Complex Analysis: Assignment-1

(due January 21, 2020)

Note. Submit solutions to 2, 5, 6, 7(c), 8. Two or three randomly chosen questions will be graded. And there will be five points for completion of the assignment.

1. Find all possible solutions to the following equations:
   (a) $z^4 = i + 1$.
   (b) $z^n = 1$.

2. Determine all values of 2^i, i^i and $(-1)^{2i}$.

3. For what values of z is e^z equal to 2, $i/2$?

4. Show that there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

   if and only if $|a| \leq |c|$. If this condition is satisfied, what are the smallest and largest values of $|z|$.

5. (a) Let $z, w \in \mathbb{C}$ such that $z\bar{w} \neq 1$. Then prove that if $|z|, |w| < 1$, then

$$\left|\frac{w - z}{1 - z\bar{w}}\right| < 1.$$

   Moreover

$$\left|\frac{w - z}{1 - z\bar{w}}\right| = 1$$

   if and only if either $|z| = 1$ or $|w| = 1$.

   (b) Now, fix a $w \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, and consider the mapping $\phi_w : \mathbb{D} \to \mathbb{C}$,

$$\phi_w(z) := \frac{w - z}{1 - z\bar{w}}.$$

   Prove that ϕ_w has the following properties:
   1. ϕ_w is a holomorphic map of $\mathbb{D}$ into itself.
   2. ϕ_w interchanges 0 and w. That is, $\phi_w(0) = w$ and $\phi_w(w) = 0$.
   3. ϕ_w is a biholomorphism. **Hint.** Compute $\phi_w \circ \phi_w$.

6. Prove that $f(z)$ is holomorphic if and only if $\overline{f(\bar{z})}$ is holomorphic. How are the two complex derivatives related?

7. Find the radius of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} (\log n)^2 z^n$.

(b) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} z^n$.

(c) The Bessel function:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}.$$

8. For what values of $z \in \mathbb{C}$ is the series

$$\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$$

convergent. In the regions that the series does converge, is it absolutely and/or uniformly convergent?