LECTURE-4

VED V. DATAR*

1. The exponential and trigonometric functions

Last lecture, we defined the *complex exponential function* by the power series

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We saw that the power series has infinite radius of convergence, and hence defines a function on the entire complex plain. In fact by the theorem from last lecture, we now know that the exponential function is holomorphic on the entire plane. Such functions, that are holomorphic on the entire complex plane, are called *entire functions*. As we saw in the previous lecture, to find the complex derivative, it is enough to differentiate term-wise:

$$\frac{d}{dz}\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d}{dz}z^n = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

To see the last equality just replace $n-1$ by n in the penultimate term. So we see that

$$\frac{d}{dz}\exp(z) = \exp(z).$$

In fact we'll see later that this property characterizes the exponential function. But first, we collect some important properties of the exponential function.

Theorem 1.1. (1) $\exp(0) = 1$.
(2) *For any complex numbers* z, w *we have that*
$$\exp(z+w) = \exp(z)\exp(w),$$
and in particular $\exp(-z) = [\exp(z)]^{-1}$.
(3) $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$.
(4) *The restriction of* $\exp(z)$ *to* $\mathbb{R}$ *is a positive strictly increasing function. In particular, for* $x \in \mathbb{R}$, $e^x = 1$ *if and only if* $x = 0$. *Moreover,* $\lim_{x\to\infty}\exp(x) = \infty$ *and* $\lim_{x\to-\infty}\exp(x) = 0$.

Proof. (1) This is trivial from the definition.

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(2) This follows from the product formula for power series and the binomial theorem. The left hand side of the equation is

$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}.$$

But then

$$\frac{(z+w)^n}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{k=0}^{n} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}.$$

This is exactly the n^{th} coefficient of the Cauchy product, and the result follows.

(3) Suppose $\exp(p) = 0$ for some $p \in \mathbb{C}$. Since $f(z) = \exp(z)$ is an analytic function with an infinite radius of convergence at 0, we also have the following representation formula on all of $\mathbb{C}$:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n.$$

But since $f^{(n)}(p) = \exp(p) = 0$ for all n, this would mean that $\exp(z) = 0$ for all z, clearly contradicting (1).

(4) Since $\exp(z) \neq 0$ and $\exp(0) = 1$, by the intermediate value theorem the function $f(x) = \exp(x)$ defined on $\mathbb{R}$ is strictly positive. Moreover, since $f(0) = 1$ and $f'(x) = e^x > 0$, the function $f(x)$ is strictly increasing, and hence $f(x) > 1$ for all $x > 0$. Then by the mean value theorem $f(x) - 1 > x$ for all $x > 0$. From this it follows that $\lim_{x\to\infty} f(x) = \infty$. Then property (2) implies that $\lim_{x\to-\infty} f(x) = 0$.

□

Remark 1.1. *Due to property (2) and our familiarity with working with exponents from high-school, from now one we adopt the more suggestive notation* $\exp(z) = e^z$.

Theorem 1.2. *There is a unique holomorphic function* $f : \mathbb{C} \to \mathbb{C}$ *satisfying*

$$\begin{cases} f'(z) = f(z) \\ f(0) = 1. \end{cases}$$

Proof. The proof uses the fact that if a holomorphic function has complex derivative identically zero, then the function has to be a constant. We will prove this fact later in the course. Assuming this, consider the function

$$g(z) = e^{-z} f(z).$$

Then by the Chain rule, since $f'(z) = f(z)$ we see that

$$g'(z) = e^{-z}(-f(z) + f'(z)) = 0.$$

Hence $g(z)$ is a constant. But by the initial condition we see that $g(0) = 1$. On the other hand by the first property in Theorem 1.1, $e^{-z} = 1/\exp(z)$, and so $f(z) = \exp(z)$. □

1.1. **Trigonometric functions.** We can now analogously define the functions *sine* and *cosine* using power series:

$$\cos z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$$
$$\sin z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!}$$

It is easy to check that the radius of convergence of both the power series is infinity, and hence they define entire functions, just like the exponential function. In fact an easy computation also shows that

$$\frac{d}{dz} \cos z = -\sin z, \ \frac{d}{dz} \sin z = \cos z.$$

The same computation then gives the following generalized Euler identity.

Proposition 1.1 (Generalized Euler identity)**.** *For any $z \in \mathbb{C}$,*

$$e^{iz} = \cos z + i \sin z.$$

Proof. This follows trivially from the following observations:

$$i^n = \begin{cases} (-1)^m, \ n = 2m \\ (-1)^m i, \ n = 2m+1. \end{cases}$$

So then by definition

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!}.$$

Euler's identity then follows from the observation that the two series on the right are simply the Maclaurin series for sine and cosine respectively. □

Remark 1.2. Polar coordinates. *The Euler identity can be used to give a third representation of complex numbers in terms of the exponential function. Namely, for $z \in \mathbb{C}$, let $r = |z|$ and $\theta = \arg z$. Then we have seen that*

$$z = r \cos \theta + ir \sin \theta.$$

So by the Euler identity, we have the representation

$$z = re^{i\theta}.$$

This is sometimes very useful in computations.

Next, we collect some properties of the sine and cosine functions. These can be proved using the generalized Euler identity, and the analogous properties of the exponential function.

Theorem 1.3. *The sine and cosine function satisfy the following.*

(1) $\sin(0) = 0$, $\cos(0) = 1$, *and for all* $z \in \mathbb{C}$ *we have*

$$\sin(-z) = -\sin z \text{ and } \cos(-z) = \cos(z).$$

(2) *For* $z, w \in \mathbb{C}$,

$$\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w$$
$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w.$$

(3) *For all* $z \in \mathbb{C}$,

$$\sin^2 z + \cos^2 z = 1.$$

Proof. (1) This follows easily from the definitions.

(2) It follows from the definition and property (1) above that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The sum-angle fomrulae then follow from this and property (2) in Theorem 1.1.

(3) This follows from the sum angle properties and property (1) above. An independent proof can be given by observing that the derivative of $f(z) = \sin^2 z + \cos^2 z$ is identically zero, hence $f(z)$ is a constant function, and hence equal to $f(0) = 1$.

□

One can also define the other trigonmetric functions $\tan z$, $\cot z$, $\sec z$ and $\csc z$ in the usual way.

1.2. **Periodicity and the definition of π.**

Theorem 1.4. *There exists a smallest positive real number* π *such that* $e^{2\pi i} = 1$. *Moreover,* $e^{iz} = 1$ *if and only if* $z = 2n\pi$ *for some* $n \in \mathbb{Z}$.

As an immediate consequence of the above theorem and Euler's identity, we have the following.

Corollary 1.1. *For all* $z \in \mathbb{C}$ *and all* $n \in \mathbb{Z}$,

$$e^{z+2n\pi i} = \exp(z), \ \sin(z + 2n\pi) = \sin(z), \text{ and } \cos(z + 2n\pi) = \cos z.$$

In particular,

$$\sin(2n\pi) = \cos\left(\frac{2n+1}{2}\pi\right) = 0$$

for all n.

Proof of Theorem 1.4. We first prove that there exists a real number τ such that $e^{i\tau} = 1$. To see this, note that by property (3) above, $-1 \leq \sin(x), \cos(x) \leq 1$ for all $x \in \mathbb{R}$. Then by the mean value theorem, it is easy to see that for all $x > 0$,

$$\sin x < x, \text{ and } \cos x > 1 - \frac{x^2}{2}.$$

Once again by an application of the mean value theorem, since $(\sin x)' = \cos(x) > 1 - x^2/2$, we see that

$$\sin(x) > x - \frac{x^3}{6}.$$

Finally, by yet another application of the mean value theorem we obtain

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

for all $x > 0$. Putting $x = \sqrt{3}$, we see that $\cos\sqrt{3} < 0$, and hence by the intermediate value theorem, there exists a $x_0 \in (0, \sqrt{3})$ such that $\cos(x_0) = 0$. Let $\tau = 4x_0$. Then by the sum, angle formulae, $\cos(\tau) = 1$ and $\sin(\tau) = 0$, and hence by the Euler identity, $e^{i\tau} = 1$.

Next, we argue that *any* period of e^{iz} has to be a real number. For, note that if $p \in \mathbb{C}$ is a period, that is if $e^{iz+ip} = e^{iz}$ for all $z \in \mathbb{C}$, then $e^{ip} = 1$. But then, if $p = a + ib$, then $1 = |e^{ip}| = e^{-b}$, and so by part(4) in Theorem 1.1, we see that $b = 0$, or that p has to be real.

Finally, we show that any period is an integral multiple of τ. We first show that τ is the smallest positive period. To see this, note that if $0 < x < x_0 (< \sqrt{3})$, then

$$\sin x > x\Big(1 - \frac{x^2}{6}\Big) > \frac{x}{2} > 0.$$

Since $(\cos x)' = -\sin x$, this shows that $\cos x$ is strictly decreasing in $[0, x_0]$. Then from the identity $\sin^2 x + \cos^2 x = 1$, since $\sin x > 0$, we see that $\sin x$ is strictly increasing in $[0, x_0]$. In particular, $0 < \sin x < 1$, and $e^{ix} \neq \pm 1$ or $\pm i$. Hence $e^{4ix} \neq 1$, and τ is indeed the smallest positive period. Now, if p is any other period of e^{iz}, then we can write $p/\tau = n + c$, where $n \in \mathbb{Z}$ and $c \in [0, 1)$. Then $1 = e^{ip} = e^{i\tau c}$. By our discussion above, if $c > 0$, then $\tau c \geq \tau$, which is a contradiction. Hence $c = 0$, and p is an integral multiple of τ. The proof of the theorem is complete with $\pi := \tau/2$. □

The logarithm function and complex powers

For functions of one real variable, the logarithm is the inverse function of the exponential function. We would like to generalize this to complex numbers. In particular, we would like to have a definition for logarithm that makes it a holomorphic function. An immediate difficulty is that while on real line, the exponential function is strictly increasing, and hence one-one, on the complex plane, we have already seen that the exponential function is not one-one. For instance $e^0 = e^{2\pi in} = 1$ for all $n \in \mathbb{Z}$. So to define an inverse function, one has to make a choice of the pre-image. For instance, we can choose to $\log 1 = 0$ or indeed any of $2\pi in$, for $n = 1, 2, 3, \cdots$.

In fact, writing in polar coordinates $z = re^{i\theta}$, $f(z)$ satisfies $e^{f(z)} = f(\exp(z)) = z$ on any open connected set if and only if

$$f(z) = \log r + i\theta + 2\pi in, \; n = 0, 1, -1, 2, -2, \cdots.$$

That is, we can at best define logarithm as *multivalued* function. A choice of n corresponds to defining a *single valued* logarithm, the corresponding function is called a *branch* of the logarithm. For instance, choosing $n = 0$, and defining

$$\log z = \log r + i\theta,$$

picks out what is called as the *principal branch* of the logarithm. This might seem like a good definition until we realize that the logarithm so defined is not even continuous. To see this, suppose $z \to -1$ from the 2nd quadrant. Then $\log z$ will tend to $i\pi$. But on the other hand, as $z \to -1$ from the third quadrant, $\log z$ will tend to $-\pi$. This is not only a minor irritant that can be fixed by some trick, but as we will see later in the course, is a fundamental issue. In fact, we will see that there is actually no way to define a holomorphic logarithm on which is defined on all of $\mathbb{C} \setminus \{0\}$. The best we can do is to define it outside of a ray. In fact we have the following.

Theorem 1.5. *The function*

$$\log z := \log |z| + i \arg z$$

defines a holomorphic function on $\mathbb{C} \setminus \mathrm{Re}(z) \leq 0$ *satisfying* $e^{\log z} = z$. *Moreover, applying chain rule, we see that*

$$\frac{d}{dz} \log z = \frac{1}{z}.$$

We will see a proof of this later in the course. The logarithm function then has the property that $\log 1 = 0$ and

$$\log zw = \log z + \log w,$$

assuming it is defined at all those points.

Logarithm as power series. Since $e^0 = 1$, and definition of logarithm must satisfy $\log 1 = 0$. Let's see if we can have a definition of logarithm using power series centered at $z_0 = 1$. By the chain rule, if have a holomorphic function $\log z$ near $z_0 = 1$, then

$$\frac{d}{dz} \log z = \frac{1}{z}.$$

Iteratively, we must have

$$\frac{d^n}{dz^n}\Big|_{z=1} \log z = (-1)^{n-1}(n-1)!.$$

If $\log z$ has a power series expansion around $z_0 = 1$, then the coefficients must be given by

$$a_n = \frac{1}{n!} \frac{d^n}{dz^n}\Big|_{z=1} \log z = (-1)^{n-1}\frac{1}{n}.$$

Turning this around, we consider the power series

$$\mathrm{L}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n.$$

We then have the following

Proposition 1.2. *The holomorphic function* $\mathrm{L}(z) : D_1(1) \to \mathbb{C}$ *satisfies*

$$\mathrm{L}(z) = \log z,$$

where $\log z$ *is the principal branch of the logarithm on* $\mathbb{C} \setminus Re(z) \leq 0$ *defined above.*

Proof. We already know that $d \log z/dz = 1/z$ for the principal branch of the logarithm. Also, $\mathrm{L}(1) = \log 1 = 0$. So, similar to the above proof, all we need to show (modulo the theorem on identically zero derivatives to be covered later) is that $\mathrm{L}'(z) = 1/z$. Since $F(z)$ is a power series, by term-wise differentiation,

$$\mathrm{L}'(z) = \sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^{n-1} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n.$$

From the geometric series expansion, we know that for $|w| < 1$,

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}.$$

Putting $w = 1 - z$ (we can do this since $|z - 1| < 1$) in the above expansion

$$\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n = \mathrm{L}'(z).$$

□

Complex powers. Once logarithm is defined, one can also define the powers of complex to other complex numbers by the simple formula

$$z^w = e^{w \log z}.$$

Example 1.1. The n^{th} roots. *If* n *is an integer, then* $e^{n \log z} = (e^{\log z})^n = z^n$*, and hence the new definition of complex powers agrees with our usual definition of integer powers of complex numbers.*