LECTURE-8

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Cauchy integral formula and Analyticity of holomorphic functions

The main consequence of Cauchy's theorem for the punctured disc is the following fundamental result that will be a basis of everything that follows in the course.

Theorem 1 (Cauchy integral formula (CIF))**.** *Let $f:\Omega\to\mathbb{C}$ be a holomorphic function. If $\overline{D_r(z_0)}\subset\Omega$, then for any $z\in D_r(z_0)$,*

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Proof. For a fixed $z\in D_R(z_0)$, we define $h_z: D^* := D_R(z_0)\setminus\{z\}\to\mathbb{C}$ by

$$h_z(\zeta)=\frac{f(\zeta)-f(z)}{\zeta-z}.$$

Clearly h_z is holomorphic in D^*. Also $\lim_{\zeta\to z}(\zeta-z)h_z(\zeta)=0$, since f is holomorphic, and hence continuous at z. Then by Cauchy's theorem

$$\begin{aligned}0=\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}h_z(\zeta)&=\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{\zeta-z}\,d\zeta-f(z)\Big(\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{1}{\zeta-z}\,d\zeta\Big)\\&=\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{\zeta-z}\,d\zeta-f(z),\end{aligned}$$

since the quantity in the bracket is simply $n(C,z)$, where $C=\partial D_r(z_0)$, which is equal to 1 because $z\in D_r(z_0)$. □

An immediate consequence of the Cauchy integral formula is that holomorphic functions are analytic

Theorem 2. *Let $\Omega\subset\mathbb{C}$ open, and $f:\Omega\to\mathbb{C}$ a holomorphic function. Then f is analytic. Moreover, if $D_R(z_0)$ is* any *disc whose closure is contained in Ω, then for all $z\in D_R(z_0)$, $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$, where*

$$a_n=\frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta. \tag{0.1}$$

In particular, holomorphic functions are infinitely complex differentiable.

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Proof. By CIF, if D is a of radius R centered at a with boundary circle C_R, then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = R} \frac{f(\zeta)}{\zeta - z}\, dz.$$

Writing $\zeta - z = (\zeta - z_0) - (z - z_0)$, we see that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\Big(1 - \frac{z - z_0}{\zeta - z_0}\Big)^{-1}.$$

For $z \in D$ and $\zeta \in C_R$, $|z - z_0| < R = |\zeta - z_0|$, or equivalently $|(z - z_0)/(\zeta - z_0)| < 1$, and hence using the geometric series

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n.$$

Since power series converge uniformly, we can also integrate term-wise (see Appendix), we get that

$$\begin{aligned} 2\pi i f(z) &= \int_{C_R} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n d\zeta \\ &= \sum_{n=0}^{\infty} \int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \cdot (z - z_0)^n \\ &= 2\pi i \sum_{n=0}^{\infty} a_n (z - z_0)^n, \end{aligned}$$

where a_n is given by the formula (0.1), and this completes the proof of the theorem. Infinite complex differentiability follows from analyticity by Corollary 1.1 from Lecture-3. □

As an immediate corollary, we have the following versions of the principle of analytic continuation for holomorphic functions.

Corollary 1. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, and suppose Ω is connected.*

(1) *If there exists a $p \in \Omega$ such that $f^{(n)}(p) = 0$ for $n = 1, 2, \cdots$, then f is a constant function.*
(2) *If there exists a n open subset U such that $f\Big|_U \equiv 0$, then $f\Big|_\Omega \equiv 0$.*

This follows immediately from Theorem 2 above and Corollary 1.2 from Lecture-4. Another important consequence of the analyticity is the following criteria for holomorphicity.

Corollary 2 (Morera)**.** *Any continuous function on an open set Ω that satisfies*

$$\int_{\partial R} f(z)\, dz = 0$$

for all rectangular regions $R \subset \Omega$, is holomorphic.

Proof. Let $p \in \Omega$ and $r > 0$ such that $\overline{D_r(p)} \subset \Omega$. The given condition is equivalent to f having a primitive $F_p(z)$ on $D_r(p)$, essentially by the proof of Theorem 2.1 in Lecture-7. Note that continuity of f is crucial for this. But then by Theorem 2, $F_p'(z) = f(z)$ is also holomorphic on $D_r(p)$. In particular, f is complex differentiable at p. Since this is true for all $p \in \Omega$, $f \in \mathcal{O}(\Omega)$. □

Cauchy integral formula for derivatives

Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then by Theorem 2, the n^{th} complex derivative $f^{(n)}(z)$ exists for all $z \in \Omega$. Moreover, by equation (0.1), for any $z \in \Omega$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta,$$

where $r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$. More generally, just as for the Cauchy integral formula, one can obtain a similar formula for $f^n(z_0)$ where the integral is over a circle centred at possibly a point other than z_0.

Theorem 3 (Cauchy integral formula for derivatives)**.** *If D is a disc with boundary C whose closure is contained in Ω, then for any $z \in D$, we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \tag{0.2}$$

Remark 1. *Essentially what this theorem says is that one can differentiate the Cauchy integral formula, and take the derivative inside the integral.*

Proof. Since $f(z)$ is analytic in the neighborhood of any point $z \in \Omega$, it is automatically holomorphic in Ω. To prove the formula (0.2) we use induction. For $n = 0$, this is simply the Cauchy integral formula.

$$\begin{aligned} \frac{f^{(n-1)}(z) - f^{(n-1)}(z_0)}{z - z_0} &= \frac{(n-1)!}{2\pi i (z - z_0)} \int_C f(\zeta) \Big(\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z_0)^n} \Big) \, d\zeta \\ &= \frac{(n-1)!}{2\pi i} \int_C f(\zeta) \cdot \frac{(\zeta - z_0)^{n-1} + \cdots + (\zeta - z)^{n-1}}{(\zeta - z)^n (\zeta - z_0)^n} \, d\zeta \\ &= \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^n} \sum_{k=0}^{n-1} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} \, d\zeta. \end{aligned}$$

Suppose $D = D_R(p)$, and suppose $|f(\zeta)| < M$ on C. By choosing z sufficiently close to z_0, we can ensure that for all $\zeta \in C$,

$$|\zeta - z| > (R - |z_0 - p|)/2 := \lambda(z_0).$$

Also, trivially, $|\zeta - z|, |\zeta - z_0| < 2R$ and $|\zeta - z_0| > \lambda(z_0)$. Then for any $k < n$,

$$\Big| \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} - \frac{1}{\zeta - z_0} \Big| = \frac{|z - z_0| (|\zeta - z|^k + \cdots |\zeta - z_0|^k)}{|\zeta - z|^{k+1} |\zeta - z_0|} < n|z - z_0| \Big(\frac{2R}{\lambda(z_0)} \Big)^{n+2}.$$

Hence, given any $\varepsilon > 0$, there exists a $\delta = \delta(n, R, M, z_0) > 0$ such that for any $|z - z_0| < \delta$, and any $\zeta \in C$, we have

$$\Big| \frac{f(\zeta)}{(\zeta - z_0)^n} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} - \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \Big| < \varepsilon.$$

uniformly in the sense that the rate of convergence is independent of ζ. So if $|z - z_0| < \delta$, then for each k,

$$\Big| \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^n} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} \, d\zeta - \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \Big| \leq (n-1)! R\varepsilon,$$

and hence

$$\lim_{z \to z_0} \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^n} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} \, d\zeta = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.$$

Summing up over k, we see that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.$$

□

An extremely useful consequence is the following estimate on the derivatives of a holomorphic function.

Corollary 3. *Let f be a holomorphic function in an open set containing the closure of a disc $D_R(z_0)$. If we denote the boundary of the disc by C, then for any $z \in D_R(z_0)$,*

$$|f^{(n)}(z)| \leq \frac{n! R}{(R - |z - z_0|)^{n+1}} ||f||_C,$$

where $||f||_C := \sup_{\zeta \in C} |f(\zeta)|$. In particular,

$$f^{(n)}(z_0) \leq \frac{n!}{R^n} ||f||_C,$$

Proof. By CIF for derivatives, we have that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Applying the triangle inequality, and remembering that on C, $|\zeta - z| \geq R - |z - z_0|$,

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi} \int_C \Big| \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \Big| |d\zeta| \\ &\leq \frac{n!}{2\pi (R - |z - z_0|)^{n+1}} \sup_C |f(\zeta)| \cdot \text{len}(C) \\ &= \frac{n! R}{(R - |z - z_0|)^{n+1}} ||f||_C. \end{aligned}$$

□

Liouville Theorem

Recall that an *entire* function is a function that is holomorphic on the entire complex plane $\mathbb{C}$. We then have the following surprising fact.

Theorem 4 (Liouville)**.** *There are no bounded non-constant entire functions.*

Proof of Theorem 4. Let $f(z)$ be a bounded entire function. We show that it then has to be a constant. Suppose $|f(\zeta)| < M$ for all $\zeta \in \mathbb{C}$, and let $z \in \mathbb{C}$ be an arbitrary point. Since f is entire, it is holomorphic on any disc $D_R(z)$. Denoting the boundary by C_R, by the estimate above,

$$|f'(z)| \leq \frac{||f||_{C_R}}{R} < \frac{M}{R}.$$

Letting $R \to \infty$ the right hand side approaches zero, and hence $f'(z) = 0$ for all $z \in \mathbb{C}$. Since $\mathbb{C}$ is connected, this forces $f(z)$ to be a constant, completing the proof of Liouville's theorem. □

Remark 2. *More generally, we can show that an entire function $f(z)$ satisfying*

$$|f(z)| \leq M(1 + |z|^{\alpha}),$$

for some constants $M, \alpha > 0$ and all $z \in \mathbb{C}$, has to be a polynomial of degree at most $\lfloor \alpha \rfloor$, where $\lfloor \cdot \rfloor$ is the usual floor function. We leave this as an exercise.

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