LECTURE-3

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1. Power series

A power series centered at $z_0 \in \mathbb{C}$ is an expansion of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n, z \in \mathbb{C}$. If a_n and z are restricted to be real numbers, this is the usual power series that you are already familiar with. A priori it is only a formal expression. But for certain values of z, lying in the so called disc of convergence, this series actually converges, and the power series represents a function of z. Before we discuss this fundamental theorem of power series, let us review some basic facts about complex series, and series of complex valued functions.

1.1. Infinite series of complex numbers: A recap.

A series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n,$$

where $a_n \in \mathbb{C}$ for all n. We say that the series converges to S, and write

$$\sum_{n=0}^{\infty} a_n = S,$$

if the sequence of partial sums

$$S_N = \sum_{n=0}^{N} a_n$$

converges to S. We need the following basic fact.

Proposition 1.1. *If $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. We will show that S_N forms a Cauchy sequence if $\sum_{n=0}^{\infty} |a_n|$ converges. Then the theorem will follow from the completeness of $\mathbb{C}$. If we let

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$T_N = \sum_{n=0}^{N} |a_n|$, then $\{T_N\}$ is a Cauchy sequence by the hypothesis. Then by triangle inequality

$$|S_N - S_M| = \Big|\sum_{n=N+1}^{M} a_n\Big| \le \sum_{n=N+1}^{M} |a_n| = |T_M - T_N|.$$

Now, given $\varepsilon > 0$, there exists a $K > 0$ such that for all $N, M > K$, $|T_M - T_N| < \varepsilon$. But then $|S_M - S_N| \le \varepsilon$, and so $\{S_N\}$ is also Cauchy. □

We say that $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges. Next suppose $f_n : \Omega \to \mathbb{C}$ are complex functions, we say that $\sum_{n=0}^{\infty} f_n(z)$ converges *uniformly* if the corresponding sequence of partial sums

$$S_N(z) = \sum_{n=0}^{N} f_n(z)$$

converges uniformly.

Proposition 1.2 (Weierstrass' M-Test). *Suppose $f_n : \Omega \to \mathbb{C}$ is a sequence of complex functions, and $\{M_n\}$ is a sequence of positive real numbers such that*

- $|f_n(z)| \le M_n$ *for all n and all $z \in \Omega$.*
- $\sum_{n=0}^{\infty} M_n$ *converges.*

Then $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly.

Proof. Like before, this time we show that the sequence of partial sums $\{S_N(z)\}$ is uniformly Cauchy. But again by triangle inequality if T_N denotes the N^{th} partial sum of $\sum M_n$, then

$$|S_N(z) - S_M(z)| \le \sum_{n=N+1}^{M} |f_n(z)| \le \sum_{n=N+1}^{M} M_n = |T_M - T_N|,$$

for all $z \in \Omega$. Since the right side does no depend on z, given $\varepsilon > 0$, one can make $|S_N(z) - S_M(z)| < \varepsilon$ by choosing $N, M > K$ where K can be chosen independent of z. □

1.2. **Convergence of power series.** By convergence of the power series, we mean the following. Consider the truncations of the power series at the N^{th} term, also called the N^{th} partial sum -

$$s_N(z) = \sum_{n=0}^{N} a_n (z - z_0)^n.$$

We say that the power series converges (unifomrly) if the sequence of functions $\{s_N(z)\}$ converges (uniformly). We say that the series converges absolutely if the sequence of functions

$$\sum_{n=0}^{N} |a_n||z|^n$$

converges. It is well known, and not difficult to see, that absolute convergence implies convergence. The fundamental fact is the following.

Theorem 1.1 (Fundamental theorem of power series)**.** *There exists a* $0 \leq R \leq \infty$ *such that*

- *If* $|z - z_0| < R$*, the series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

  converges absolutely.

- *For any compact set* $K \subset D_R(z_0)$*, the absolute convergence is actually uniform.*
- *If* $|z - z_0| > R$*, then the series diverges.*

Moreover, R *can be computed using the Cauchy-Hadamard formula:*

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

The number R is called the *radius* of convergence, and the domain $D_R = \{z \mid |z - z_0| < R\}$ is called the *disc* of convergence. Recall that for a sequence $\{b_n\}$ of real numbers,

$$L = \limsup b_n$$

if the following two conditions hold

- For all $\varepsilon > 0$, and all $N > 0$, there exists $n > N$ such that

$$b_n > L - \varepsilon.$$

- For all $\varepsilon > 0$, there exists an N such that for all $n > N$,

$$b_n < L + \varepsilon.$$

Remark 1.1. *It is not difficult to show that if*

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists, then it is equal to $1/R$*. On many occasions the limiting ratio is easier to calculate.*

Remark 1.2. *Suppose* $0 < R < \infty$ *is the radius of convergence of the above power series. The by the theorem, the series converges on the* open *disc* $|z - z_0| < R$*. The behavior of the series at points on boundary however is subtle as the examples below indicate.*

Example 1.1. *The power series*

$$\sum_{n=0}^{\infty} z^n$$

has radius of convergence 1. In fact it easy to see that on $|z| < 1$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

Observe that on the unit disc $|z^n| = |z|^n = 1$, *and so by the divergence test, the series cannot converge at any boundary point. On the other hand, the left hand side is defined and holomorphic at all points* $z \neq 1$ *even though the power series is only defined inside the unit disc. We then say that the holomorphic function* $1/1-z$ *is an analytic continuation of the power series to the domain* $\mathbb{C} \setminus \{z = 1\}$. *We will say more about analytic continuation towards the end of the course. We remark that the following misleading formula often appears in popular culture (most notably in a video on the youtube channel - Numberphile, an otherwise decent math channel), many times accompanied with a quote with the effect that "Oh look - math is magical!":*

$$1 - 1 + 1 - 1..... = \frac{1}{2}.$$

The formula as stated is of course junk since the left hand side is clearly a divergent series. But there are ways of interpreting the left hand side. For instance, the left hand side is in fact Cesaro summable, which is a generalization of usual infinite summation in that a convergent series is also Cesaro summable and the Cesaro sum equals the sum of the series. In this case, the Cesaro sum does turn out to be $1/2$. *A more fundamental way (at least in my opinion) of interpreting the left hand side, as precisely the analytic continuation of* $\sum_{n=0}^{\infty} z^n$ *to* $z = -1$. *Then as remarked above, this analytic continuation is given by* $1/(1-z)$ *which of course equals* $1/2$ *at* $z = -1$. *Another example of such misleading propogation of math, especially in India, is that Ramanujan proved the "miraculous" identity that*

$$1 + 2 + 3 + \cdots = \frac{-1}{12}.$$

We'll see later in the course that the left hand side should in fact be replaced by the analytic continuation of the series $\sum_{n=1}^{\infty} n^{-s}$ *to* $s = -1$. *The infinite seres is a priori only defined on the region* $Re(s) > 1$, *but can be analytically continued to* $\mathbb{C} \setminus \{1\}$, *and this if of course the famous* $\zeta(s)$ *of Riemann.. We'll then compute that* $\zeta(-1) = -1/12$*!*

Example 1.2. *Consider the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n}.$$

Again, it is easy to see that the radius of convergence is 1. At $z = 1$ *this is the usual harmonic series, and is divergent. It turns out in fact, that this series converges for all other points on* $|z| = 1,\ z \neq 1$. *This follows from the following test due to Abel, which we state without proof.*

Lemma 1.1 (Abel's test)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n z^n.$$

Suppose

- $a_n \in \mathbb{R}$, $a_n \geq 0$.
- $\{a_n\}$ *is a decreasing sequence such that* $\lim_{n\to 0} a_n = 0$.

Then the power series converges on $|z| = 1$ *except possibly at* $z = 1$.

Clearly the series in the example above satisfies all the hypothesis, and hence is convergent at all points on $|z| = 1$ *except at* $z = 1$.

Example 1.3. *Next, consider the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n^2}.$$

Again, the radius of convergence is 1, and again by Abel's test the power series is convergent on $|z| = 1$ *except possibly at* $z = 1$*. But at* $z = 1$*, the series is clearly convergent, for instance by the integral test. So in this example the power series is convergent on the entire boundary.*

Example 1.4. *Finally consider the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

To find the radius of convergence, we use the ratio test. Denoting $a_n = 1/n!$*, we see that*

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So $R = \infty$*. Notice that if* z *is a real number then this is the usual Taylor expansion of the exponential function. Inspired by this, we define the complex exponential function,* $exp(z) : \mathbb{C} \to \mathbb{C}$ *by*

$$exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We will study this function in more detail in the next lecture.

1.3. **Holomorphicity of power series.** From the previous theorem, since the convergence is uniform on comapct subsets of the disc of convergence, it is clear that a power series represents a continuous function. In fact, much more is true.

Theorem 1.2. *Consider the function defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

on the disc of convergence $D_R = \{z \mid |z - z_0| < R\}$ *where* $0 < R \leq \infty$. *Then* $f(z)$ *is holomorphic on* D_R *with*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^n.$$

Proof. Without loss of generality we can assume $z_0 = 0$. Firstly, observe that since $\lim_{n\to\infty} n^{1/n} = 1$, the power series $\sum_{n=1}^{\infty} n a_n (z - z_0)^n$ also has radius of convergence R. To prove the theorem, we need to show that for any $p \in D_R(0)$,

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h} = \sum_{n=1}^{\infty} n a_n p^n.$$

Or equivalently, given any $\varepsilon > 0$, we need to find a $\delta > 0$ such that

$$|h| < \delta \implies \left| \frac{f(p+h) - f(p)}{h} - \sum_{n=1}^{\infty} n a_n p^n \right| < \varepsilon \tag{1.1}$$

Let us denote by

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \;\; E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n,$$

the N^{th} partial sum, and the N^{th} error term respectively, so that

$$f(z) = S_N(z) + E_N(z).$$

Then since the partial sums are polynomials, they are holomorphic, and in fact

$$\lim_{N\to\infty} S_N'(z) = \sum_{n=1}^{\infty} n a_n z^n, \tag{1.2}$$

where the convergence is uniform on compact subsets of $D_R(0)$ Now suppose $|p| < r < R$, then for any N, we can break the difference that we need to estimate into three parts -

$$\Big| \frac{f(p+h) - f(p)}{h} - \sum_{n=1}^{\infty} n a_n p^n \Big| = \Big| \frac{S_N(p+h) - S_N(p)}{h} - S_N'(p) \Big|$$
$$+ |S_N'(p) - \sum_{n=1}^{\infty} n a_n p^n| + \Big| \frac{E_N(p+h) - E_N(p)}{h} \Big|$$

Since S_N, being a polynomial, is holomorphic, there exists a $\delta > 0$ so that for $|h| < \delta$, the first term is smaller than $\varepsilon/3$. Similarly, by equation (1.2), the second term can be made smaller than $\varepsilon/3$ by choosing N big enough. So all that remains is to control the error term. Using the factorization

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$, we have

$$\Big|\frac{E_N(p+h) - E_N(p)}{h}\Big| \leq \sum_{n=N+1}^{\infty} |a_n| \Big|\frac{(p+h)^n - p^n}{h}\Big|$$
$$\leq \sum_{n=N+1}^{\infty} |a_n| |((p+h)^{n-1} + \cdots + p^{n-1})|$$

But if $|h| < \delta$ for sufficiently small δ (in particular if $\delta < r - |p|$), then $|p+h| \leq |p| + |h| < r$, and so

$$\Big|\frac{E_N(p+h) - E_N(p)}{h}\Big| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.$$

But this is the tail of the series $\sum n|a_n|r^{n-1}$ which converges for $r < R$, so we can also make this term smaller than $\varepsilon/3$ by choosing N big enough. This shows that we can find δ small enough so that (1.1) is satisfied.

□

Notice that the derivative is again a power series with the same radius of convergence. So applying the above theorem inductively we obtain -

Corollary 1.1. *A power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is infinitely complex differentiable in it's disc of convergence. Moreover, the derivatives can be computed by successive term-wise differentiation:*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}.$$

In particular, the coefficients of the power series are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

We say that a function $f : \Omega \to \mathbb{C}$ is *analytic* if for every $p \in \Omega$, there exists an $r = r(p) > 0$ and a sequence of numbers $\{a_n = a_n(p)\}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n$$

for every $z \in D_r(p)$. A priori, it is not quite clear that if a function is represented by a power series expansion on a disc of convergence $D_R(p)$ then it is automatically analytic. For instance, it is not clear if there should be a power series expansion around any other point $q \in D_R(p)$. The next proposition answers this question in the afirmative.

Proposition 1.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence R. Then f is analytic on $D_R(z_0)$. In fact, for every*

$p \in D_R(z_0)$, and every $z \in D_r(p)$ where $r := R - |p - z_0|$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n.$$

Proof. We use the binomial expansion. As before, without loss of generality, we assume that $z_0 = 0$. Writing $z = z - p + p$, and applying the binomial theorem, we see that on $|z - p| < R - |p|$ (since $|z| < R$), we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-p+p)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} p^{n-k}(z-p)^k \\ &= \sum_{k=0}^{\infty} b_k (z-p)^k, \end{aligned}$$

where

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n p^{n-k} = \frac{f^{(k)}(p)}{k!}$$

by Corollary 1.1. □

Remark 1.3. *In some cases the power series on the right in the conclusion of Proposition 1.3 might have a larger radius of convergence than $R - |q-p|$. In such cases, the new power will define an analytic continuation of f. For instance, let us consider the power series*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

on $|z| < 1$ with $p = 0$. Let $q = -1/2$. Then if $|z + 1/2| < 1 - 1/2 = 1/2$, we have

$$\frac{1}{1-z} = \frac{1}{3/2 - (z+1/2)} = \frac{2}{3} \cdot \frac{1}{1 - 2(z+1/2)/3} = \frac{2}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} \left(z + \frac{1}{2}\right)^n.$$

On the other hand, it can be easily seen that the power series on the right has a radius of convergence $R = 3/2$, and hence defines an extension of the original power series in the new region $|z + 1/2| < 3/2$. This was Weierstrass' method of analytically continuing holomorphic functions.

Corollary 1.2. *[Principle of analytic continuation for power series] Let f be an analytic function on a connected open set Ω. If there exists a point $p \in \Omega$ such that $f^{(n)}(p) = 0$ for all $n \in \mathbb{N}$, $f \equiv 0$ on all of Ω. In particular, the conclusion is true if there is an open set $U \subset \Omega$ such that $f\Big|_U \equiv 0$.*

Proof. Let $S = \{z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \cdots\}$. Then by the continuity of f, S is closed in Ω (ie. all limit points of S *in* Ω, are also contained in S). Also, S is non-empty by the hypothesis. Now suppose $q \in S$. Since the function is analytic, there is an open disc $D_r(q)$ on which

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(q)}{n!}(z-q)^n = 0,$$

since $q \in S$. But then $D_r(q) \subset S$, and so S is open in Ω. But then since Ω is connected, this forces $S = \Omega$, and in particular, $f \equiv 0$ on Ω. $\square$

Appendix: Multiplication and composition of power series

Given two power series

$$\sum_{n=0}^{\infty} a_n z^n, \ \sum_{n=0} b_n z^n,$$

their product can be defined, at least formally, in the following way-

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n z^n,\end{aligned}$$

where c_n is given by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This product goes by the name of *Cauchy product*. The main theorem, which we state without proof, is the following.

Theorem 1.3. *If the radius of convergence of two series centered at z_0 is R_1 and R_2 respectively, then their product power series has a radius of convergence that is at least* $\min(R_1, R_2)$.

Proof of a slightly general version, applicable for any infinite series, can be found on the wiki article `https://en.wikipedia.org/wiki/Cauchy_product#Convergence_and_Mertens.27_theorem`.