LECTURE-9

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In this lecture, we give three further applications of the Cauchy integral formula.

The fundamental Theorem of algebra

Recall that in the previous lecture we proved Liouville's theorem, namely that there are no bounded, non-constant, entire functions. As a simple and beautiful consequence of this, we can prove the fundamental theorem of algebra, namely that any polynomial can be completely factored into linear factors over complex numbers.

Theorem 1. *Any non-constant polynomial $p(z)$ has a complex root, that is there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$. As a consequence, any polynomial is completely factorable, that is we can find $\alpha_1, \cdots, \alpha_n$ (some of them might be equal to each other), and $c \in \mathbb{C}$ such that*

$$p(z) = c(z - \alpha_1)\cdots(z - \alpha_n),$$

where $n = \deg(p(z))$.

Proof. For concreteness, let

$$p(z) = a_n z^n + \cdots + a_1 z + a_0,$$

where $a_n \neq 0$. Then we claim that there exists an R such that

$$\frac{|a_n|}{2}|z|^n \leq |p(z)| \leq 2|a_n||z|^n,$$

whenever $|z| > R$. To see this, by the triangle inequality,

$$|p(z)| \leq |a_n||z|^n + \cdots |a_1||z| = |a_0| = |a_n||z|^n \Big(1 + \frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_0|}{|z|^n}\Big) \leq \frac{3}{2}|a_n||z|^n$$

if $|z| > R$ and R is sufficiently big. For the other inequality, we use the other side of the triangle inequality. That is,

$$|p(z)| \geq |a_n||z|^n \Big| 1 - \Big| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \Big| \Big|.$$

But

$$\Big| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \Big| \leq \frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_0|}{|z|^n} < \frac{1}{2},$$

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if $|z| > R$ when R is as above. So when $|z| > R$,

$$|p(z)| \geq \frac{|a_n||z|^n}{2}.$$

Now suppose, that $p(z)$ has no root in $\mathbb{C}$. Then

$$f(z) = \frac{1}{p(z)}$$

will be an entire function. Moreover, on $|z| > R$, by the lower bound above,

$$|p(z)| > |a_n||z|^n/2 > |a_n|R^n/2,$$

that is $|f(z)| \leq M$ for some M on $|z| > R$. On the other hand, we claim that there exists an $\varepsilon > 0$ such that $|p(z)| > \varepsilon$ on $|z| \leq R$. If not, then there is a sequence of points $z_k \in \overline{D_R(0)}$ such that $|p(z_k)| \to 0$. Since $\overline{D_R(0)}$ is compact, there exists a subsequence, which we continue to denote by z_k, such that $z_k \to a \in \overline{D_R(0)}$. But then by continuity, $p(a) = 0$, contradicting our assumption that there is no root. This proves the claim. The upshot is that on $|z| \leq R$, $|f(z)| \leq 1/\varepsilon$. This shows that $f(z)$ is a bounded, entire function, and hence by Liouville, must be a constant, which in turn implies that $p(z)$ must be a constant. This proves the first part of the theorem.

The second part follows from induction. If $p(z)$ is a non-constant polynomial, let α_1 be a root, which is guaranteed to exist by the first part. Then by the remainder theorem, $p(z)$ is divisible by $(z - \alpha_1)$. For a proof of the remainder theorem, see remark below. Then we define a new polynomial

$$p_1(z) = \frac{p(z)}{z - \alpha_1}.$$

The crucial observation is that $\deg(p_1)$ is *strictly* smaller than $\deg(p)$. In finitely many steps we should reach a linear polynomial which is obviously factorable. □

Remark 1. *We used the remainder theorem for the proof of the second part of the theorem which basically says that for a polynomial* $p(z)$,

$$p(z) - p(a) = (z - a)q(z)$$

for some polynomial $q(z)$ *of a strictly smaller degree. To see this, let*

$$p(z) = a_n z^n + \cdots + a_1 z + a_0,$$

and so

$$p(z) - p(a) = a_n(z^n - a^n) + \cdots + a_1(z - a).$$

Each term is of the form

$$a_k(z^k - a^k) = a_k(z - a)(z^{k-1} + z^{k-2}a + \cdots + za^{k-2} + a^{k-1}),$$

and this completes the proof. It is also easy to see that highest degree term in $q(z)$ *is* $a_n z^{n-1}$*, and hence in particular, the degree of* $q(z)$ *is strictly smaller.*

Zeroes of a holomorphic function

A complex number a is called a *zero* of a holomorphic function $f : \Omega \to \mathbb{C}$ if $f(a) = 0$. A basic fact is that zeroes of holomorphic functions are *isolated.* This follows from the following theorem.

Theorem 2. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function that is not identically zero, and let $a \in \Omega$ be a zero of f. Then there exists a disc D around a, a non vanishing holomorphic function $g : D \to \mathbb{C}$ (that is, $g(z) \neq 0$ for all $z \in D$), and a unique positive integer n such that*

$$f(z) = (z - a)^n g(z).$$

Moreover, we have that

$$n = \min\{\nu \in \mathbb{N} \mid f^{(\nu)}(a) \neq 0\}.$$

The positive integer n is called the *order* or *multiplicity* of the zero at a.

Proof. By the principle of analytic continuation, if f is not identically zero, there exists an n such that $f^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$ but $f^{(n)}(a) \neq 0$. Let $D_\varepsilon(a)$ be a disc such that $\overline{D_\varepsilon(a)} \subset \Omega$. Then f has a power series expansion in the disc centered at $z = a$ with the first n terms vanishing. So we write

$$f(z) = a_n(z - a)^n + a_{n+1}(z - a)^{n+1} + \cdots$$

with $a_n \neq 0$. The the theorem is proved with

$$g(z) = a_n + a_{n+1}(z - a) + \cdots .$$

□

As an immediate corollary to the theorem we have the following.

Corollary 1. *Let $f : \Omega \to \mathbb{C}$ holomorphic.*

(1) *The set of zeroes of f is isolated. That is, for every zero a, there exists a small disc $D_\varepsilon(a)$ centered at a such that $f(z) \neq 0$ for all $z \in D_\varepsilon(a) \setminus \{a\}$.*
(2) *(strong principle of analytic continuation) If Ω is connected, and $f, g : \Omega \to \mathbb{C}$ are holomorphic functions such that for some sequence of points $p_n \in \Omega$, $f(p_n) = g(p_n)$. Then either $f \equiv g$ or $\{p_n\}$ does not have a limit point in Ω.*

Proof. By the theorem, if $a \in \Omega$ is a root, then there exists a disc D around a, and integer m, and a holomorphic function $g : D \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = (z - a)^m g(z).$$

Since $g(a) \neq 0$, by continuity, there is a small disc $D_\varepsilon(a) \subset D$ such that $g(z) \neq 0$ for any $z \in D_\varepsilon(a)$. But then on this disc $(z - a)$ is also not zero anywhere except at a, and hence for any $z \in D_\varepsilon(a) \setminus \{a\}$, $f(z) \neq 0$ exactly what we wished to prove. Part (2) is a trivial consequence of the Lemma. □

Example 1. *Even though the zeroes are isolated, they could converge to the boundary. For instance, consider the holomorphic function*

$$f(z) = \sin\left(\frac{\pi}{z}\right)$$

on $\mathbb{C}^*$. Clearly $z = 1/n$ is a sequence of zeroes. They are isolated, but converge to $z = 0$ which is not in the domain.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then by Corollary 1 any disc D such that $\overline{D} \subset \Omega$ contains only finitely many zeroes of the function in the interior. The next proposition allows one to calculate the number of zeroes counted with multiplicity.

Corollary 2. *Let $f : \Omega \to \mathbb{C}$ be holomorphic, and let $D \subset \Omega$ be a disc such that $\overline{D} \subset \Omega$, and $C := \partial D$ contains no zeroes of f. Let $p_1. \cdots, p_k$ are the zeroes of f in D with multiplicity $n_1. \cdots, n_k$.*

(1) *There exists a no-where vanishing holomorphic function $g : D \to \mathbb{C}$ such that*

$$f(z) = (z - p_1)^{n_1} \cdots (z - p_k)^{n_k} g(z).$$

(2) *The total number of roots (counted with multiplicity) is given by*

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{k} n_j.$$

Proof. (1) For each $j \in \{1, \cdots, k\}$, there exist a radius $r_j > 0$ and a nowhere vanishing holomorphic function $g_j : D_{r_j}(p_j) \to \mathbb{C}$ such that for all $z \in D_{r_j}(p_j)$,

$$f(z) = (z - p_j)^{n_j} g_j(z).$$

We can choose r_j small enough so that the discs have mutually disjoint closures. Let $U := D \setminus \left(\cup_{j=1}^{k} \overline{D_{r_j/2}(p_j)}\right)$. Define the function

$$g(z) := \begin{cases} \frac{g_j(z)}{\Pi_{i\neq j}(z-p_i)^{n_i}}, \ z \in D_{r_j}(p_j) \text{ where } j = 1, \cdots, k \\ \frac{f(z)}{\Pi_j (z-p_j)^{n_j}}, \ z \in U. \end{cases}$$

Clearly $g(z)$ satisfies all the required properties.

(2) With $g(z)$ as above, for any $z \neq p_j$, a simple computation shows that

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{k} \frac{n_j}{z - p_j} + \frac{g'(z)}{g(z)}.$$

Since $g(z)$ is nowhere vanishing, $g'(z)/g(z)$ is holomorphic on D, and hence by Cauchy's theorem it's integral on C vanishes. It then follows that

$$\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{k} n_j \int_C \frac{dz}{z-p_j} = \sum_j n_j.$$

$\square$

Sequences of holomorphic functions

Recall that if a sequence of differentiable functions $f_n : I \to \mathbb{R}$ converges uniformly to $f : I \to \mathbb{R}$, for some interval $I \subset \mathbb{R}$, it does not necessarily imply that f is also differentiable. For instance, consider $f_n : [-1,1] \to \mathbb{R}$ defined by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

Then it is not difficult to see that $f_n \to |x|$ uniformly, but of course $|x|$ is not differentiable at $x = 0$. It turns out that holomorphic function behave much better under uniform convergence. We say that a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ converges to $f : \Omega \to \mathbb{C}$ *compactly* if the convergence is uniform over compact subsets $K \subset \Omega$. That is, for all $\varepsilon > 0$ and $K \subset \Omega$ compact, there exists a $N = N(\varepsilon, K)$ such that

$$|f_n(x) - f(x)| < \varepsilon,$$

for all $n > N$ and all $x \in K$.

Theorem 3. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge compactly to $f : \Omega \to \mathbb{C}$, then $f(z)$ is holomorphic. Moreover*

$$f_n^{(k)} \to f^{(k)}$$

compactly on Ω for all $k \in \mathbb{N}$.

Proof. Fix a $a \in \Omega$, and let D be a disc around a such that it's closure is also in Ω. Then for any triangle $T \subset D$, by Goursat's theorem

$$\int_T f_n(z)\,dz = 0.$$

Since $\overline{D}$ is compact, $f_n \to f$ converges uniformly on D, and hence

$$\int_T f(z)\,dz = \lim_{n\to\infty} \int_T f_n(z)\,dz = 0.$$

But this is true for all triangles in D, and hence by Morera's theorem f is holomorphic in D and in particular at a. Since a is arbitrary, this shows that f is holomorphic on Ω.

We prove that $f_n' \to f'$ compactly. For $k > 1$, the result will follow from induction. There is no loss of generality in assuming that $\overline{\Omega} \subset \mathbb{C}$ is compact. First we define

$$\Omega_r = \{z \in \Omega \mid \overline{D_r(z)} \subset \Omega\}.$$

Geometrically, this is the set of all points in Ω that are at least a distance r away from the boundary of Ω. Given any compact set K, there exists a $r > 0$ such that $K \subset \Omega_r$, and hence it suffices to show that $f_n' \to f'$ uniformly on Ω_r. The key point is the following estimate.

Claim 1. *Let $F : \Omega \to \mathbb{C}$ be holomorphic. Then for any $r > 0$,*

$$\sup_{z \in \Omega_r} |F'(\zeta)| \leq \frac{2}{r} \sup_{\zeta \in \Omega_{r/2}} |F(\zeta)|.$$

First notice that if $z \in \Omega_r$ then $\overline{D_{r/2}(z)} \subset \Omega_r$. To see this, let $w \in \overline{D_{r/2}(z)}$ i.e. $|z - w| \leq r/2$, and we need to show that $w \in \Omega_{r/2}$ or equivalently that $\overline{D_{r/2}(w)} \subset \Omega$. But for any $w' \in \overline{D_{r/2}(w)}$, $|w' - w| \leq r/2$ and hence by the triangle inequality $|w' - z| \leq r$, that is $w' \in \overline{D_r(z)}$ which is contained in Ω by definition, since $z \in \Omega_r$. This shows that $\overline{D_{r/2}(w)} \subset \Omega$ and hence that $\overline{D_{r/2}(z)} \subset \Omega_{r/2}$. Now by Cauchy's integral formula, if we denote the boundary of $D_r(z)$ by $C_r(z)$, then for any $z \in \Omega_{2r}$,

$$F'(z) = \frac{1}{2\pi i} \int_{C_{r/2}(z)} \frac{F(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

By the above observation, if $z \in \Omega_r$ then $C_{r/2}(z) \subset \Omega_{r/2}$, and hence by triangle inequality, for all $z \in \Omega_r$, since $|\zeta - z| = r/2$ for $\zeta \in C_{r/2}(z)$ we have

$$\begin{aligned} |F'(z)| &\leq \frac{1}{2\pi r^2} \sup_{\zeta \in C_{r/2}(z)} |F(\zeta)| \mathrm{len}(C_{r/2}(z)) \\ &\leq \frac{2}{r} \sup_{\zeta \in \Omega_r} |F(\zeta)| \end{aligned}$$

This proves the claim. Now given any $\varepsilon > 0$, since $\overline{\Omega_{r/2}} \subset \Omega$ is compact (remember we are assuming without any loss of generality that Ω is bounded), there exists an $N = N(r, \varepsilon)$ such that for all $n > N$ and all $\zeta \in \Omega_{r/2}$,

$$|f_n(\zeta) - f(\zeta)| < \frac{\varepsilon r}{2}.$$

But then by the claim, for $n > N$ and for all $z \in \Omega_r$, we have the estimate

$$|f(z) - f_n(z)| \leq \frac{2}{r} \cdot \frac{\varepsilon r}{2} = \varepsilon,$$

proving that $f_n \to f$ uniformly on Ω_r, and this completes the proof of the theorem.

□

Recall that Weiestrass' theorem states that any continuous function on a compact interval is the uniform limit of polynomials. On the other, by the above theorem, a continuous non-holomorphic function cannot be the uniform limit of polynomials. Instead we have the following, which we state without a proof.

Theorem 4 (Runge's thoerem)**.** *Let* $K \subset \mathbb{C}$ *and let* f *be a function that is holomorphic in a neighbourhood of* K*.*

(1) *There exists a sequence of rational functions* $R_n(z)$ *such that* $R_n \xrightarrow{u.c} f$ *on* K*, and such that the singularities of the rational functions all lie in* K^c*.*

(2) *If* K^c *is connected, then one there exists a sequence of polynomials* $p_n(z)$ *such that* $p_n \xrightarrow{u.c} f$*.*

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