LECTURE-5

VED V. DATAR*

1. A review of multivariable calculus

Let $\Omega \subset \mathbb{R}^2$ be an open set, and $p = (a, b) \in \Omega$. Then a function $f : \Omega \to \mathbb{R}^2$ is said to be *(totally) differentiable* at p, if there exists a linear map $Df_p : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{|h| \to 0} \frac{f(p+h) - f(p) - Df_p(h)}{|h|} = 0.$$

The linear map Df_p is called the *(total) derviative* of f at p. If f is differentiable at p, then f is also continuous, and moreover the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at p. In fact, if

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are the standard basis vectors for $\mathbb{R}^2$, then

$$\frac{\partial f}{\partial x}(p) = Df_p(\vec{e_1}), \text{ and } \frac{\partial f}{\partial y}(p) = Df_p(\vec{e_2}).$$

In particular if $f = (u, v)$, the matrix for the linear map Df_p in terms of the standard basis, called the *Jacobian matrix*, is given by

$$\mathbf{J_f}(p) := \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

The determinant of the Jacobian matrix is called simply the *Jacobian*, and we will denote it by $J_f(p)$.

Remark 1. *Note that the mere existence of partial derivatives is not sufficient for the function to be differentiable. On the other hand, if the partial derivatives exist and are continuous, then the function is indeed differentiable.*

Date: 24 August 2016.

2. Cauchy-Riemann equations

Recall that a function $f : \Omega \to \mathbb{C}$ is holomorphic if

$$f'(z) = \frac{d}{dz} f(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite at all points $z \in \Omega$. Putting $z = x + iy$ and decomposing f into it's real and imaginary parts, we can write

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where $u, v : \Omega \to \mathbb{R}$ are real valued functions of two real variables.

Question 1. *What restrictions does holomorphicity of f put on the functions u and v?*

So suppose f is holomorphic. In the limit above, suppose h goes to zero along the the real axis ie. $h = h+i0$ is a real number. Then one can re-write the difference quotient as

$$\begin{aligned}
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x+h, y) - v(x, y)}{h} \\
&= \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y).
\end{aligned}$$

Notice that since the limit on the let exists, by definition of the limits of complex values functions, the individual limits of the real and imaginary parts also exist, and hence the two limits on the right exist. In other words, if f is holomorphic, then the partials of u and v with respect to x exist. On the other hand, if h goes to zero along the imaginary axis ie. $h = ik$, where $k \in \mathbb{R}$ goes to zero, then

$$\begin{aligned}
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{k \to 0} \frac{u(x, y+k) - u(x, y)}{ik} + i \lim_{k \to 0} \frac{v(x, y+k) - v(x, y)}{ik} \\
&= \lim_{k \to 0} \frac{v(x, y+k) - v(x, y)}{k} - i \lim_{k \to 0} \frac{u(x, y+k) - u(x, y)}{k} \\
&= \frac{\partial v}{\partial y}(x, y) - i \frac{\partial u}{\partial y}(x, y).
\end{aligned}$$

Again, the fact that f is holomorphic implies that the partials of u and v with respect to y also exist. But of course, if f is holomorphic, then these two limits must coincide. Setting the real and imaginary parts equal to each other we obtain the so called *Cauchy-Riemann equations.* In fact we have the following fundamental characterisation of holomorphicity. To state it, we first define "complex" multiplication of vectors in $\mathbb{R}^2$ as the map $J : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} -b \\ a \end{bmatrix}.$$

This is of course, simply multiplication by i, once we identify $\mathbb{R}^2$ with $\mathbb{C}$. It is easy to check that this is a linear isomorphism satisfying $J^2 = id$, and that the matrix with respect to the standard basis is

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$ be an open set and $f = u+iv : \Omega \to \mathbb{C}$ a function. Then the following are equivalent.*

(1) *$f = u + iv$ is complex differentiable at a point $z \in \Omega$.*

(2) *Df_p exists and u and v satisfy the Cauchy-Riemann (CR) equations:*

$$\text{(CR)} \qquad \begin{cases} \frac{\partial u}{\partial x}(x,y) & = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & = -\frac{\partial u}{\partial y}(x,y) \end{cases}$$

(3) *Df_p exists and is $\mathbb{C}$-linear in the sense that for any vector $\vec{v} \in \mathbb{R}^2$,*

$$Df_p(J(\vec{v})) = J(Df_p(\vec{v})).$$

Moreover, if f is complex differentiable at $p = a + ib$, then

$$f'(p) = \frac{\partial f}{\partial x}(a,b) = i\frac{\partial f}{\partial y}(a,b). \tag{2.1}$$

Proof. • **Proof of** (1) $\implies$ (2)**.** Firstly, the remarks above prove that if f is complex differentiable at $p = a + ib$, then the partial derivatives of u and v exist at (a,b) and satisfy the Cauchy-Riemann equations. Moreover, $f'(p)$ can be computed using the formula (2.1) above. All it is remains to be shown is that f is in fact differentiable as a vector valued function of two variables. From the definition of differentiability, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\Big|\frac{f(p+h) - f(p) - f'(p)h}{h}\Big| < \varepsilon,$$

whenever $|h| < \delta$. By the Cauchy Riemann equations, the Jacobian matrix at p is given by

$$\mathbf{J_f}(p) = \begin{pmatrix} u_x(a,b) & u_y(a,b) \\ -u_y(a,b) & u_x(a,b) \end{pmatrix}.$$

If Df_p is the linear transformation associated to the Jacobian, then an easy computation shows that for any complex number $h = \lambda + i\mu$,

$$Df_p(h) = (\lambda u_x(a,b) + \mu u_y(a,b)) + i(-\lambda u_y(a,b) + \mu u_x(a,b)) = f'(p)h.$$

Here the left hand side is a vector in $\mathbb{R}^2$, while the right hand side is a complex number, and the identification is the usual one. So give $\varepsilon > 0$, for the $\delta > 0$ chosen above, we have that

$$\Big|\frac{f(p+h) - f(p) - Df_p(h)}{|h|}\Big| = \Big|\frac{f(p+h) - f(p) - f'(p)h}{h}\Big| < \varepsilon,$$

whenever $|h| < \delta$. Hence f is differentiable at p with derivative given by Df_p.

- **Proof of** (2) $\iff$ (3)**.** Let $\vec{e_1}$ and $\vec{e_2}$ be the standard basis vectors for $\mathbb{R}^2$ as above. Note that $J(\vec{e_1}) = \vec{e_2}$ and $J(\vec{e_2}) = -\vec{e_1}$. Then

$$Df_p(J(\vec{e_1})) = Df_p(\vec{e_2}) = \begin{bmatrix} \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial y}(p) \end{bmatrix}$$

$$J(Df_p(\vec{e_1})) = J\Big(\begin{bmatrix} \frac{\partial u}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) \end{bmatrix} \Big) = \begin{bmatrix} -\frac{\partial v}{\partial x}(p) \\ \frac{\partial u}{\partial x}(p) \end{bmatrix}.$$

So the Cauchy-Riemann equations are satisfied if and only if $Df_p(J(\vec{v})) = J(Df_p(\vec{v}))$.

- **Proof of** (2) $\implies$ (1)**.** Let $p = a + ib \in \Omega$ be a point, and let

$$\begin{cases} A &= \frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \\ B &= \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b). \end{cases}$$

By the definition of differentiability,

$$u(a+h, b+k) = u(a,b) + hA - kB + \varepsilon_1(h,k)$$
$$v(a+h, b+k) = v(a,b) + hB + kA + \varepsilon_2(h,k),$$

where $\varepsilon_1(h,k)/\sqrt{h^2+k^2} \to 0$ and $\varepsilon_2(h,k)/\sqrt{h^2+k^2} \to 0$ as $(h,k) \to 0$. So

$$\frac{f(a+ib+(h+ik)) - f(a+ib)}{h+ik} = \frac{h(A+iB) - k(B-iA)}{h+ik} + \frac{\varepsilon_1(h,k) + \varepsilon_2(h,k)}{h+ik}.$$

For the first term, multiplying and dividing by the conjugate $h - ik$ and simplifying, we see that

$$\frac{(hA - kB) + i(hB + kA)}{h+ik} = A + iB.$$

For the second term, using triangle inequality, we see that

$$\Big|\frac{\varepsilon_1(h,k) + \varepsilon_2(h,k)}{h+ik}\Big| \le \Big|\frac{|\varepsilon_1(h,k)|}{\sqrt{h^2+k^2}}\Big| + \Big|\frac{|\varepsilon_2(h,k)|}{\sqrt{h^2+k^2}}\Big| \to 0$$

as $(h,k) \to 0$. And so we see that

$$\lim_{h+ik \to 0} \frac{f(a+ib+(h+ik)) - f(a+ib)}{h+ik}$$

exists, and is in fact equal to $A + iB$. This proves that f is holomorphic, with

$$f'(p) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b) = i\frac{\partial u}{\partial y}(a,b) - \frac{\partial v}{\partial y}(a,b).$$

$\square$

As a consequence, we have the following useful observation.

Corollary 1. *Let $f = u + iv : \Omega \to \mathbb{R}$ be holomorphic at $p \in \Omega$. Then, $J_f(p) = |f'(z)|^2$. In particular, the Jacobian of is always positive, and hence any holomorphic map is orientation preserving.*

Proof. By the Cauchy Riemann equations and (2.1),

$$J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2.$$

□

Corollary 2. *Let $\Omega \subset \mathbb{C}$ be a connected, open subset, and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $f'(z) \equiv 0$, then f is a constant.*

Proof. If $f'(z) \equiv 0$, then by the Cauchy Riemann equations, and (2.1), $Df_p \equiv 0$. By the mean value theorem (applied to line segments in $\mathbb{R}^2$), we see that f has to be a constant. □

The Cauchy-Riemann equations help us compute complex derivatives, or rule out the possibilities of some functions being holomorphic.

Example 1. *The function $f(z) = \bar{z}$ cannot be holomorphic. To see this, note that $f(z) = x - iy$. Applying the above theorem with $u(x, y) = x$ and $v(x, y) = -y$, we see that $u_x = 1$ while $v_y = -1$, and so the above equations are not satisfied. For this function $\bar{f}(z) = z$ is holomorphic. Functions such as these whose conjugates are holomorphic, are called anti-holomorphic functions. It is easy to prove that the only holomorphic and anti-holomorphic functions are the constants.*

Example 2. *We have already seen that $f(z) = |z|$ is not holomorphic. Using the above theorem it is easy to see that in fact, any function, defined on an open connected set, that takes only real values cannot be holomorphic, unless it is a constant function. This is because from the above theorem (since v=0) we get that $u_x = u_y = 0$. That is, the gradient of u is zero in an open connected set in $\mathbb{R}^2$. But then, by a standard fact from multivariable calculus, u (and hence f) has to be a constant.*

Example 3. *The function $f(x+iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin. On the other hand, one can show that the function is not holomorphic at $z = 0$. The problem is of course that Df_0 does not exist.*

2.1. **New notation.** A more compact way to write down the Cauchy-Riemann equations is to introduce the

$$\frac{\partial}{\partial z}, \ \frac{\partial}{\partial \bar{z}}$$

operators. These are the analogs of the partial derivative operators $\partial/\partial x, \partial/\partial y$ in the complex setting. To see how to define these operators, note that any point in the plane (x, y) can be described using the $(z, \bar{z})$ variables via the formulas

$$x = \frac{z + \bar{z}}{2}, \ y = \frac{z - \bar{z}}{2i}.$$

Formally, using chain rule from multivariable calculus (any sensible definition of these operators should satisfy chain rule after all!), and treating z and $\bar{z}$ as independent variables, we see that

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}\\ &= \frac{1}{2}\Big(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\Big)\end{aligned}$$

Similarly we see that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\Big).$$

Motivated by these formulae, we define the two operators as

$$\begin{aligned}\frac{\partial}{\partial z} &:= \frac{1}{2}\Big(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\Big)\\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\Big)\end{aligned}$$

Then for a holomorphic function $f(z)$ the Cauchy-Riemann equations can be re-written as

$$\frac{\partial f}{\partial \bar{z}} = 0,\ \frac{\partial f}{\partial z} = f'(z).$$

So holomorphicity implies that the function is independent of the $\bar{z}$ variable. Note that for any complex valued two variable function, we then have

$$\frac{\partial f}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}.$$

Remark 2. (Chain rule in the new notation). *Let f and g be function from domains in $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose f is differentiable at $p = (a, b)$, g is defined in a neighborhood of $f(p)$ and is differentiable at $f(p)$, then $g \circ f$ is differentiable at p. Then the chain rule from multivariable calculus is equivalent to*

$$\begin{aligned}\frac{\partial g\circ f}{\partial z}(p) &= \frac{\partial g}{\partial w}(f(p))\cdot\frac{\partial f}{\partial z}(p) + \frac{\partial g}{\partial \bar{w}}(f(p))\cdot\frac{\partial \bar{f}}{\partial z}(p)\\ \frac{\partial g\circ f}{\partial \bar{z}}(p) &= \frac{\partial g}{\partial w}(f(p))\cdot\frac{\partial f}{\partial \bar{z}}(p) + \frac{\partial g}{\partial \bar{w}}(f(p))\cdot\frac{\partial \bar{f}}{\partial \bar{z}}(p).\end{aligned}$$

Heuristically, this would be the "obvious" chain rule one might write down, if we consider $(z, \bar{z})$ as two independent variables of f (and resp. $(w, \bar{w})$ two independent variables of g). The proof of course requires an interpretation of the chain rule in terms of multiplication of Jacobian matrices, and the expression above of the Jacobian matrix in terms of the holomorphic and anti-holomorphic derivatives. In particular, if f and g are both holomorphic, then

$$\frac{\partial g\circ f}{\partial z}(p) = g'(f(p))\cdot f'(p),\ \frac{\partial g\circ f}{\partial \bar{z}}(p) = 0.$$

Harmonic functions

For a function of two variables $u(x, y)$ whose first two partials exist and are continuous, the *Laplacian* is defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The function is said to be *harmonic* if $\Delta u = 0$ at all points. Harmonic functions show up almost everywhere in physics, and most prominently in electrostatics. For instance the electric potential in a charge free region is harmonic function! In two dimensions, the study of harmonic functions is equivalent to the study of holomorphic functions via the following theorem.

Proposition 2.1. *Let $f = u + iv : \Omega \to \mathbb{C}$ be a holomorphic function, Suppose u and v have continuous second partial derivatives, then u and v are both harmonic functions.*

Proof. The proof is an easy consequence of the Cauchy-Riemann equations. By CR equations, $u_x = v_y$, and $u_y = -v_x$. differentiating the first with respect to 'x' and the second equation with respect to 'y', and recalling that since v has continuous second partials, the mixed partials commute, we see that

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

□

Remark 3. *Later, we will see that the real and imaginary parts of any holomorphic function are infinitely differentiable. In particular they have continuous second partials. So the hypothesis in the above theorem is superfluous. That is, for any holomorphic function, the real and imaginary parts are always harmonic functions.*

Given a harmonic function $u : \Omega \to \mathbb{R}$, a function $v : \Omega \to \mathbb{R}$ is said to be a *conjugate harmonic* function if $f = u + iv$ is a holomorphic function. As a consequence of the Cauchy-Riemann equations we then have the following.

Proposition 2.2. *Let v_1 and v_2 be conjugate harmonic functions, and let Ω be connected. Then $v_1 - v_2$ is a constant.*

∗ Department of Mathematics, Indian Institute of Science
Email address: `vvdatar@iisc,ac.in`