LECTURE-7

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1. Theorems of Cauchy and Goursat

In the previous lecture, we saw that if f has a primitive in an open set, then

$$\int_\gamma f\,dz = 0$$

for all closed curves γ in the domain. This was a simple application of the fundamental theorem of calculus. It is somewhat remarkable, that in many situations the converse also holds true. In the next few lectures we will explore this theme, and prove theorems that will form the basis of all that we will accomplish in the rest of the course. The starting point is the following.

Theorem 1.1 (Cauchy)**.** *Let D be a disc in the complex plane. If $f : D \to \mathbb{C}$ is holomorphic, then*

$$\int_\gamma f\,dz = 0$$

for all closed curves γ contained in D.

We will prove this, by showing that all holomorphic functions in the disc have a primitive. The key technical result we need is *Goursat's theorem.*

Theorem 1.2 (Goursat)**.** *Let $\Omega \subset \mathbb{C}$ be an open subset, and $T \subset \Omega$ be a triangle contained inside Ω. If $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_T f(z)\,dz = 0.$$

Remark 1.1. Relation to Green's theorem. *If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field, such that P and Q have continuous first partials, then for any close curve*

$$\int_R (Q_x - P_y)\,dx\,dy = \int_{\partial R} P\,dx + Q\,dy.$$

Now, suppose that $f = u + iv$ and that u and v have continuous partials. Then

$$\int_\gamma f\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx).$$

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Consider the vector fields $(P_1, Q_1) = (u, -v)$ and $(P_2, Q_2) = (v, u)$. Then for each $j = 1, 2$, by the Cauchy Riemann equations

$$\frac{\partial Q_j}{\partial x} = \frac{\partial P_j}{\partial y}.$$

Then by Green's theorem, the line integral is zero. The key point is our assumption that u and v have continuous partials, while in Cauchy's theorem we only assume holomorphicity which only guarantees the existence of the partial derivatives. Later in the course, we will in fact show that holomorphicity implies continuous partials (actually infinite differentiability) as a consequence *of Cauchy's theorem.*

2. Proof of Cauchy's theorem assuming Goursat's theorem

Cauchy's theorem follows immediately from the theorem below, and the fundamental theorem for complex integrals.

Theorem 2.1. *Let $f : D \to \mathbb{C}$ be a holomorphic function. Then $f(z)$ has a primitive on D.*

Proof. We first observe that By translation, we can assume without loss of generality that the disc D is centered at the origin. For any points $z, w \in \mathbb{C}$, we denote by $l_{z,w}$ the straight line segment from z to w. For any $z = (x, y) \in D$, let γ_z denote the path from the origin to z consisting of a horizontal segment from 0 to $(x, 0)$ followed by a vertical segment from $(x, 0)$ to (x, y). We then define

$$F(z) = \int_{\gamma_z} f(w)\, dw,$$

and claim that $F(z)$ is holomorphic with $F'(z) = f(z)$. To see this, first note that if $z_1 = (x, 0)$, then for any small $h \in \mathbb{R}$,

$$\int_{\gamma_{z+h}} f(w)\, dw - \int_{\gamma_z} f(w)\, dw = \int_{l_{z_1,z_1+h}} f(w)\, dw + \int_{l_{z_1+h,z+h}} f(w)\, dw - \int_{l_{z_1,z}} f(w)\, dw.$$

On the other hand, by Theorem 1.2,

$$\int_{l_{z_1,z_1+h}} f(w)\, dw + \int_{l_{z_1+h,z+h}} f(w)\, dw + \int_{l_{z+h,z}} f(w)\, dw + \int_{l_{z,z_1}} f(w)\, dw = 0,$$

and so we have the key identity:

$$F(z+h) - F(z) = \int_{l_{z,z+h}} f(w)\, dw. \tag{2.1}$$

Intuitively, if $|h| << 1$, then $f(w) \approx f(z)$ on $l_{z,z+h}$, and so the integral is approximately $f(z)h$. To make this rigorous, we write

$$\begin{aligned}\int_{l_{z,z+h}} f(w)\, dw &= \int_{l_{z,z+h}} f(z)\, dw + \int_{l_{z,z+h}} (f(w) - f(z))\, dw \\ &= f(z)h + \int_{l_{z,z+h}} (f(w) - f(z))\, dw,\end{aligned}$$

and so

$$\frac{F(z+h)-F(z)}{h} = f(z) + \frac{1}{h}\int_{l_{z,z+h}} (f(w)-f(z))\,dw.$$

Dividing by h and using triangle inequality,

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| = \Big|\frac{1}{h}\int_{l_{z,z+h}} (f(w)-f(z))\,dw\Big| \leq \frac{1}{|h|}\int_{l_{z,z+h}} |f(w)-f(z)|\,|dw|.$$

By the continuity of f, given any $\varepsilon > 0$ there exists a δ such that if $|h| < \delta$, then for all $w \in l_{z,z+h}$

$$|f(w)-f(z)| \leq \varepsilon.$$

Using this in the above estimate, and remembering that $\text{length}(l_{z,z+h}) = |h|$ we see that

$$\Big|\frac{F(z+h)-F(z)}{h} - f(z)\Big| \leq \varepsilon,$$

so long as $|h| \leq \delta$. This shows that

$$\frac{\partial F}{\partial x}(z) := \lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z),$$

Next, consider a path σ_z consisting of a vertical line segment from 0 to iy followed by a horizontal segment from iy to z. By Theorem 1.2,

$$F(z) = \int_{\sigma_z} f(w)\,dw.$$

By an argument similar to the one above, we can prove that $\partial F/\partial y$ exists, and that

$$\frac{\partial F}{\partial y}(z) = \lim_{k\to 0}\frac{f(z+ik)-f(z)}{k} = if(z).$$

The analog of the key identity is that

$$F(z+ik)-F(z) = \int_{l_{z,z+ik}} f(w)\,dw,$$

which is approximately $if(z)k$. (integrating in the vertical direction incurs an i). In any case, this shows that the partials of F exist *and* are continuous, and hence F is a (totally) differentiable map from D to $\mathbb{R}^2$. On the other hand, since

$$\frac{\partial F}{\partial y}(z) = i\frac{\partial F}{\partial x}(z),$$

the partials also satisfy the Cauchy-Riemann equations. Hence F is complex differentiable at z, and moreover, $F'(z) = f(z)$. □

3. Proof of Goursat's theorem

We first prove the theorem assuming f is holomorphic on all of Ω. The proof consists of choosing a nested sequence of rectangles $R^{(n)}$ starting with $R^{(0)} = R$. Note that when we say triangle we mean the one-dimensional object, and not the region inside the triangle. Suppose we have already constructed the triangle $R^{(n-1)}$. The first step in the construction of $R^{(n)}$ is to bisect each side of $R^{(n-1)}$. This results in four new triangles which we label $R_1^{(n-1)}, R_2^{(n-1)}, R_3^{(n-1)}$ and $R_4^{(n-1)}$. We also give them the an orientation consistent with the original triangle (see figure) so that the integrals over the common boundaries cancel, and we have that

$$\int_{R^{(n-1)}} f(z)\,dz = \sum_{j=1}^{4} \int_{R_j^{(n-1)}} f(z)\,dz.$$

By triangle inequality,

$$\Big|\int_{R^{(n-1)}} f(z)\,dz\Big| \leq \sum_{j=1}^{4} \Big|\int_{R_j^{(n-1)}} f(z)\,dz\Big|,$$

and so for at least one triangle $R_j^{(n-1)}$,

$$\Big|\int_{R^{(n-1)}} f(z)\,dz\Big| \leq 4\Big|\int_{R_j^{(n-1)}} f(z)\,dz\Big|.$$

Choose any one such triangle, and label it $R^{(n)}$. The inductively, we will have

$$\Big|\int_{R} f(z)\,dz\Big| \leq 4^n\Big|\int_{R^{(n)}} f(z)\,dz\Big|. \tag{3.2}$$

Recall that the diameter of a subset of $\mathbb{C}$ is the maximum distance between any two points in that subset. We then have the following elementary observation.

Lemma 3.1. *If $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of the triangle $R^{(n)}$ respectively, then*

$$d^{(n)} = 2^{-n}d^{(0)}, \text{ and } p^{(n)} = 2^{-n}p^{(0)}.$$

Moreover, there exists a unique $p \in \cap R^{(n)}$.

Proof. The diameter of a rectangle is the maximum over the distances between the vertices, and hence equal to the diagonal length.. Since $R_1^{(n-1)}$,$R_2^{(n-1)}$, $R_3^{(n-1)}$ and $R_4^{(n-1)}$ are all congruent rectangles with side lengths that are half of the side lengths of $R^{(n-1)}$, clearly $d^{(n)} = 2^{-1}d^{(n-1)}$ and $p^{(n)} = 2^{-1}p^{(n-1)}$, and we obtain the result by induction. Now, let $x_n \in R^{(n)}$ be any point. Then since the sequence $\{x_n\}$ is contained in $R^{(0)}$, a compact subset, there exists a limit point $p \in R^{(0)}$. $\square$

Continuing with the proof of Goursat's theorem, since f is holomorphic at $z = p$, we can write

$$f(z) = f(p) + f'(p)(z - p) + \psi(z)(z - p),$$

where $\psi(z) \to 0$ as $z \to p$. Now the constant function $f(p)$ and the linear function $f'(p)(z-p)$ both have primitives; $f(p)z$ and $f'(p)(z-p)^2/2$ respectively. So by the fundamental theorem, their line integrals on $R^{(n)}$ are zero. So

$$\int_{R^{(n)}} f(z)\, dz = \int_{R^{(n)}} \psi(z)(z - p)\, dz.$$

Now since $d^{(n)} \to 0$ as $n \to \infty$, given any $\varepsilon > 0$ there exists a n such that

$$|\psi(z)| < \varepsilon$$

on $R^{(n)}$. But then

$$\Big| \int_{R^{(n)}} f(z)\, dz \Big| \leq \varepsilon \sup_{R^{(n)}} |z - p| p^{(n)} = \varepsilon d^{(n)} p^{(n)} \leq 4^{-n} \varepsilon d^{(0)} p^{(0)}.$$

Then by the inequality in (3.2)

$$\Big| \int_R f(z)\, dz \Big| \leq \varepsilon d^{(0)} p^{(0)},$$

which can be made arbitrarily small, since $d^{(0)}$ and $p^{(0)}$ are fixed. Hence $\int_R f(z)\, dz = 0$, and this proves the theorem in the case that f is holomorphic everywhere.

4. Extension of Cauchy's and Goursat's theorems to punctured domains

For applications, we need a slightly stronger version of Cauchy's theorem, which in turn relies on a slightly stronger version of Goursat's theorem. For any open set $\Omega \subset \mathbb{C}$, and any $p \in \Omega$ we denote $\Omega_p^* := \Omega \setminus \{p\}$.

Theorem 4.1. *Let D be a disc, and $p \in D$. Let $f : D_p^* \to \mathbb{C}$ be a holomorphic function such that $\lim_{z \to p}(z - p) f(z) = 0$. Then for any <u>closed curve</u> $\gamma \subset D_p^*$,*

$$\int_\gamma f(z)\, dz = 0.$$

As before, the kye technical input is a version of the theorem for rectangles.

Theorem 4.2. *Let Ω be any open subset of $\mathbb{C}$. For some $p \in \Omega$, let $f : \Omega_p^* \to \mathbb{C}$ be a holomorphic function satisfying $\lim_{z \to p}(z - p) f(z) = 0$. Then for any rectangle $R \subset \Omega$ with p not on the boundary for the triangle,*

$$\int_{\partial R} f(z)\, dz = 0.$$

Proof. Let $\varepsilon > 0$, and let $\delta > 0$ such that

$$|f(z)| \leq \frac{\varepsilon}{|z-p|},$$

whenever $|z-p| < \delta$. Let R_0 be a small square of side length δ with p at it's centre. Note that $|z-p| > \delta/2$ for all $z \in \partial R$. By extending the sides of R_0, divide R into nine rectangles $R_0, \cdots, R_8$. Clearly

$$\int_{R_j} f(z)\,dz = 0$$

for $j = 1, 2, \cdots, 8$ by Theorem 1.2. Since the integrals over the common boundaries cancel out if we choose the correct (ie. anti-clockwise) orientations

$$\int_R f(z)\,dz = \int_{R_0} f(z)\,dz.$$

Next, we estimate the integral over R_0,

$$\Big|\int_{\partial R_0} f(z)\,dz\Big| \leq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z-p|} < \frac{2\varepsilon}{\delta}\mathrm{len}(\partial R_0) = 8\varepsilon.$$

Since ε is arbitrary, this shows that $\int_{\partial R} f(z)\,dz = 0$.

□

Proof of Theorem 4.1. The idea of the proof is to again show that $f(z)$ has a primitive on D_p^*, and we proceed as in the proof of Theorem 2.1. Let $p = (a, b)$. Pick a point $z_0 = (x_0, y_0)$ such that $x_0 \neq a$ and $y_0 \neq b$. Let $z + (x, y) \in D_p^*$. If $x \neq a$, we let $\gamma_{z_0,z}$ be the path consisting of the line segment (x_0, y_0) to (x, y_0), followed by a vertical line from (x, y_0) to (x, y). On the other hand, if $x = a$, then we let $\gamma_{z_0,z}$ consist of three segments - A vertical segment from (x_0, y_0) to (x_0, y_η) followed by a horizontal segment from (x_0, y_η) to (x, y_η) followed by another vertical segment from (x, y_η) to (x, y). We then define $F : D_p^* \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_{z_0,z}} f(w)\,dw.$$

First, we claim that $\partial F/\partial x = f(z)$ for every $z \in D_p^*$. Note that the key step in the proof of Theorem 2.1 is to obtain the identity (2.1). We obtain the same identity in this new situation. Let $h \in \mathbb{R}$ be a small number. If $x \neq a$, then the same argument as before will imply that $\partial F/\partial x$ exists at (x, y) and equals $f(z)$. So suppose $x = a$. Let R be the rectangle with vertices z_0, $(a+h, y_0)$, $(a+h, y_\eta)$ and (x_0, y_η) and let R' be the rectangle with vertices (a, y_η), $(a+h, y_\eta)$, $z+h = (a+h, y)$ and $z = (a, y)$. Note that p lies either in the interior or the exterior of R (but not on the boundary), and so by Theorem 1.2 or Theorem 4.2,

$$\int_{\partial R} f(w)\,dw = 0.$$

On the other hand, p lies in the exterior of R', and so by Theorem (1.2),

$$\int_{\partial R'} f(w)\,dw = 0.$$

It is easy to check that

$$F(z+h) - F(z) = \int_{l_{z,z+h}} f(w)\,dw.$$

Now the argument proceeds exactly as in the proof of Theorem 2.1. Next, as before, one proves that $\partial F/\partial y$ exists everywhere on D_p^* and equals $if(z)$ using a suitable modification of the path σ.

□

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