4. *The Cayley transform is a biholomorphism from the upper half plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$. Moreover, it maps the boundary $\partial\mathbb{H}$ bijectively to $\partial\mathbb{D}$.*

Proof.

- $\beta(z)$ **maps $\mathbb{H}$ into $\mathbb{D}$ and $\partial\mathbb{H}$ to $\partial\mathbb{D}$.** To see this, we compute

$$|\beta(z)|^2 = \frac{1+|z|^2+i(z-\bar{z})}{1+|z|^2-i(z-\bar{z})} = \frac{1+|z|^2-2Im(z)}{1+|z|^2+2Im(z)} < 1$$

  if $Im(z) > 0$ i.e. if $z \in \mathbb{H}$. On the other hand it is also clear from the computation that $|\beta(z)| = 1$ if and only if $Im(z) = 0$.
- $\beta(z)$ **is surjective from $\mathbb{H}$ onto $\mathbb{D}$.** It is easy to see by direct computation that $\beta^{-1}(w)$ given by the above formula is an inverse. It is a nice exercise to check that indeed $\beta^{-1}(w)$ is in the upper half plane if $w \in \mathbb{D}$.

□

A survey of elementary mappings

Given two regions Ω_1 and Ω_2, we wish to construct a conformal map one to the other. A reasonable strategy is to first try to map Ω_1 into the unit disc, and then to map the unit disc into Ω_2. A priori, t his strategy might seem limiting. After all why would there exist such a conformal map from Ω_1 into the unit disc. In a couple of lectures time, we'll prove a deep fact - The Riemann mapping theorem - that for a large class of regions, namely simply connected strict subsets of $\mathbb{C}$, such a mapping always exists. In view of this it is important to build up a toolkit of familiar mappings, so that more complicated mapping can be constructed by taking products, compositions etc. So in this section we'll familiarise ourselves with the mapping properties of complex powers, exponentials and logarithms. A good strategy in finding the image of a certain region under a conformal mapping is to find the image of the boundary. Our convention is to use $z = x + iy$ as the complex coordinate in the domain, and $w = u + iv$ as the complex coordinate on the image.

Rotations and dilations. Clearly rotations $R_\theta(z) = e^{i\theta}z$, dilations and $T_\lambda(z) = \lambda z$ are examples of conformal maps. In fact these form a subgroup of the full group of Mobius transformations.

Complex powers. Complex powers are useful in mapping sectors and half planes to each each other. We illustrate this using two examples.

- **Case-1:** $f(z) = z^n,\ n \in \mathbb{N}$. An example of such a mapping is in Figure 2
- **Case-1:** $f(z) = z^n,\ n \in \mathbb{N}$. An example of such a mapping is in Figure 3

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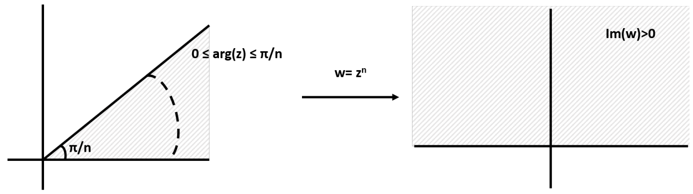

FIGURE 2. integer powers turn a sector into a half plane

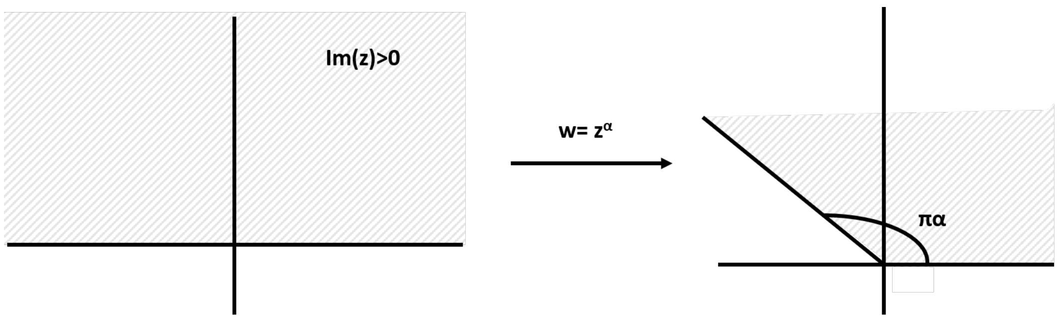

FIGURE 3. fractional powers can turn a half plane into a sector

The logarithm. If $L(z) = \log z$ is a branch of the logarithm, then $L'(z) = 1/z \neq 0$ on it's domain of definition. Hence it defines a conformal map. As an illustration, in the Figures 4 and 5, we consider the branch of the logarithm

$$\log z = \log|z| + i\arg(z),$$

where $\arg(z) \in (-\pi/2, 3\pi/2)$. That is, our branch cut is the line $\{z = iy \mid y < 0\}$ or the $-ve$ y axis.

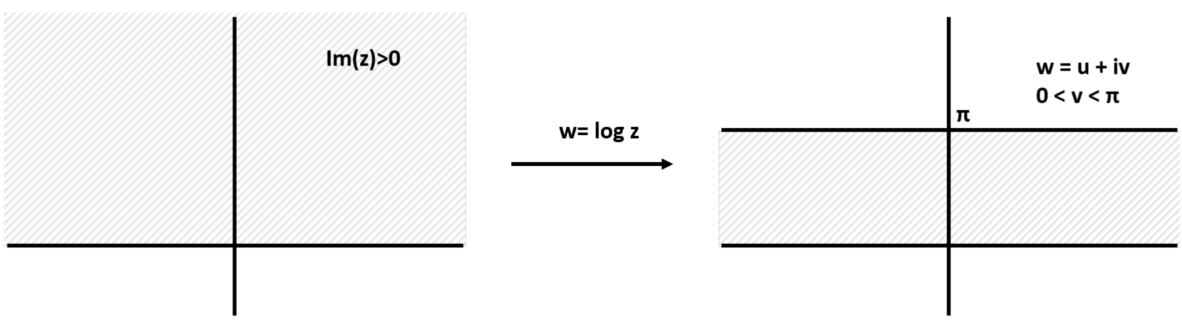

FIGURE 4. The logarithm mapping the $\mathbb{H}$ into an infinite horizontal strip

In the Figure 4, the boundary of the domain $\mathbb{H}$ contained in the domain of definition of $\log z$ consists of two components, namely the negative and positive x (or real) axis. The negative axis consists of points with $\arg(z) = \pi$, and hence is mapped via the log to the line $v = \pi$. Similarly the positive

x-axis has $\arg(z) = 0$, and hence is mapped to $v = 0$. Hence $\mathbb{H}$ is mapped by the above branch of logarithm to the infinite strip $\{0 < v < \pi\}$. The reader should similarly work out the mapping in Figure 5.

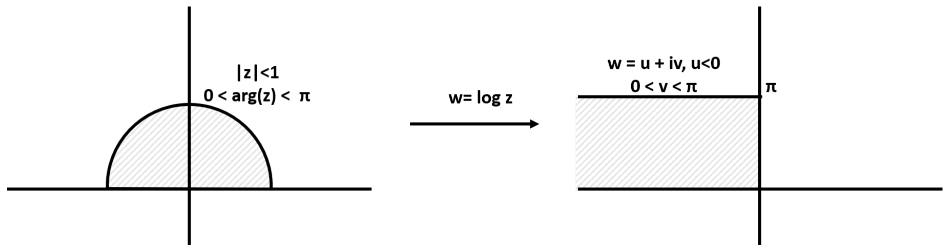

FIGURE 5. The logarithm mapping a semi-circular region into a half infinite strip

The exponential map. The map $f(z) = e^z$ is clearly a conformal map by Proposition 1 since it's derivative never vanishes. Figures 6 and 7 illustrate some of the mapping properties of the exponential. The reader should try to work out why the images of the two mappings below are given by the figures on the left. As in the discussion above, the trick is to work out the images of the various boundary components.

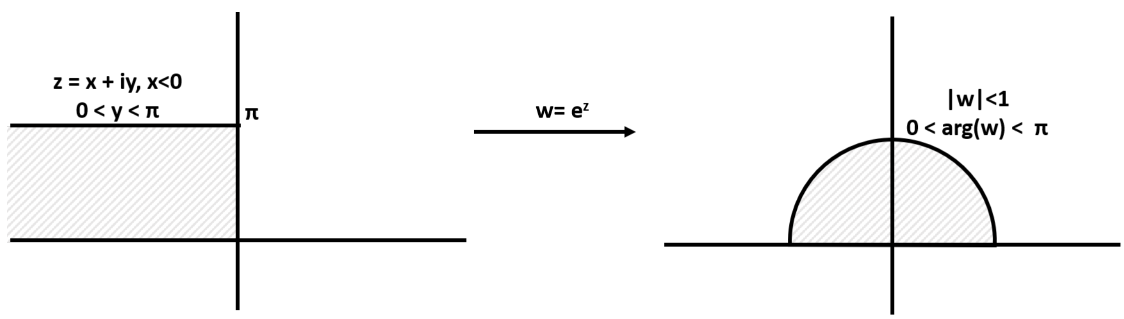

FIGURE 6. The exponential mapping a half infinite strip to a semi-circular region.

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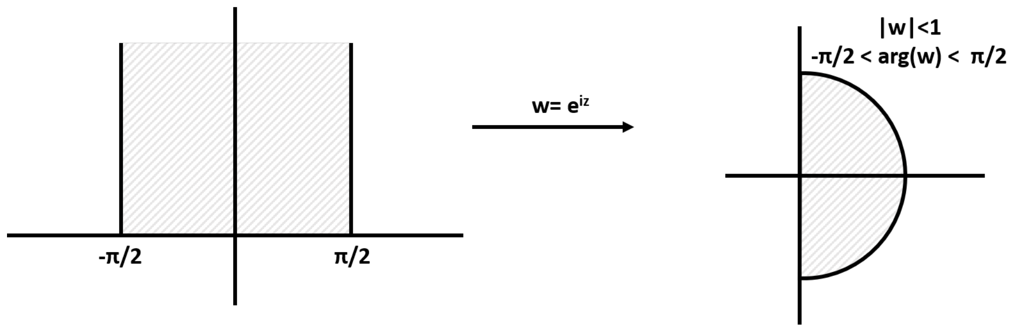

FIGURE 7. e^{iz} mapping a half infinite strip to a semi-circle.