LECTURE-24

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Recall that two domains are called conformally equivalent or biholomorphic if there exists a holomorphic bijection from one to the other. This automatically implies that there is an inverse holomorphic function. The aim of this lecture is to prove the following deep theorem due to Riemann. Denote by $\mathbb{D}$ the unit disc centered at the origin.

Theorem 0.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected set that is not all of $\mathbb{C}$. Then for any $z_0 \in \Omega$, there exists a unique biholomorphism $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0, \quad \textit{and } F'(z_0) > 0.$$

Here $F'(z_0) > 0$ stands for $F'(z_0)$ being real and positive, and can be thought of as a normalization, to ensure that the above map is unique. The precise normalization by itself is not very important. The reader should try to test her/his understanding of the proof by coming up with other normalizations that work, and also some that do not work (for instance, you might not be able to impose that $F'(z_0) = 1$). Note that by Liouville's theorem, such a statement is patently false if $\Omega = \mathbb{C}$, and so the hypothesis that Ω is a *proper* subset is a necessary condition. As a consequence of the Theorem, we have the following corollary.

Corollary 0.1. *Any two proper, simply connected subsets for $\mathbb{C}$ are conformally equivalent.*

Proof of uniqueness in Theorem 0.1. Let $F_1 : \Omega \to \mathbb{D}$ and $F_2 : \Omega \to \mathbb{D}$ be two such mappings. Then $f = F_2 \circ F_1^{-1}$ satisfies the following properties

- $f : \mathbb{D} \to \mathbb{D}$ is injective and onto.
- $f(0) = 0$.
- $f'(0) > 0$.
- f^{-1} also satisfies both these properties.

By Schwarz lemma, $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f^{-1}(w)| \leq |w|$ for all $w \in \mathbb{D}$. Let $w = f(z)$, then second inequality gives $|z| \leq |f(z)|$, and hence $|z| = |f(z)|$. But then by the equality part of Schwarz lemma, we see that $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. But then $f'(0) = a$, which forces $a = 1$ (since $f'(0) > 0$). Hence $f(z) = z$ for all $z \in \mathbb{D}$ or equivalently $F_2(w) = F_1(w)$ for all $w \in \Omega$.

Date: 24 August 2016.

Montel's and Hurwitz's theorems

The proof relies on two theorems on sequences of holomorphic functions. Recall that we say that a sequence of functions f_n converges compactly on Ω to f if it converges uniformly on any compact subset $K \subset \Omega$. More precisely, for every compact set $K \subset \Omega$ and $\varepsilon > 0$, there exists an $N = N(\varepsilon, K)$ such that

$$\sup_{z \in K} |f_n(z) - f(z)| < \varepsilon$$

whenever $n > N$. In Lecture 9 we proved the following theorem:

Theorem 0.2. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge compactly to $f : \Omega \to \mathbb{C}$, then $f(z)$ is holomorphic. Moreover*

$$f_n^{(k)} \to f^{(k)}$$

compactly on Ω for all $k \in \mathbb{N}$.

We say that family of continuous functions $\mathcal{F}$ on an open set Ω is *normal* if every sequence of functions in $\mathcal{F}$ has a subsequence that converges compactly on Ω. Note that the definition does not require the limiting function to be contained in $\mathcal{F}$. On the other hand, by Theorem 0.2 above, the limiting function will certainly be holomorphic. The family is said to be *locally uniformly bounded* if for any $K \subset \Omega$ compact, there exists a constant M_K such that

$$\sup_{z \in K} |f(z)| < M_K$$

for all $f \in \mathcal{F}$.

Theorem 0.3 (Montel's theorem)**.** *A family $\mathcal{F}$ of holomorphic functions on Ω is normal if and only if it is locally uniformly bounded.*

To prove this, we first recall the Arzela-Ascoli theorem. Recall that family $\mathcal{F}$ of continuous functions on Ω is said to *locally equicontinuous* if for all $a \in \Omega$ and all $\varepsilon > 0$ there exists a $\delta = \delta(a, \varepsilon)$ such that

$$z, w \in D_\delta(a) \implies |f(z) - f(w)| < \varepsilon,$$

for all $f \in \mathcal{F}$. Then we have the following basic theorem, which we state without proof.

Theorem 0.4 (Arzela-Ascoli)**.** *If a family of functions is locally equicontinuous and locally uniformly bounded, then for every sequence of functions $\{f_n\} \in \mathcal{F}$, there exists a continuous function f and a subsequence $\{f_{n_k}\}$ which converges to f compactly on Ω.*

Remark 0.1. *Generally, Arzela-Ascoli is stated for compact sets assuming equicontinuity. One can prove the above theorem by taking an exhaustion of Ω by compact sets ie. $K_1 \subset K_2 \cdots \subset K_n \cdots$ such that $\Omega = \cup K_n$. Local equicontinuity will then imply that the family is genuinely equicontinuous on each compact set K_n. Then apply the standard Arzela-Ascoli to $\mathcal{F}$ restricted to each K_n, and use Cantor diagonalization argument.*

Proof of Montel's theorem. First, suppose that $\mathcal{F}$ is a locally uniformly bounded family of holomorphic functions. By the Arzela-Ascoli theorem, if we show that $\mathcal{F}$ is automatically locally equicontinuous, then for every sequence, there will exist a subsequence which converges compactly on Ω to a continuous function f. By Theorem 0.2, the limit function will then be holomorphic, and hence $\mathcal{F}$ would be a normal family.

Hence it is enough to show that the family $\mathcal{F}$ is locally equicontinuous. To do this, we use the Cauchy integral formula. Fix an $a \in \Omega$ and $\varepsilon > 0$. We need to choose a delta that works. Let $r > 0$ be such that $\overline{D_{2r}(a)} \subset \Omega$, and let M_r such that

$$|f(\zeta)| \leq M_r,$$

for all $\zeta \in \overline{D_{2r}(a)}$ and all $f \in \mathcal{F}$. By the Cauchy estimates (see Corollary 3 from Lecture 8), we have that for any $\zeta \in D_r(a)$,

$$|f'(\zeta)| \leq \frac{2M_r}{r}.$$

Then by the fundamental theorem for complex integrals, for any $z, w \in D_r(a)$,

$$|f(z) - f(w)| = \Big| \int_{l_{w,z}} f'(\zeta)\, d\zeta \Big| \leq \frac{2M_r}{r} |z - w|.$$

Given an $\varepsilon > 0$, let us pick $\delta < 2M_r \varepsilon / r$. Then whenever $|z - w| < \delta$, we have $|f(z) - f(w)| < \varepsilon$. This proves local equicontinuity.

Conversely, suppose $\mathcal{F}$ is a normal family, but not locally uniformly bounded. Then there exists a compact set $K \subset \Omega$, and a sequence of functions $f_n \in \mathcal{F}$ such that

$$\sup_{z \in K} |f_n(z)| \geq n.$$

Since the family is normal, there exists a subsequence f_{n_k} which converges uniformly on K. But then $\sup_{z \in K} |f_{n_k}|$ would be a bounded sequence which is a contradiction. $\square$

We also need the following theorem due to Hurwitz on the limit of injective holomorphic functions.

Theorem 0.5 (Hurwitz). *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic, injective functions on an open connected subset, which converge uniformly on compact subsets to $F : \Omega \to \mathbb{C}$. Then either F is injective, or is a constant.*

Proof. We argue by contradiction. So suppose F is non constant and not injective. Then for some $w \in \mathbb{C}$, there exists $a, b \in \Omega$ such that $F(a) = F(b) = w$. Let $f_n(a) = w_n$, then $w_n \to w$. Choose an $r > 0$ small enough so that there does not exist any $z \in \overline{D_r(b)}$ such that $F(z) = w$. This is possible by the principle of analytic continuation since we are assuming that F is non-constant. In particular $a \notin \overline{D_r(b)}$. Since f_n is injective for any n, there exists no solution to

$$f_n(z) = w_n$$

in the closure of the disc $D_r(b)$, and so by the argument principle applied to $f_n(z) - w_n$, we see that

$$\frac{1}{2\pi i}\int_{|\zeta - b| = r} \frac{f_n'(\zeta)}{f_n(\zeta) - w_n}\, d\zeta = 0.$$

But since $f_n \to F$ uniformly on compact sets, in particular, on the compact set $\overline{D_r(a)}$ we have $f_n'(\zeta) \to F'(\zeta)$ and $f_n(\zeta) - w_n \to F(\zeta) - w$ uniformly. Hence the integral also converges uniformly, and from this we conclude that

$$\frac{1}{2\pi i}\int_{|\zeta - b| = r} \frac{F'(\zeta)}{F(\zeta) - w}\, d\zeta = 0.$$

This integral calculates the number of zeroes of $F(\zeta) - w = 0$ in $D_r(b)$ which we know is at least one (counting multiplicity) since $F(b) = w$. This is a contradiction, and hence if F is non-constant, it has to be injective. □

Proof of Riemann mapping theorem

For a fixed $z_0 \in \Omega$, we define a family $\mathcal{F}$ of holomorphic function functions by

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ holomorphic and injective, } f(z_0) = 0\}.$$

The required biholomorphic map will be obtained by maximizing the modulus of the derivative at z_0, amongst all functions in this family. We first show that this family is non-empty.

Lemma 0.1. *There is an injective holomorphic function $f : \Omega \to \mathbb{D}$ such $f(z_0) = 0$. That is, $\mathcal{F} \neq \phi$.*

Proof. Since $\Omega \neq \mathbb{C}$, there is an $a \in \mathbb{C} \setminus \Omega$. Then $z - a$ is never zero on Ω. Since Ω is simply connected, we can choose a holomorphic branch of $\log(z - a)$, or in other words, there is a holomorphic function $l : \Omega \to \mathbb{C}$ such that

$$e^{l(z)} = z - a$$

for all $z \in \Omega$. Clearly $l(z)$ is injective. Moreover, if $z_1, z_2 \in \Omega$ and $z_1 \neq z_2$ then $l(z_2) - l(z_1) \notin 2\pi i\mathbb{Z}$, i.e their difference cannot be an integral multiple of $2\pi i$. In particular, $l(z) \neq l(z_0) + 2\pi i$. We in fact claim that $|f(z) - (f(z_0) + 2\pi i)|$ is bounded strictly away from zero. That is,

Claim. There exists an $\varepsilon > 0$ such that $|l(z) - (l(z_0) + 2\pi i)| > \varepsilon$ for all $z \in \Omega$.

To see this, assume the claim is false. Then there is a sequence $\{z_n\} \in \Omega$ such that $l(z_n) \to l(z_0) + 2\pi i$. But then exponentiating, since the exponential function is continuous, we see that $z_n \to z_0$. But then, since $l(z)$ is continuous, this implies that $l(z_n) \to l(z_0)$ contradicting the assumption that $l(z_n) \to l(z_0) + 2\pi i$. This proves the claim.

Now consider the function

$$\tilde{f}(z) = \frac{1}{l(z) - l(z_0) - 2\pi i}.$$

By the claim, this is a bounded, injective and holomorphic function on Ω, and hence $\tilde{f} : \Omega \to D_R(0)$, where R can be taken to be $R = 1/\varepsilon$ where ε is from the claim above. Suppose $\tilde{f}(z_0) = a$, then

$$f(z) = \frac{\tilde{f}(z) - a}{R + |a|}$$

is the required function. $\square$

Next, let

$$\lambda = \sup_{f \in \mathcal{F}} |f'(z_0)|.$$

We claim that $\lambda > 0$. To see this, consider the $f \in \mathcal{F}$ constructed above. Since $f(z)$ is injective, by Corollary 0.3 from Lecture 19, $|f'(z_0)| > 0$ and hence $\lambda > 0$.

Lemma 0.2. *There is a function $F \in \mathcal{F}$ such that $|F'(z_0)| = \lambda$. In particular, λ is also finite.*

Proof. Let $f_n \in \mathcal{F}$ be a sequence of functions that maximize $|f'(z_0)|$; that is

$$\lim_{n\to\infty} |f_n'(z_0)| = \lambda.$$

Since $|f_n(z)| < 1$, by Montel's theorem there is a subsequence that converges uniformly on compact sets to a holomorphic function F satisfying $|F(z)| \leq 1$, and $F(z_0) = 0$. Moreover since the derivatives also converge, we must have $|F'(z_0)| = \lambda \neq 0$. In particular, F cannot be a constant. Then by the maximum modulus principle, $|F(z)| < 1$ for all $z \in \Omega$, since otherwise, there will be a point $z \in \Omega$ with $|F(z)| = 1$, and hence will be an interior maximum point for $|F|$. Finally to show that $F \in \mathcal{F}$, we need to show that F is injective. But this follows from Hurwitz's theorem since F is non-constant and all $f_n \in \mathcal{F}$ are injective. $\square$

Proof of Riemann mapping. Since $F'(z_0) \neq 0$, by composing with a suitable rotation, we can assume that $F'(z_0)$ is real and positive. We claim that this F is the required bi-holomorphism. We already know that $F : \Omega \to \mathbb{D}$, $F(z_0) = 0$ and F is injective. To complete the proof, we need to show that F is surjective. If not, then there exists a $\alpha \in \mathbb{D}$ such that $F(z) = \alpha$ has no solution in Ω. We then exhibit a $G \in \mathcal{F}$ with $|G'(z_0)| > |F'(z_0)|$ contradicting the choice of F. To do this, consider $\psi_\alpha : \mathbb{D} \to \mathbb{D}$ defined by

$$\psi_\alpha(w) = \frac{\alpha - z}{1 - \bar{\alpha} z},$$

and let

$$g(z) = \sqrt{\psi_\alpha \circ F(z)}.$$

Since $\psi_\alpha(z) = 0$ if and only if $z = \alpha$, we see that $\psi_\alpha \circ F$ is always zero free, and so a holomorphic branch of $\log \psi_\alpha \circ F$ can be defined since Ω is simply connected. We can then choose a holomorphic branch for $g(z)$ by letting

$$g(z) = e^{\frac{1}{2}\log \psi_\alpha \circ F(z)}.$$

Note that $g(z_0) = \sqrt{\alpha}$. To construct a member of the family, we need to bring this back to the origin, and hence we define

$$G(z) = \psi_{\sqrt{\alpha}} \circ g(z).$$

Then $G(z_0) = 0$. Moreover, $G(z)$ is also injective since $\psi_{\sqrt{\alpha}}$ and $g(z)$ are injective, and so $G \in \mathcal{F}$.

Claim. $|G'(z_0)| > |F'(z_0)|$.

To see this, observe that

$$F(z) = \psi_\alpha^{-1} \circ s \circ \psi_{\sqrt{\alpha}}^{-1} \circ G(z) = \Phi \circ G(z),$$

where $s(w) = w^2$ is the squaring function and $\Phi = \psi_\alpha^{-1} \circ s \circ \psi_{\sqrt{\alpha}}^{-1} : \mathbb{D} \to \mathbb{D}$. We compute that $\Phi(0) = \psi_\alpha^{-1}(s(\psi_{\sqrt{\alpha}}^{-1}(0)) = \psi_\alpha^{-1}(s(\sqrt{\alpha})) = \psi_\alpha^{-1}(\alpha) = 0$. By Schwarz lemma, $|\Phi(z)| \leq |z|$, and so

$$|\Phi'(0)| \leq 1.$$

We claim that $|\Phi'(0)| < 1$. Suppose, $|\Phi'(0)| = 1$, then by the second part of Schwarz lemma, $\Phi(z) = az$ for some unit complex number a. In particular, $\Phi(z)$ is injective. But Φ cannot be injective since $s(z)$ is a $2-1$ function ie. sends two points to a single point, and ψ_α and $\psi_{\sqrt{\alpha}}$ are injective. This shows that $|\Phi'(0)| < 1$. But then $F'(z_0) = \Phi'(G(z_0)) \cdot G'(z_0) = \Phi'(0) \cdot G'(z_0)$, and hence $|F'(z_0)| < |G'(z_0)|$ which proves the claim, and completes the proof of the theorem.

□

Green's functions and a generalization of the Riemann mapping theorem

For the purposes of this section, we assume that Ω is bounded and has a sufficiently nice boundary $\partial\Omega$ (for concreteness, assume that $\partial\Omega$ is a union of piecewise regular curves). Often one is interested in solving the Dirichlet problem on Ω: Namely, given any smooth (real valued) function f on Ω and a continuous function u_0 on $\partial\Omega$, to find a smooth function u such that

$$\begin{cases} \Delta u = f \\ u\Big|_{\partial\Omega} = u_0. \end{cases}$$

This problem arises in many (seemingly) different areas in mathematics and physics. For instance, one interpretation of solutions of the above problem, is that u represents the voltage distribution on a conductor Ω with charge distribution given by $f(z)$ and the boundary being held at voltage u_0.

A *Green's function* for Ω based at $z_0 \in \Omega$ is a function $G_{z_0} : \Omega \to \mathbb{R}$ such that

(1) $G_{z_0}(z)$ is a harmonic function on $\Omega \setminus \{z_0\}$.

(2) $G_{z_0}\Big|_{\partial\Omega} \equiv 0$.
(3) $G_{z_0}(z) - \frac{1}{2\pi}\log|z-z_0|$ is bounded in a neighbourhood of z_0.

Note that $\log|z-z_0|$ is a harmonic function in a punctured neighbourhood of z_0. By an analog of the Riemann removable singularity theorem for harmonic functions, condition (3) is equivalent to $G_{z_0}(z) - \frac{1}{2\pi}\log|z-z_0|$ extending as a harmonic function in a disc $D_r(z_0)$. It is often useful to think of the Green's function as a function of two variables $G : \Omega \times \Omega \to \mathbb{R}$, where we set

$$G(z,w) := G_w(z).$$

It turns out that the function G is actually symmetric in the two variables. The reason why Green's function is important is that it is a *fundamental solution* to the Dirichlet problem on Ω. That is, with f and u_0 as above, a solution to the Dirichlet poblem is given by

$$u(z) = \int_\Omega G(z,w)f(w)\,dwd\bar{w} - \int_{\partial\Omega} \frac{dG}{d\nu}(z,w)g(w)\,d\sigma,$$

where $dG/d\nu$ is the outward normal derivative of G on the boundary (where differentiation is with respect to the variable w), $dwd\bar{w}$ is the usual Lebesgue (or Euclidean) measure on Ω, and $d\sigma$ is the surface measure on $\partial\Omega$.

Conversely, if one can Dirichlet problems with continuous data, then one can construct a Green's function. The idea is to simply find a harmonic function $H_{z_0}(z)$ with boundary data $u_0 = -\log|z-z_0|$. The Green's function $G_{z_0}(z)$, will then be

$$G_{z_0}(z) = \frac{1}{2\pi}\log|z-z_0| + \frac{1}{2\pi}H_{z_0}(z).$$

Now suppose that Ω is simply connected. Fix a $z_0 \in \Omega$, and as above, let $H_{z_0}(z) := 2\pi G_{z_0}(z) - \log|z-z_0|$, which is harmonic by property (3) above. Since Ω is simply connected, $H_{z_0}(z)$ has a *harmonic conjugate*, that is a function $H^*_{z_0}(z) : \Omega \to \mathbb{R}$ which is harmonic, and such that $f(z) = H_{z_0(}z) + iH^*_{z_0}(z)$ is a holomorphic function on Ω (The proof is essentially the same as that for a disc, and this was an exam problem on the midterm). We let $F(z) = (z-z_0)e^{f(z)}$. For any $z \in \partial\Omega$, $G_{z_0}(z) = 0$, $H_{z_0}(z) = -\log|z-z_0|$, and so $|F(z)| = 1$. By the maximum principle, $|F(z)| < 1$ for all $z \in \Omega$. Hence F is a map from Ω to the unit disc $\mathbb{D}$. Furthermore, $F(z)$ has only one zero in Ω, and that too a simple one, namely at $z = z_0$. Next, let $w_0 \in \mathbb{D}$ such that $|w_0| < 1-\varepsilon < 1$, and let γ be the curve given by $|F(z)| = 1-\varepsilon$ and $\Gamma = F \circ \gamma$. Then Γ is of course the circle $|w| = 1-\varepsilon$ (but possibly traversed multiple times). Since $F(z) = 0$ has only one solution in Ω, by the argument principle (rather the index version of it), we see that

$$\int_\gamma \frac{F'(z)}{F(z)-w_0}\,dz = n(\Gamma, w_0) = n(\Gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)} = 1.$$

This shows that $F(z) = w_0$ has a unique solution for any $w_0 \in \mathbb{D}$, and hence shows both surjectivity and injectivity of $F(z)$. Finally, we can compose F with a rotation to ensure that $F'(z_0)$ is real and positive.

The reader is encouraged to read a detailed and complete account of a proof of the Riemann mapping theorem along the lines of Riemann's original "proof" from `https://link.springer.com/article/10.1186/s40627-016-0009-7`.

We end with a vastly more general Riemann mapping theorem. Recall that a domain Ω is said to be n-connected, if $\hat{\mathbb{C}} \setminus \Omega$ has n connected components. For instance, Ω is 1-connected if and only if it is simply connected.

Theorem 0.6. *Let Ω be an n-connected domain in $\mathbb{C}$ such that no component of $\hat{\mathbb{C}} \setminus \Omega$ consists of a single point. Then there exists a biholomorphism $F : \Omega \to \mathcal{D}$, where*

1. *$\mathcal{D} = \mathbb{D}$ if $n = 1$,*
2. *$\mathcal{D}$ is an annulus $A_{r,R}(0) = \{z \in \mathbb{C} \mid r < |z| < R\}$ if $n = 2$,*
3. *$\mathcal{D}$ is $A_{r,R}(0) \setminus \cup_{i=1}^{n-2} Supp(\gamma_i)$ if $n > 2$, where γ_i are concentric arcs lying on circles $|\zeta| = r_i$, with $r < r_i < R$.*

A historical note. One of the first pushes towards making Dirichlet's problem and harmonic functions a part of mainstream mathematics arose out of Riemann's (faulty) proof of his theorem on conformal mappings into the disc. In fact the first systematic and rigorous study of the Dirichlet problem was to fix the error in Riemann's original proof. By the turn of the twentieth century, the vastly more general uniformization theorem had also been proved using similar methods, and elliptic partial differential equations and calculus of variations (of which the above problem is the simplest example) had become a part of mainstream mathematics. So much so that, they were the subject of two of Hilbert's problems in his 1900 address to the Congress of mathematicians. Finally, this whole circle of ideas of using solutions of partial differential equations to say something about the topology continues to be a fruitful area of mathematical research. Some of the spectacular successes include Hodge theory (characterizing cohomology groups via harmonic forms) and Donaldson theory (characterising smooth structures on four manifolds via solving Yang-Mills equations, which are a non-linear generalization of Dirichlet's problem).

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