LECTURE-25

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In this lecture, we study two important functions, namely the Gamma function and the Zeta function. Each function is initially defined in a certain region in the complex plain; the Gamma function by an integral and the Zeta function by an infinite series. Both the functions are then extended to obtain meromorphic functions on the entire complex plain. The key techinical tool is the principal of analytic continuation.

Revisiting the principle of analytic continuation

Recall that the principal of analytic continuation says that if two functions agree on some open set, then they must agree on the entire connected component containing the open set. Given a holomorphic function $f : \Omega \to \mathbb{C}$ it is natural to ask for the biggest possible open set Ω' containing Ω on which f has a holomorphic extension. It is in fact much more natural to ask for *meromorphic* extensions. So we pose the following question.

Question 1. *Given $f : \Omega \to \mathbb{C}$ holomorphic, what is the biggest Ω' containing Ω such that there exists a meromorphic function $F : \Omega' \to \hat{\mathbb{C}}$ such that*

$$F_{|\Omega} = f.$$

Is the extension unique.

The uniqueness part is answered in the affirmative by the following extension of the principle of analytic continuation to meromorphic functions.

Lemma 0.1. *Let Ω be a connected open set, and $f, g : \Omega \to \hat{\mathbb{C}}$ be meromorphic functions with poles at isolated sets S_f and S_g respectively. Let $S = S_f \cup S_g$. Suppose there is a sequence of pairwise distinct points $z_n \in \Omega \setminus S$ such that $z_n \to z_0 \in \Omega$ and $f(z_n) = g(z_n)$ for all n, then $S_f = S_g = S$, the poles of f and g are of the same order and the Laurent series expansions match up, and $f = g$ as meromorphic functions.*

Proof. Since $f, g \in \mathcal{O}(\Omega \setminus S)$, by the usual principle of analytic continuation,

$$f\Big|_{\Omega\setminus S} = g\Big|_{\Omega\setminus S}.$$

To complete the proof, we need to show that $S_f = S_g = S$. Let $p \in S_f$. Then there is an $\varepsilon > 0$ such that $D_\varepsilon(p)$ does not contain any point of S apart from p. That is, $D_\varepsilon(p) \setminus \{p\} \subset \Omega \setminus S$, and hence $f(z) = g(z)$ for all

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$z \in D_\varepsilon(p) \setminus \{p\}$. Since p is a pole of f, $f(z) \to \infty$ as $z \to p$, and hence $g(z) \to \infty$ as $z \to p$. This shows that $S_f \subset S_g$. By symmetry we get the reverse inclusion and this proves that $S_f = S_g$. Since f and g are equal in the complement, it is also clear that the poles will be of the same order, and the Laurent series expansions match up. Hence $f = g$ as meromorphic functions. $\square$

Example 0.1. *Consider the function defined by the power series*

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

This defines a holomorphic function on the unit disc $\mathbb{D}$. Moreover, by the summation formula for geometric series, the function is precisely $f(z) = (1-z)^{-1}$. On the other hand, the function $F(z) = (1-z)^{-1}$ is meromorphic on the entire complex plane with a single pole of order one at $z = 1$. So $F(z)$ defines a meromorphic extension of $f(z)$ to $\mathbb{C}$ and an analytic continuation to $\mathbb{C} \setminus \{1\}$. Note that $F(-1) = 1/2$, since $F(z)$ is an analytic continuation of $f(z)$, naively (and rather thoughtlessly) one might be tempted to plug in $z = -1$ in the power series and write

$$1 - 1 + 1 - 1 + \cdots \text{ “} = \text{” } \frac{1}{2}. \tag{0.1}$$

Of course as stated the above equality is meaningless since the series on the left is a divergent series. The correct way to make sense out of this is by analytic continuation. There are other ways to make sense out of the series, for instance by using Cesaro summability. G.H Hardy wrote an entire book on divergent series, and this was quite a hot topic for research in Britain in the early 20th century.

Analytic continuation: The model case

In this section we answer the above question completely in the following model case, for the (truncated) *Mellin transform*. Recall that a function $\varphi : [-1, 1] \to \mathbb{R}$ is called smooth, if derivatives of all orders exist on $(-1, 1)$, and are continuous on $[-1, 1]$.

Theorem 0.1. *Let $\varphi(t)$ be a smooth function on the unit interval $[-1, 1]$, and consider the function*

$$f(z) = \int_0^1 x^z \varphi(x)\, dx,$$

where we use the principal branch, namely $x^z = e^{z \ln x}$. Then

1. *$f(z)$ defines a holomorphic function on $Re(z) > -1$.*
2. *$f(z)$ admits a unique extension as a meromorphic function on $\mathbb{C}$ with at most simple poles at the negative integers with*

$$\mathrm{Res}_{z=-n} f(z) = \frac{\varphi^{(n-1)}(0)}{(n-1)!},$$

where $\varphi^{(k)}(0)$ as usual denotes the k^{th} derivative.

Note that the usual Mellin transform involves the integral over all of $(0,\infty)$, and hence we call the above a truncated Mellin transform (possibly non standard terminology).

Proof. To prove this, we first show that the integral is absolutely convergent for $\mathrm{Re}(z) > -1$. We only need to worry about convergence near $x = 0$. It is easy to see that $|x^z| = |e^{z\ln x}| = x^{\mathrm{Re}(z)}$, and so if $|\varphi(x)| < M$ on $[0,1]$, then $|x^z\varphi(x)| < Mx^{\mathrm{Re}(z)}$, which, by the p-test, is integrable near $x = 0$ if $\mathrm{Re}(z) > -1$. So by the comparison theorem, $f(z)$ is well defined for $\mathrm{Re}(z) > -1$. To show that it is holomorphic, we look at the difference quotient. Note that x^z is holomorphic for all $x \in (0,1)$ with derivative

$$\frac{dx^z}{dz} = x^z \ln x.$$

We claim

$$f'(z) = \int_0^1 x^z\varphi(x)\ln x\,dx.$$

To prove this, it is enough to show the following.

Claim. For all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\Big|\frac{f(z+h)-f(z)}{h} - \int_0^1 x^z\varphi(x)\ln x\,dx\Big| < \varepsilon,$$

whenever $|h| < \delta$.

Note that

$$\begin{aligned}\Big|\frac{f(z+h)-f(z)}{h} - \int_0^1 x^z\varphi(x)\ln x\,dx\Big| &= \Big|\int_0^1 (\frac{x^{z+h}-x^z}{h} - x^z\ln x)\varphi(x)\,dx\Big| \\ &\le \int_0^1 |x^z\varphi(x)|\Big|\frac{e^{h\ln x}-1}{h} - \ln x\Big|\,dx. \qquad (0.2)\end{aligned}$$

By the power series expansion of e^z, we have that

$$\frac{e^{h\ln x}-1}{h} - \ln x = h(\ln x)^2\sum_{n=0}^{\infty}\frac{h^n(\ln x)^n}{(n+2)!}.$$

Now the infinite series on the right is convergent. In fact we have that

$$\Big|\sum_{n=0}^{\infty}\frac{h^n(\ln x)^n}{(n+2)!}\Big| \le \sum_{n=0}^{\infty}\Big|\frac{h^n(\ln x)^n}{n!}\Big| < e^{|h||\ln x|} = x^{-|h|},$$

where we used the fact that $|\ln x| = \ln(1/x)$ since $x \in (0,1)$. From the power series expansion above, we then have the estimate

$$\Big|\frac{e^{h\ln x}-1}{h} - \ln x\Big| \le h(\ln x)^2 x^{-|h|} \le C_\eta h x^{-|h|-\eta},$$

for any $\eta > 0$ and $|h|$ small. Here C_η is a constant that possibly depends on η but is independent of h. This estimate holds because $\lim_{x\to 0} x^\eta (\log x)^2$ for any $\eta > 0$.

Now suppose $M = \sup_{x\in[0,1]} |\varphi(x)|$, then going back to the integral estimate in (0.2) we see that if $|h| < Re(z)$, then

$$\int_0^1 |x^z\varphi(x)|\Big|\frac{e^{h\ln x}-1}{h} - \ln x\Big|\, dx \leq hC_\eta M\int_0^1 x^{Re(z)-|h|-\eta}\, dx.$$

We choose $\eta > 0$ small enough $Re(z) - 2\eta > -1$. Suppose $|h| < \eta$, then

$$\int_0^1 x^{Re(z)-|h|-\eta}\, dx \leq \int_0^1 x^{Re(z)-2\eta}\, dx := A_\eta.$$

Note that A_η of course depends on z, but z is fixed throughout this argument, and hence we hide the dependence of A_η on z. So putting all of this together with (0.2)

$$\Big|\frac{f(z+h)-f(z)}{h} - \int_0^1 x^z\varphi(x)\ln x\, dx\Big| \leq MA_\eta C_\eta|h| < \varepsilon,$$

if $|h| < \varepsilon/MC_\eta A_\eta$. So the claim is proved by choosing

$$\delta = \min\Big(\frac{\varepsilon}{MC_\eta A_\eta}, \eta\Big).$$

Next, we show that a meromorphic extension exists on all of $\mathbb{C}$. For any given integer $N > 0$, we can write the Taylor expansion of φ around $x = 0$ as

$$\varphi(x) = \sum_{j=0}^{N-1} \frac{\varphi^j(0)}{j!}x^j + E_N(x),$$

where $E_N(x)$ is a smooth function on $[-1,1]$ such that $|E_N(x)| \leq C|x|^N$ for some constant $C > 0$. So for $\mathrm{Re}(z) > -1$,

$$\begin{aligned} f(z) &= \sum_{k=0}^{N-1}\int_0^1 \frac{\varphi^k(0)}{k!}x^{k+z} + \int_0^1 E_N(x)x^z\, dx \\ &= \sum_{k=0}^{N-1} \frac{\varphi^k(0)}{k!}\cdot\frac{1}{z+k+1} + \int_0^1 E_N(x)x^z\, dx \end{aligned}$$

Since $|E_N(x)| \leq C|x|^N$, $\int_0^1 E_N(x)x^z\, dx$ is convergent for $\mathrm{Re}(z+N) > -1$, and hence defines a holomorphic function on $\mathrm{Re}(z) > -(N+1)$ by the first part. On the other hand, putting $k+1 = j$, the first term on the right defines a meromorphic function on all of $\mathbb{C}$ with simple poles at $z = -j$, $j = 1, 2, \cdots, N$ with residue $\varphi^{j-1}(0)/(j-1)!$. So the right hand side defines a meromorphic function $f_N(z)$ on $\mathrm{Re}(z) > -(N+1)$, which restricts to $f(z)$ on $\mathrm{Re}(z) > -1$. By uniqueness of meromorphic extensions, for $M > N$, the restriction of f_N and f_M to $\mathrm{Re}(z) > -(N+1)$ are equal, and hence letting

$N \to \infty$, f_N converges to a meromorphic function on all of $\mathbb{C}$ with simple poles at $z = -n$ with residue $\varphi^{(n-1)}(0)/(n-1)!$. $\square$

Remark 0.1. *Note that if $\varphi^{(n-1)}(0) = 0$ for some n, then the residue of $f(z)$ at $z = -n$ would be zero. And since $z = -n$ can at most be a simple pole, this would imply that $z = -n$ is in fact not a pole at all, but is a removable singularity.*

The Gamma function

The Gamma function $\Gamma(s)$ is defined on $\mathrm{Re}(s) > 0$ as the Mellin transform of e^{-x}. That is,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx.$$

It is easy to see (exercise!) that the integral is convergent on $\mathrm{Re}(s) > 0$ and hence is well defined and finite on this region. Note also that $\Gamma(1) = 1$.

Theorem 0.2. *With $\Gamma(s)$ defined as above for $\mathrm{Re}(s) > 0$, we have the following.*

(1) *There exists a meromorphic extension of $\Gamma(s)$ on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \cdots$ with residue*

$$\mathrm{Res}_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}.$$

(2) *(Functional equation) For $s \neq -n$, $n = 0, 1, 2, \cdots$,*

$$\Gamma(s+1) = s\Gamma(s),$$

and hence for integers n, $\Gamma(n+1) = n!$.

Proof. For $\mathrm{Re}(s) > 0$, we can write

$$\Gamma(s) = \int_0^1 e^{-x} x^{s-1}\, dx + \int_1^\infty e^{-x} x^{s-1}\, dx.$$

The second integral is convergent for all s, and hence defines an entire function by similar arguments as in the proof of the first part of Theorem 0.1. The first integral, by Theorem 0.1 (applied to $s - 1 = z$), can be extended to a meromorphic function with simple poles at $s = -n$, for $n = 0, 1, 2, \cdots$. The residue also comes from the first term. Applying theorem 0.1 with $\varphi = e^{-x}$, and $s - 1 = z$, we see that the residue at $s = -n$ (or $z = -(n+1)$) is given by

$$\mathrm{Res}_{s=-n}\Gamma(s) = \frac{1}{n!} \cdot \frac{d^n}{dx^n}\Big|_{x=0} e^{-x} = \frac{(-1)^n}{n!}.$$

We first prove part (2) when $\mathrm{Re}(s) > 0$. In this range we can use the integral formula,

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty e^{-x}x^s\,dx \\ &= -\int_0^\infty x^{s-1}\,de^{-x} \\ &= e^{-x}x^s\Big|_{x=0}^{x=\infty} + \int_0^\infty e^{-x}\,dx^s \\ &= s\int_0^\infty e^{-x}x^{s-1}\,dx = s\Gamma(s).\end{aligned}$$

To prove the equality over the entire complex plane, consider $F(s) = \Gamma(s+1) - s\Gamma(s)$.

Claim. $F(s)$ extends to an entire function.

Assuming this, since $F(s) \equiv 0$ on $Re(s) > 0$, by the principle of analytic continuation, $F(s)$ is identically zero, and we are done.

Proof of the Claim. Clearly $F(s)$ is holomorphic everywhere except possibly at the negative integers and $s = 0$. Moreover, F can only have simple poles at these points. At $s = 0$, $\Gamma(s+1)$ is holomorphic, and so is $s\Gamma(s)$ since Γ has a simple pole at $s = 0$. So F can have pole at only negative integers. To rule this out, let us calculate the residue. For $n \in \mathbb{N}$,

$$\mathrm{Res}_{s=-n}\Gamma(s+1) = \mathrm{Res}_{z=-(n-1)}\Gamma(z) = \frac{(-1)^{n-1}}{(n-1)!}.$$

On the other hand,

$$\mathrm{Res}_{s=-n}s\Gamma(s) = \lim_{s\to -n} s(s+n)\Gamma(s) = -n\mathrm{Res}_{s=-n}\Gamma(s) = -\frac{(-1)^n}{(n-1)!},$$

and hence $\mathrm{Res}_{s=-n}F(s) = 0$. Since $s = -n$ is a simple pole, this implies that $s = -n$ is a removable singularity. $\square$

Theorem 0.3 (Euler reflection formula)**.** *The Gamma function satisfies the identity*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Proof. By analytic continuation, it enough to prove the identity for $s \in \mathbb{R} \cap (0,1)$. Recall that in Example-2 in Lecture-21, we proved the following identity: For $0 < a < 1$,

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv = \frac{\pi}{\sin \pi s}.$$

We now rewrite

$$\Gamma(1-s) = \int_0^\infty e^{-x}x^{-s}\,dx = t\int_0^\infty e^{-vt}(vt)^{-s}\,dv,$$

where we made the change of variables $x = vt$. Note that the above formula for $\Gamma(1-s)$ is valid for all $t \geq 0$. Now we compute

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t} t^{s-1} \Gamma(1-s)\, dt \\
&= \int_0^\infty e^{-t} t^{s-1} \left(t^{1-s} \int_0^\infty e^{-vt} v^{-s}\, dv \right) dt \\
&= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s}\, dt\, dv \\
&= \int_0^\infty \frac{v^{-s}}{1+v}\, dv \\
&= \frac{\pi}{\sin \pi s}.
\end{aligned}$$

□

The Zeta function

For $Re(s) > 1$, we define the *zeta function* by the infinite series

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s},$$

where as before, $n^s = e^{s \ln(n)}$. This time by comparison test for series, since $|n^s| = n^{\mathrm{Re}(s)}$, this is a convergent series for $\mathrm{Re}(s) > 1$, and by the Weierstrass M-test defines a holomorphic function on $Re(s) > 1$.

Theorem 0.4. *The zeta function above satisfies the following properties*

(1) *For $Re(s) > 1$, we have the identity,*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t - 1} t^{s-1}\, dt.$$

That is, the zeta function (upto a factor of $\Gamma(s)$) is the Mellin transform of $(e^t - 1)^{-1}$.

(2) *$\zeta(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $z = 1$ and holomorphic on $\mathbb{C} \setminus \{1\}$. Moreover, we have*

$$\mathrm{Res}_{z=1} \zeta(s) = 1.$$

(3) *$\zeta(s) = 0$ whenever $s = -2n$ for some $n \in \mathbb{N}$. These are the so called* trivial zeroes *of the zeta function.*

The Riemann hypothesis conjectures that in fact *all* other zeroes (the so called non-trivial zeroes) lie on the line $Re(s) = 1/2$.

Proof. For the first identity, we observe that

$$\frac{\Gamma(s)}{n^s} = \frac{1}{n} \int_0^\infty e^{-x} (x/n)^{s-1}\, dx = \int_0^\infty e^{-nt} t^{s-1}\, dt.$$

where we changed variables $x = nt$ in the second equality. Summing up we obtain

$$\zeta(s)\Gamma(s) = \int_0^\infty \Big(\sum_{n=1}^\infty e^{-nt}\Big) t^{s-1}\, dt = \int_0^\infty \Big(\frac{1}{1-e^{-t}} - 1\Big) t^{s-1}\, dt = \int_0^\infty \frac{1}{e^t - 1} t^{s-1}\, dt.$$

Now let $f(s) = \int_0^\infty \frac{1}{e^t-1} t^{s-1}\, dt$. We can write

$$f(s) = \int_0^\infty \varphi(t) t^{s-2},$$

where $\varphi(t) = t/(e^t - 1)$. From the Tayor expansion of e^t we can see that $\varphi(t)$ is smooth on $[-1, 1]$, and so by Theorem 0.1 with $z = s - 2$, $f(s)$ is holomorphic on $\mathrm{Re}(s-2) > -1$ or equivalently on $\mathrm{Re}(s) > 1$, and admits a meromorphic extension with simple poles at $\mathrm{Re}(s) = 1, 0, -1, -2, \cdots$. But $\Gamma(s)$ itself has simple poles at $s = 0, -1, -2, \cdots$, and hence the meromorphic extension $\zeta(s) = f(s)/\Gamma(s)$ will have a pole only at $s = 1$. When $s = 1$, $s - 2 = -1$, and so from Theorem 0.1, $\mathrm{Res}_{s=1} f(s) = \varphi(0)$. But

$$\frac{t}{e^t - 1} = \frac{t}{t + t^2/2 + \cdots} = \frac{1}{1 + t/2 + \cdots},$$

and so $\varphi(0) = 1$, and hence $\mathrm{Res}_{s=1} f(s) = 1$. But since $\Gamma(1) = 1$, we then have that $\mathrm{Res}_{s=1}\zeta(s) = 1$. It follows from Problem-7 in Assignment-4 that

$$\varphi(t) = \frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n=1}^\infty (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n},$$

where B_n is the n^{th} Bernoulli number. In particular, $\varphi^{(2n+1)}(0) = 0$, for all $n = 1, 2, \cdots$, and hence

$$\mathrm{Res}_{s=-2n} f(s) = \mathrm{Res}_{z=-2n-2} f(z) = \frac{\varphi^{(2n+1)}}{(2n+1)!} = 0.$$

So $f(s)$ has a removable singularity at $s = -2n$. But since $\Gamma(s)$ has a simple pole at $s = -2n$, it follows that $\zeta(-2n) = 0$ for all $n \in \mathbb{N}$. □

Example 0.2. *Let us calculate $\zeta(0)$. First, note that if two functions $f(z)$ and $g(z)$ have a simple pole at $z = 0$, then $h(z) = f(z)/g(z)$ has a removable singularity at $z = 0$. Moreover the extension, which we also denote by $h(z)$, satisfies $h(0) = Res_{z=0} f(z)/Res_{z=0} g(z)$. We apply this to the meromorphic extension of*

$$f(s) = \int_0^\infty \frac{1}{e^t - 1} t^{s-1}\, dt,$$

and $g(s) = \Gamma(s)$. For $\mathrm{Re}(s) > 1$, we re-write

$$f(s) = \int_0^1 \frac{t}{e^t - 1} t^{s-2}\, dt + \int_1^\infty \frac{1}{e^t - 1} t^{s-1}\, dt.$$

The second part, by the argument above is an entire function. So the residue comes from the meromorphic extension of the first integral. We apply Theorem 0.1 with $\varphi(t) = t/(e^t - 1)$ and $z = s - 2$. To find the residue at $s = 0$

we apply the second part of Theorem 0.1 (with $n = 2$, since $z = -2$ is the same as $s = 0$)

$$\mathrm{Res}_{s=0} f(s) = \frac{\varphi'(0)}{1!}.$$

But we can write down the Taylor expansion of

$$\varphi(t) = \frac{t}{e^t - 1} = \frac{t}{t + t^2/2 + \cdots} = \frac{1}{1 + t/2 + \cdots} = 1 - \frac{t}{2} + \textit{higher order terms},$$

and so $\varphi'(0) = -1/2$. On the other hand the residue of the Gamma function is given by Theorem 0.2, and we see that $\mathrm{Res}_{s=0}\Gamma(s) = 1$, and putting everything together, we obtain that

$$\zeta(0) = -\frac{1}{2}.$$

Example 0.3. *By working a bit harder, we can compute $\zeta(-1)$. Once again applying Theorem 0.1, this time with $n = 3$ (since $s = -1$ is $z = -3$) we have*

$$\mathrm{Res}_{s=-1} f(s) = \frac{\varphi^{(2)}(0)}{2!}.$$

Computing the next term in the Taylor expansion,

$$\varphi(t) = \frac{1}{1 + t/2 + t^2/6 + O(t^3)} = 1 - \frac{t}{2} - \frac{t^2}{6} + \frac{t^2}{4} + O(t^3) = 1 - \frac{t}{2} + \frac{t^2}{12} + O(t^3),$$

and so $\mathrm{Res}_{s=-1} f(s) = 1/12$. On the other hand, $\mathrm{Res}_{s=-1}\Gamma(s) = (-1)^1/1 = -1$, and hence

$$\zeta(-1) = -\frac{1}{12}.$$

Remark 0.2. *Recall that for $Re(s) > 1$,*

$$\zeta(s) = \sum_{n=0}^{\infty} n^{-s}.$$

We can then formally (and formally is the key word here) "plug in" $s = -1$, and write

$$1 + 2 + \cdots \text{“} = \text{”} \zeta(-1) = -\frac{1}{12}. \tag{0.3}$$

Of course this does not make any "real" sense since $1+2+3\cdots$ is a divergent series and $\zeta(-1)$ is a finite number since $\zeta(s)$ is holomorphic at $s = -1$. The equation (0.3) is found in one of Ramanujan's notebooks. Apparently Ramanujan had stumbled upon a way of summing up certain divergent series, and being unaware of analytic continuation, used the rather crude notation that seems to suggest that the sum of all natural numbers is not only a finite number but also negative!

There is a particularly misleading video posted by numpherphile, an otherwise decent youtube math channel, on this "astounding" identity - `https://www.youtube.com/watch?v=w-I6XTVZXww`, which gives a "derivation" of *the above "identity" using other misleading identities such as* (0.1). *Following the barrage of criticism that this video received, other channels made better videos. For instance this video -* `https://www.youtube.com/watch?v=jcKRGpMiVTw` *by mathlogger clarifies the identity using Cesaro summability. There is also a very beautiful video on the analytic continuation of the zeta function by 3Blue1Brown -* `https://www.youtube.com/watch?v=sDONjbwqlYw`.

We end this lecture, with the beautiful functional equation of Riemann's which we state without proof.

Theorem 0.5. *[Functional equation]*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Example 0.4. *As an application we can calculate $\zeta(2)$. Namely,*

$$\zeta(2) = 4\pi\zeta(-1)\lim_{s\to 2}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s).$$

We can compute the limit directly by using the Residue of $\Gamma(1-s)$ at $s=2$. Alternately, by Theorem 0.3

$$\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = \frac{\sin\pi s}{2\cos(\pi s/2)}\Gamma(1-s) = \frac{\pi}{2\Gamma(s)\cos\pi s/2},$$

and so

$$\zeta(2) = -\frac{\pi^2}{3\Gamma(2)(-2)} = \frac{\pi^2}{6}.$$

This gives a third derivation of the famous Basel-Euler identity:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

and is a good place to end.

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