LECTURE-6

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Curves in the complex plane

A *parametrized curve* (or simply a curve) in a domain $\Omega \subset \mathbb{C}$ is a continuous function $z(t) : [a,b] \to \Omega$. Writing

$$z(t) = x(t) + iy(t),$$

we say that $z(t)$ is differentiable with derivative $z'(t)$, if $x'(t)$ and $y'(t)$ exist for all $t \in (a,b)$, and then we set

$$z'(t) = x'(t) + iy'(t).$$

We call it *regular* if $z'(t)$ exists and $z'(t) \neq 0$ for all $t \in (a,b)$. Geometrically, the vector

$$x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$$

gives the tangent vector to the curve at the point $(x(t), y(t))$, and so the complex number $z'(t)$ encodes the information of the tangent vector. Moreover

$$|z'(t)| = \sqrt{x'(t)^2 + y'(t)^2},$$

measures the *speed* at which the curve is traversed.

For a parametrised curve as above, the points $z(a)$ and $z(b)$ are called the *initial* and *final* points of the curve respectively, and together they are referred to as the end points of the curve. The curve is said to be *closed* if $z(a) = z(b)$, and is called *simple* if $z(t)$ is injective on the interval (a,b).

Orientation. The choice of a parametrization fixes the orientation of the curve. Given a parametrization $z(t) : [a,b] \to \mathbb{C}$ of the curve C, we say that $w(s) : [c,d] \to \mathbb{C}$ is an *orientation preserving* re-parametrization or that $z(t)$ and $w(s)$ are *equivalent* parametrizations, if there exists a strictly increasing function $\alpha : [c,d] \to [a,b]$ with $\alpha(c) = a$ and $\alpha(d) = b$, such that

$$w(s) = z(\alpha(s)).$$

Example 1. *Consider the curve defined by* $z : [0, 2\pi] \to \mathbb{C}$ *where*

$$z(t) = R(\cos t + i \sin t).$$

The image of the curve is of course a circle of radius R. *The tangent vector is given by*

$$z'(t) = R(-\sin t + i \cos t),$$

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and so the speed is $|z'(t)| = 1$. *A parametrization is given by* $w : [0, \pi] \to \mathbb{C}$,

$$w(t) = R(\cos 2t + i \sin 2t),$$

which traverses the same circle, but with double the speed. On the other hand the parametrization

$$z(t) = R(\cos t - i \sin t)$$

for $t \in (0, 2\pi)$ *also describes the same circle with the same speed, but traversed in a clock-wise direction. A circle is said to be positively oriented if the parametrization traverses the circle in the anti-clockwise direction, while negatively oriented otherwise.*

We will need to consider slightly more general curves. A curve $z(t) : [a, b]$ is said to be *piecewise regular* if there is a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that $z(t)$ restricted to each (t_{i-1}, t_i) is a regular curve.

Notation and conventions. If the image of restriction of the curve $z(t)$ to the interval (t_{i-1}, t_i) is denoted by C_i, and the image of the full curve is denoted by C, we then write

$$C = \sum_{i=1}^{n} C_n.$$

We denote by $-C$, the curve C traced in the opposite direction. For instance, if $z(t) : [a, b] \to \mathbb{C}$ is a parametrization for C, a parametrization for $-C$ is given by $z^-(s) : [a, b] \to \mathbb{C}$ where

$$z^-(s) = z(a + b - s).$$

For a positive integer $a > 0$, we denote by aC to be the curve C traversed 'a' times. A circle $C_R(p)$ or $|z - p| = R$, unless otherwise specified, will always mean a circle of radius R centred at p traversed once in the anti-clockwise direction.

Complex integration

For a continuous function $f = u + iv : [a, b] \to \mathbb{C}$ of one *real* variable, we can extend the definition of integration by defining

$$\int_a^b f(t)\, dt := \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt.$$

Now, suppose we are given a smooth curve as above, and a function $f : \Omega \to \mathbb{C}$, we then define the *complex integral along the curve* by

$$\int_C f(z)\, dz := \int_a^b f(z(t)) z'(t)\, dt,$$

where C denotes the image of the curve. Note that the multiplication above is the complex multiplication. It is convenient to think of dz as a complex differential, representing an infinitesimal complex change, and given by

$$dz = dx + idy.$$

So if $f = u + iv$, then

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx),$$

where the integrals on the right are now the usual line integrals from multivariable calculus.

Remark 1. *Recall that the differential forms dx and dy are defined to be duals of the vector fields $\partial/\partial x$ and $\partial/\partial y$ in the xy-plane. That is,*

$$dx\Big(\frac{\partial}{\partial x}\Big) = dy\Big(\frac{\partial}{\partial x}\Big) = 1,\ dx\Big(\frac{\partial}{\partial y}\Big) = dy\Big(\frac{\partial}{\partial x}\Big) = 0.$$

Then it is easy to compute that $dz = dx + idy$ and $d\bar{z} = dx - idy$ are dual complex valued *differential forms to the complex valued vector fields $\partial/\partial z$ and $\partial/\partial\bar{z}$ vector fields defined in the previous lecture.*

To make sure the integral is well defined, we need to show that it is independent of orientation preserving parametrizations.

Lemma 1. *Let C be a curve with parametrization $z(t)$. Let $w(s) = z(\alpha(s))$ be another* orientation preserving *parametrization, where $\alpha : [a, b] \to [c, d]$. Then*

$$\int_C f(w)\,dw = \int_C f(z)\,dz.$$

Proof. By the chain rule, since $w'(s) = z'(\alpha(s))\alpha'(s)$, we see that

$$\int_C f(w)\,dw = \int_a^b f(w(s))w'(s)\,ds = \int_a^b f(z(\alpha(s))z'(\alpha(s))\alpha'(s)\,ds.$$

Putting $t = \alpha(s)$, we see that $\alpha'(s)ds = dt$, and hence

$$\int_C f(w)\,dw = \int_a^b f(z(t))z'(t)\,dt = \int_C f(z)\,dz.$$

This shows that the definition of integration is independent of the parametrization chosen. □

We can now extend the definition of complex integrals to piecewise smooth curves by linearity. That is, if $C = C_1 + \cdots + C_n$ is a piecewise smooth curve with C_j smooth curve for all $j = 1, \cdots, n$, then we define

$$\int_C f(z)\,dz := \sum_{j=1}^n \int_{C_j} f(z)\,dz.$$

Other integrals. Similar to our definition of $\int_C f(z)\,dz$ we can define the integral with respect to $d\bar{z}$ by

$$\int_C f(z,\bar{z})\,d\bar{z} := \overline{\int_C \overline{f(z)}\,dz}.$$

We can also define the integral with respect to arc-length. For a complex or real valued function $f(z)$ and a curve $z = z(t) : [a,b] \to \mathbb{C}$ we also define

$$\int_C f(z)\,|dz| := \int_a^b f(z)|z'(t)|dt.$$

We then define the *length* of the curve by

$$\mathrm{len}(C) = \int_C |dz|.$$

We next state, without proof, some basic properties of the complex line integral. The proofs follow from the definition of the complex integral and corresponding properties of the Riemann integral.

Proposition 1. *Let C be a parametric piecewise regular curve in an open set $\Omega \subset \mathbb{C}$.*

(1) *For any complex numbers a, b and any complex valued functions f and g we have that*

$$\int_C [af + bg](z)\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz.$$

(2) *For integers $a_j \in \mathbb{Z}$ and piecewise smooth curves C_j, if we denote by $C = a_1C_1 + \cdots + a_nC_n$, we have*

$$\int_C f(z)\,dz = \sum_{j=1}^n a_j \int_{C_j} f(z)\,dz.$$

In particular,

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

(3) *(Triangle inequality)*

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz| \le \sup_{z\in C}|f(z)|\cdot \mathrm{length}(C),$$

with all inequalities replaced with equality if and only if f is a constant real number.

(4) *If $f_n \xrightarrow{u.c} f$, then*

$$\lim_{n\to\infty}\int_C f_n(z)\,dz = \int_C f(z)\,dz.$$

A fundamental computation

We now compute the integrals

$$\int_{C_R} z^n \, dz,$$

where $C_R = \{z \mid |z| = R\}$ is the circle of radius R centered at the origin positively oriented and traversed once. This computation, although elementary, will play a fundamental role in the rest of course. A parametrization of the circle is given by $z(t) = Re^{i\theta},\ \theta \in [0, 2\pi]$. Then $dz = iRe^{i\theta} d\theta$, and so

$$\begin{aligned}\int_{C_R} z^n, \, dz &= \int_0^{2\pi} R^n e^{in\theta}(iRe^{i\theta}) \, d\theta \\ &= iR^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta\end{aligned}$$

Now if $n + 1 \neq 0$, then

$$\int_0^{2\pi} e^{i(n+1)\theta} \, d\theta = \frac{1}{(n+1)i} e^{i(n+1)\theta}\Big|_{\theta=0}^{\theta=2\pi} = 0,$$

since $e^{i\theta}$ is periodic with period 2π. On the other hand, if $n = -1$, then

$$\int_{C_R} z^n, \, dz = iR^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta = i \int_0^{2\pi} d\theta = 2\pi i.$$

So summarizing, we have the following:

$$\frac{1}{2\pi i} \int_{|z|=R} z^n \, dz = \begin{cases} 0, & n \neq -1 \\ 1, & n = -1. \end{cases}$$

In particular, the integral is independent of the radius R. More generally, we have the following.

Proposition 2. *Let D be any disc in $\mathbb{C}$, and let p be a point not lying on the boundary circle $C = \partial D$. If C is traversed only once with positive orientation, then*

$$n(C; p) := \frac{1}{2\pi i} \int_C \frac{1}{z - p} \, dz = \begin{cases} 1, & p \in D \\ 0, & p \notin \overline{D}. \end{cases} \quad .$$

Moreover, if $n \neq -1$ and $n \in \mathbb{Z}$, then

$$\int_C (z - p)^n \, dz = 0.$$

The number $n(C, p)$ is called the *index* or the *winding number* of the circle C around p.

Proof. Without loss of generality, we can assume that $z_0 = 0$. Now suppose $p \in D$. Then for any $z \in C$, $|z| > |p|$. Then by the geometric series expansion, for $z \in C$ we have

$$\frac{1}{z-p} = \frac{1}{z} \cdot \frac{1}{1 - p/z} = \frac{1}{z} + \sum_{n=2}^{\infty} \frac{p^{n-1}}{z^n}.$$

Integrating both sides (this can be done since convergence is uniform), and using the above computation, we see that $n(C,p) = 1$. On the other hand, if $p \notin \overline{D}$, then $|z| < |p|$ for all $z \in C$, and hence

$$\frac{1}{z-p} = -\frac{1}{p} \cdot \frac{1}{1 - z/p} = -\frac{1}{p} \sum_{n=0}^{\infty} \frac{z^n}{p^n}.$$

Again integrating both sides, we see that $n(C,p) = 0$, since there are only positive powers of z on the right. For the second part, if $n > 0$, then the integrand is a polynomial and hence the integral is zero by the above computations. If $n = -m < 0$, then we can write

$$(z-p)^n = \Big(\frac{1}{z-p}\Big)^m.$$

Once again using the geometric series expansions above, in both cases, there will no terms with exponent -1. And hence by the computation above, the integral will be zero. □

Remark 2. *Later in the course, we will define the index $n(\gamma, p)$ of a general curve γ around a point p by a similar formula, and we shall prove (rather indirectly) that the index of* any *closed curve is always an integer. Assuming this, we can provide a more conceptual explanation of the above result. It is clear that the $n(C,p)$, as a function of p defined on the open set $\mathbb{C} \setminus C$ is a continuous function. But then being integer valued, it must be locally constant. From our elementary observation, $n(C,0) = 1$, and hence $n(C,p) = 1$ for all $p \in D$. On the other hand, clearly as $|p| \to \infty$, $n(C,p)$ approaches 0. Again by virtue of being locally constant, this implies that $n(C,p) = 0$ for all $p \in \mathbb{C} \setminus \overline{D}$.*

Primitives

We then have the following theorem, which is a generalization of the fundamental theorem for line integrals from multivariable calculus.

Proposition 3 (Fundamental theorem for complex integrals)**.** *If C is any curve joining the point p to q, then*

$$\int_C F'(z)\, dz = F(q) - F(p).$$

Proof. Let $z(t) : [0,1] \to \mathbb{C}$ be a parametrization for C such that $p = z(0)$ and $q = z(1)$. Then

$$\int_C F'(z) = \int_0^1 F'(z(t))z'(t)\,dt.$$

But by Chain rule, if we let $g(t) = F(z(t))$, then $g'(t) = F'(z(t))z'(t)$, and so

$$\int_C F'(z)\,dz = \int_0^1 \frac{dg}{dt}\,dt = g(1) - g(0),$$

where we use the usual one variable fundamental theorem of calculus. But $g(0) = F(z(0)) = F(p)$ and $g(1) = F(q)$, and this completes the proof. $\square$

Recall that an open set is called *connected* is any two points can be joined by a continuous curve lying completely inside the open set. An important and immediate consequence of the fundamental theorem is the following.

Corollary 1. *Let $\Omega \subset \mathbb{C}$ be an open connected subset, and $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f'(z) = 0$ for all $z \in \Omega$ if and only if $f(z)$ is a constant.*

For a domain $\Omega \subset \mathbb{C}$, and $F, f : \Omega \to \mathbb{C}$ complex valued functions, we say that $F(z)$ is a *primitive* of $f(z)$ if

$$F'(z) = f(z)$$

for all $z \in \Omega$. Then another direct corollary of the above theorem is the following.

Corollary 2. *Suppose $f : \Omega \to \mathbb{C}$ has a primitive on F, then*

$$\int_C f(z)\,dz = 0$$

for every closed curve $C \subset \Omega$.

Using the above corollary, we can explain the results of the calculation above. Recall that for any integer $n \neq -1$, we have that

$$\Big(\frac{1}{n+1}\Big)\frac{d}{dz}z^{n+1} = z^n.$$

Or in other words, for $n \neq -1$, z^n has a primitive at least on $\mathbb{C} \setminus \{0\}$. Hence the integral on any closed loop not passing through $z = 0$, is always zero. In particular, the integrals around $|z| = R$ are zero. That leaves the case when $n = -1$. We have already seen (as a consequence of the chain rule) that if a holomorphic logarithm $\log z$ can be defined, then it is a natural primitive for $1/z$. But we saw some lectures back, we saw that going around a circle centered at the origin, makes the logarithm function discontinuous, leave alone non-holomorphic. In fact, combining the corollary with the calculations above, we have managed to *prove* the following.

Proposition 4. *Let $\Omega \subset \mathbb{C} \setminus \{0\}$ be an open set containing at least one circle $|z| = r$. Then there is no holomorphic function $F : \Omega \to \mathbb{C}$ such that*

$$e^{F(z)} = z.$$

In particular, there cannot be a holomorphic logarithm defined on all of $\mathbb{C}^*$, or indeed on any punctured neighbourhood $D_r(0) \setminus \{0\}$ of 0.

In fact, the theorem can also be used to explain the fact that $n(\partial D, p) = 0$ when $p \notin \overline{D}$. Again without loss of generality, we assume that $D = D_R(0)$. Recall that $\log w$ can indeed be defined via a power series in the region $|w - 1| < 1$, namely

$$\log w = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(w-1)^n,$$

and it satisfies $(\log w)' = w^{-1}$. Now putting $w = z/p - 1$, $f(z) = \log(z/p - 1)$ is a holomorphic function in the region $|z| < |p|$ with

$$f'(z) = \frac{1}{z-p}.$$

Since C is completely contained in the region $|z| < |p|$, by Corollary 2, $n(C, p) = 0$.

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