Complex Analysis: Assignment-2

(due Februaury 04, 2020)

Note. Please submit solutions to 2, 3(a) and (c), 4, 6 and 7.

1. Prove that the function $f(x+iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin, and yet, using only the definition, prove that the function is not complex differentiable at 0.

2. Suppose f is holomorphic on a region Ω. If any one of the following holds:

   1. $Re(f)$ is a constant,
   2. $Im(z)$ is a constant,
   3. $|f|$ is a constant,

   prove that f is a constant function.

3. Let $\Omega \subset \mathbb{C}$ be open and $f = u + iv : \Omega \to \mathbb{C}$ be a smooth map.

   (a) If h is a smooth map in a neighborhood of $f(p)$, then prove that

   $$\frac{\partial h \circ f}{\partial z}(p) = \frac{\partial h}{\partial w}(f(p)) \cdot \frac{\partial f}{\partial z}(p) + \frac{\partial h}{\partial \bar{w}} f(p) \cdot \frac{\partial \bar{f}}{\partial z}(p)$$
   $$\frac{\partial h \circ f}{\partial \bar{z}}(p) = \frac{\partial h}{\partial w}(f(p)) \cdot \frac{\partial f}{\partial \bar{z}}(p) + \frac{\partial h}{\partial \bar{w}} f(p) \cdot \frac{\partial \bar{f}}{\partial \bar{z}}(p)$$

   (b) If f is holomorphic, prove that $\det J_f(z) = |f'(z)|^2$ for all $z \in \Omega$.

   (c) Using the usual inverse function theorem from multivariable calculus, prove the following holomorphic inverse function theorem: If f is holomorphic at p with $f'(p) \neq 0$, then there there exists open sets U and V around p and $f(p)$ respectively, such that $f : U \to V$ has a holomorphic inverse $f^{-1} : V \to U$.

4. Let $\Omega \subset \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$ a holomorphic function satisfying $|f(z) - 1| < 1$ in Ω. Prove that for any closed regular curve γ in Ω,

   $$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0.$$

5. If $P(z)$ is a polynomial and $R > 0$ prove that

   $$\int_{|z-a|=R} P(z) d\bar{z} = -2\pi i R^2 P'(a).$$

6. Let f be an entire function, and $\alpha, C > 0$ be constants. Suppose

$$|f(z)| \leq C(1 + |z|^{\alpha})$$

for all $z \in \mathbb{C}$. Prove that $f(z)$ is a polynomial of degree at most $[\alpha]$, where $[\alpha]$ denotes the greatest integer less than or equal to α.

7. Compute the following integrals. The circles are traversed once in the anti-clockwise direction.
   (a) $\int_{|z|=1} e^z z^{-n}\, dz$.
   (b) $\int_{|z|=2} \frac{dz}{1+z^2}$.
   (c) $\int_{|z|=r} \frac{|dz|}{|z-a|^2}$, where $|a| \neq r$. **Hint.** First prove that $|dz| = -ir dz/z$ on the circle $|z| = r$.

8. Is there a holomorphic function f on a domain Ω such that there exists and integer $N \in \mathbb{N}$ and a complex number $z \in \Omega$ such that $|f^{(n)}(z)| > n! n^n$ for all $n > N$? Can you formulate a sharper theorem of the same kind?

9. Let $\Omega \subset \mathbb{C}$ be open, and let $f : \Omega \to \Omega$ be a holomorphic function. Suppose there exists a point $z_0 \in \Omega$ such that

$$f(z_0) = z_0, \;\; f'(z_0) = 1.$$

   Prove that f is linear, that is, there exists $a, b \in \Omega$ such that $f(z) = az + b$. **Hint.** First prove that one can assume without loss of generality that $z_0 = 0$. If f is not linear, then $f(z) = z + a_n z^n + O(z^{n+1})$ for some $a_n \neq 0$ and $n > 1$. If f_k denote f composed with itself k-times, prove that $f_k(z) = z + k a_n z^n + O(z^{n+1})$. Now, apply Cauchy inequalities and let $k \to \infty$ to conclude the desired result.

10. Suppose $f : \mathbb{D} := D_1(0) \to \mathbb{C}$ is holomorphic. Prove that the diameter of the image $d := \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ satisfies the estimate

$$d \geq 2|f'(0)|,$$

   and that equality holds if and only if the function is linear.