LECTURE-23

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In this lecture, we compute the automorphism groups of the disc and the complex plane.

Biholomorphisms and automorphisms

A holomorphic function $f : \Omega \to \Omega'$ is said to be a *bi-holomorphism* if it is a bijective function, and f^{-1} is also holomorphic.

Lemma 1. *A holomorphic function $f : \Omega \to \Omega'$ is a bi-holomorphism if and only if it is bijective.*

Proof. If f is a bi-holomorphism, then it is automatically bijective from the definition. Conversely suppose $f : \Omega \to \Omega'$ is a bijective, holomorphic function. Injectivity implies that $f'(z) \neq 0$ for all $z \in \Omega$. Then the holomorphic inverse function theorem implies that the inverse is holomorphic. $\square$

For any domain $\Omega \subset \mathbb{C}$, a bi-holomorphic function $f : \Omega \to \Omega$ is called an *automorphism* of Ω.

Lemma 2. *The set of automorphisms of a domain Ω*

$$\mathrm{Aut}(\Omega) := \{f : \Omega \to \Omega \mid f \text{ is holomorphic and bijective}\},$$

forms a group under the law of composition.

We call $\mathrm{Aut}(\Omega)$ the automorphism group of Ω. Note that if Ω and Ω' are bi-holomorphic, then $\mathrm{Aut}(\Omega)$ and $\mathrm{Aut}(\Omega')$ are isomorphic as groups. In fact, if $\varphi : \Omega \to \Omega'$ is a biholomorphism, then $\Phi : \mathrm{Aut}(\Omega) \to \mathrm{Aut}(\Omega')$ defined by

$$\Phi(f) = \varphi \circ f \circ \varphi^{-1},$$

is the required group isomorphism.

The aim of this note is to compute the automorphism groups of the complex plane, the punctured plane and the disc (and hence the upper half plane).

1. Automorphism group of the disc

In this section, we compute $\mathrm{Aut}(\mathbb{D})$.Recall that in Problem-5 from Assignment-1, you were asked to prove that for any $|\alpha| < 1$,

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z}$$

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is a biholomorphism from the disc to itself. Our main theorem is that upto rotation these are the only automorphisms. En route, we'll also provide a short proof that ψ_α is indeed an element in the automorphism group, using some of the tools we have developed over the past few months, but which were of course unavailable when you were asked to solve the problem!

Theorem 1. *The automorphism group of the disc is given by*

$$\mathrm{Aut}(\mathbb{D}) = \{\psi_{\alpha,\theta}(z) = e^{i\theta}\frac{\alpha - z}{1 - z\bar{\alpha}} \mid \alpha \in \mathbb{D},\ \theta \in [0, 2\pi)\}.$$

Moreover, the automorphism $\psi_{\alpha,\theta}$ is precisely the automorphism that takes $z = 0$ to $z = \alpha$. In particular, the automorphisms of the disc fixing the origin are all given by $z \to e^{i\theta}z$ for some fixed θ.

The key tool in the proof is the Schwarz lemma, whose utility extends well beyond the computation of the automorphism groups of the disc.

Lemma 3 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map such that $f(0) = 0$. Then*

- $|f(z)| \le |z|$ *for all* $z \in \mathbb{D}$.
- $|f'(0)| \le 1$.

Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$

Proof. Since $f(0) = 0$, there exists a holomorphic function

$$g : \mathbb{D} \to \mathbb{D}$$

such that

$$f(z) = zg(z).$$

For any fixed $z \in \mathbb{D}$, let $1 > r > |z|$. Then by the maximum modulus principle,

$$|g(z)| \le \max_{|w|=r} \frac{|f(w)|}{r} < \frac{1}{r}.$$

Letting $r \to 1^-$ we see that $|g(z)| < 1$ for all $z \in \mathbb{D}$, and hence

$$|f(z)| < |z|$$

for all $z \in \mathbb{D}$. This directly implies that $|f'(0)| \le 1$. Now, suppose that $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D} \setminus \{0\}$. Then $|g(z_0)| = 1$, and hence by the maximum modulus principle, $g(z)$ must be a constant, and hence $f(z) = az$ for some a with $|a| = 1$.

Finally, suppose $|f'(0)| = 1$. Then there exists a sequence $z_n \to 0$, $z_n \neq 0$ such that

$$1 - \frac{1}{n} \le \frac{|f(z_n)|}{|z_n|} \le 1.$$

By the definition of g, we then have that $1 - 1/n \le |g(z_n)| \le 1$, and hence by continuity, $|g(0)| = 1$. Then 0 is an interior maximum point for $|g(z)|$, and hence again by maximum modulus principle, $g(z)$ is a constant. That is, $g(z) = a$, where $a = g(0)$ satisfies $|a| = 1$, and once again $f(z) = az$.

□

Proof of the theorem. There are two steps in the proof. We let $\psi_\alpha := \psi_{\alpha,0}$ as above (ie. ψ_α is the map with rotation by zero angle). Note that ψ_α exchanges 0 and α. That is, $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$.

- We'll first prove that each $\psi_{\alpha,\theta}$ is actually an automorphism. To see this, first we observe that

$$|\psi_{\alpha,\theta}(z)|^2 = \psi_{\alpha,\theta}(z)\overline{\psi_{\alpha,\theta}}(z) = \frac{|\alpha|^2 + |z|^2 - z\bar{\alpha} - \bar{z}\alpha}{1 + |\alpha|^2 - z\bar{\alpha} - \bar{z}\alpha},$$

  and so $|\psi_{\alpha,\theta}(z)| = 1$ if $|z| = 1$. But then by the maximum principle, since $\psi_{\alpha,z}$ is clearly non-constant, $|\psi_{\alpha,\theta}(z)| < 1$ for all $z \in \mathbb{D}$. Hence $\psi_{\alpha,\theta}$ maps $\mathbb{D}$ into itself. Next, consider $\varphi_\alpha := \psi_\alpha \circ \psi_\alpha$. Clearly, $\varphi_{\alpha,\theta}(0) = 0$ and $\varphi_\alpha(\alpha) = \alpha$, and φ_α maps $\mathbb{D}$ into itself. By equality in Schwarz lemma, $\varphi_\alpha(z) = z$ for all $z \in \mathbb{D}$. In particular, ψ_α is surjective and injective, and hence a biholomorphism, with inverse function given by itself ie. $\psi_\alpha^{-1} = \psi_\alpha$. But then $\psi_{\alpha,\theta} = e^{i\theta}\psi_\alpha$ is also clearly a biholomorphism.
- It remains to show that these are the only automorphisms. Let $F \in \mathrm{Aut}(\mathbb{D})$ such that $F(0) = 0$. By the Schwarz lemma, $|F(z)| \leq |z|$. But the same also holds true for $F^{-1}(z)$, and so

$$|z| = |F^{-1}(F(z))| \leq |F(z)| \leq |z|,$$

  and hence all inequalities must be equalities. That is, $|F(z)| = |z|$ for all $z \in \mathbb{D}$. By the equality part of Schwarz lemma, we have that $F(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$. Now suppose $F \in \mathrm{Aut}(\mathbb{D})$ such that $F(\alpha) = 0$. Consider the automorphism $F_\alpha(z) := F(\psi_\alpha(z))$. Then $F_\alpha(0) = 0$, and hence by the above argument, $F_\alpha(z) = e^{i\theta}z$ for some θ. But then, since $\psi_\alpha^{-1} = \psi_\alpha$, we have that

$$F(z) = F_\alpha(\psi_\alpha^{-1}(z)) = e^{i\theta}\psi_\alpha(z) = \psi_{\alpha,\theta}(z).$$

This completes the proof of the theorem. □

2. Automorphism groups of $\mathbb{C}$ and $\mathbb{C}^*$

Theorem 2. *Let $f : \mathbb{C} \to \mathbb{C}$ be an injective holomorphic map. Then*

$$f(z) = az + b$$

for some $a, b \in \mathbb{C}$ with $a \neq 0$.Since any such map is automatically surjective, we have that

$$\mathrm{Aut}(\mathbb{C}) = \{az + b \mid a, b \in \mathbb{C}, a \neq 0\}.$$

Theorem 3. *Any injective holomorphic map $f : \mathbb{C}^* \to \mathbb{C}^*$ is given by either*

$$f(z) = az \text{ or } f(z) = \frac{a}{z},$$

for some $a \in \mathbb{C}$ and $a \neq 0$. Since any such map is automatically surjective, we have that

$$\mathrm{Aut}(\mathbb{C}^*) = \{az, \frac{a}{z} \mid a \in \mathbb{C}, a \neq 0\}.$$

The key lemma needed to compute both the automorphism groups is the following.

Lemma 4. *Let $g : \mathbb{C}^* \to \mathbb{C}$ be holomorphic and injective. Then g cannot have an essential singularity at $z = 0$.*

Proof. Suppose, for the sake of contradiction, g has an essential singularity at $z = 0$. Consider the disc $D_1(2)$ of radius 1 around the point $z = 2$, and the unit disc $\mathbb{D}$. Note that both these discs are disjoint from each other. By the open mapping theorem, $V = g(D_1(2))$ is an open neighborhood of $g(2)$. By the Casorati-Weierstrass theorem, $g(\mathbb{D})$ is dense in $\mathbb{C}$. That means there is some $w \in g(D_1(2))$ and some $z_1 \in \mathbb{D}$ such that

$$g(z_1) = w.$$

But since $w \in g(D_1(2))$, there is already a $z_2 \in D_1(2)$ such that

$$g(z_2) = w.$$

Since the open discs $D_1(2)$ and $\mathbb{D}$ are disjoint sets, $z_1 \neq z_2$, but $g(z_1) = w = g(z_2)$. This contradicts the injectivity of g. Hence $z = 0$ cannot be an essential singularity. □

We also need an elementary generalization of Liouville's theorem, which was a homework problem sometime back. We provide a proof for the sake of completeness.

Lemma 5. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that*

$$|f(z)| \leq M(1 + |z|^n)$$

for some $M > 0$ and all $z \in \mathbb{C}$. Then f is a polynomial of degree less than or equal to n.

Proof. By the Cauchy estimates (Corollary 3 in Lecture-8), if $R > 1$, we have that,

$$|f^{(k)}(0)| \leq \frac{k!M(1 + R^n)}{R^k}. \leq 2Mk!R^{n-k}.$$

But this holds no matter what R is chosen. So letting $R \to \infty$, if $k > n$, the right hand side goes to zero. Hence

$$f^{(k)}(0) = 0$$

for all $k = n + 1, n + 2, \cdots$. But since f is entire, it has a power series expansion whose coefficients are given by

$$a_k = \frac{f^{(k)}(0)}{k!} = 0,$$

for $k > n$. Hence the power series terminates, and f is a polynomial of degree less than or equal to n. □

We are now ready to compute the automorphism groups of $\mathbb{C}$ and $\mathbb{C}^*$.

Proof of Theorem 2. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective, holomorphic map. The key idea is to study the function at infinity. That is, define a holomorphic function $g : \mathbb{C}^* \to \mathbb{C}$ by

$$g(z) := f\Big(\frac{1}{z}\Big).$$

Then it is easy to see that g is also injective. Applying lemma 4, g either has a zero or a pole at $z = 0$. In any case, this means that there exists a constant $M > 0$ and integer $n > 0$ such that $|z|^n|g(z)| < \frac{1}{M}$ on $|z| < 1$. Transferring the estimates to f, we see that

$$|f(z)| < M|z|^n,$$

for $|z| > 1$. On the other hand, since $|z| \leq 1$ is compact, we actually get that (possibly by choosing a bigger M), that

$$|f(z)| \leq M(1 + |z|^n),$$

for all $z \in \mathbb{C}$. Then by lemma 5, $f(z)$ is a polynomial. By the fundamental theorem of algebra, $f(z)$ has at most n roots. We claim that $n = 1$. To see this, note that by injectivity, all the roots have to be identical, or equivalently, $f(z) = a(z - \alpha)^n$ for some $a, \alpha \in \mathbb{C}$ with $c \neq 0$. If $n > 1$, then $f'(\alpha) = 0$, and hence $f(z)$ cannot even be locally injective (see Theorem 1 or Corollary 0.3 in Lecture 19), and hence we must have $n = 1$. But then clearly $f(z) = az + b$, where we put $b = -a\alpha$. This proves the first part of the theorem. For the second part, notice that any linear polynomial is surjective, and hence the f above will automatically be surjective, and hence give an automorphism. □

Proof of Theorem 3. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be an injective map. Then by lemma 4, $z = 0$ is either a removable singularity or a pole.

Case 1. Suppose $z = 0$ is a removable singularity. Then f extends to $\tilde{f} : \mathbb{C} \to \mathbb{C}$. We then claim that $\tilde{f}$ is also injective. Suppose not, then since f is injective, the only possibility is that there are $z_0, w_0 \in \mathbb{C}$ with $z_0 \neq 0$ such that

$$\tilde{f}(z_0) = \tilde{f}(0) = w_0.$$

By the argument principle there is a neighborhood U of w_0 and disjoint discs $D_r(z_0)$, $D_\rho(0)$ around z_0 and 0 respectively such that for any $w \in U$, $w \neq w_0$ there are solutions $z_1 \in D_r(z_0)$ and $z_2 \in D_\rho(0)$ to

$$\tilde{f}(z) = w.$$

. But $z_2 \neq 0$ since $w \neq w_0$. So the two distinct solutions are actually solutions to

$$f(z) = w,$$

contradicting the injectivity of f. This proves that the extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is an injective holomorphic map. But then by Theorem 2, $\tilde{f}(z) = az + b$ for some $a, b \in \mathbb{C}$ and $a \neq 0$. All that is needed now is to show that $b = 0$. If not, then $0 = \tilde{f}(-b/a) = f(-b/a)$ which is a contradiction since f takes non-zero values being a map from $\mathbb{C}^*$ into $\mathbb{C}^*$. To sum up, in this case $f(z) = az$.

Case-2: Suppose $z = 0$ is a pole. Then if we define

$$h(z) = \frac{1}{f(z)},$$

then h extends to an holomorphic function $\tilde{h} : \mathbb{C} \to \mathbb{C}$. Then by the proof in the first case, we can see that

$$\tilde{h}(z) = cz,$$

for some $c \neq 0$. But then this shows that $f(z) = z/c$, and proves the theorem with $a = 1/c$. $\square$

3. Automorphism group of the extended complex plane

. Recall that a map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called holomorphic if

(1) If $F\Big|_{\mathbb{C}}$ is a meromorphic function and
(2) $G(z) = F(1/z)$ is holomorphic in a neighbourhood of $z = 0$.

Theorem 4. *The automorphism group of the extended complex plane is given by*

$$\mathrm{Aut}(\hat{\mathbb{C}}) = \Big\{T(z) = \frac{az+b}{cz+d} \mid ad - bc = 1\Big\},$$

and hence $\mathrm{Aut}(\hat{\mathbb{C}}) \cong PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\pm I$.

Proof. From our discussion in the previous lecture, it is clear that any $T(z)$ defined as above is an automorphism of $\hat{C}$, and hence the set on the right is contained in $\mathrm{Aut}(\hat{\mathbb{C}})$. To show the reverse containment, let $T \in \mathrm{Aut}(\hat{\mathbb{C}})$. Then there is a unique point $z_0 \in \hat{\mathbb{C}}$ such that $T(z_0) = \infty$, and a unique point w_0 such that $T(\infty) = w_0$.

- **Case-1:** $z_0 = \infty$**.** In this case, $w_0 = \infty$, and so $F(z) = T\Big|_{\mathbb{C}}$ is a bi-holomorphism of $\mathbb{C}$, and by Theorem 2, $F(z) = az + b$ for some a, b and $a \neq 0$. But then extending this to infinity, $T(z) = az + b$ and hence we are done.
- **Case-2:** $z_0 \neq \infty$**.** In this case, since T is one-one, $w_0 \neq \infty$ (else it will map both z_0 and ∞ to ∞, contradicting the one-one property). Now consider the function $F : \mathbb{C}^* \to \mathbb{C}$ defined by

  $$F(z) = T(z + z_0) - w_0.$$

  It is easy to check that F maps $\mathbb{C}^*$ to $\mathbb{C}^*$, and is infact an automorphism of $\mathbb{C}^*$ is (this follows essentially from the fact that F has a zero at infinity). Moreover, since $z = 0$ is a pole for F, by Theorem

3, there exists an $a \in \mathbb{C}^*$ such that $F(z) = a/z$. But then solving for z,

$$T(z) = \frac{a}{z - z_0} + w_0,$$

and is a Mobius transformation.

$\square$

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