Complex Analysis: Assignment-3

(due February 25, 2020)

Note. Please submit solutions to 2, 4, 5(a), 6 and 8.

1. Let γ be a simple closed curve, and $a \notin \text{Supp}(\gamma)$. The prove that

$$n(\gamma, a) = \begin{cases} \pm 1, & a \in int(\gamma) \\ 0, & a \in ext(\gamma). \end{cases}$$

2. Are there holomorphic functions $f(z)$ and $g(z)$ in a neighbourhood of 0 such that for $n = 1, 2 \cdots$ we have
   (a) $f(1/n) = f(-1/n) = 1/n^2$,
   (b) $g(1/n) = g(-1/n) = 1/n^3$?

3. Determine all holomorphic functions on the unit disc $\mathbb{D} := D_1(0)$ such that

$$f''\Big(\frac{1}{n}\Big) + f\Big(\frac{1}{n}\Big) = 0,$$

   for all $n = 2, 3, \cdots$.

4. For any open set $\Omega \subset \mathbb{C}$ and any complex valued function, the L^2-norm on U is defined to be

$$||f||_{L^2(\Omega)} := \Big(\int_\Omega |f(x,y)|^2 \, dx \, dy \Big)^{1/2}$$

   if it is finite. We then define the space $H(\Omega)$ by

$$H(\Omega) := \{ f \in \mathcal{O}(\Omega) \mid ||f||_{L^2(\Omega)} < \infty \}.$$

   We equip it with the norm $|| \cdot ||_{L^2(\Omega)}$ defined above.

   (a) Let $z_0 \in \Omega$ and $\overline{D_r(z_0)} \subset \Omega$. For any $0 < s < r$, prove that

$$\sup_{z \in D_s(z_0)} |f(z)| \leq \frac{1}{\sqrt{\pi}(r-s)} ||f||_{L^2(D_r(z_0))}.$$

   Hint. Write $f(z)$ in terms of an integral on $\partial D_r(z_0)$ by the Cauchy integral formula and use polar coordinates.

   (b) Prove that if $\{f_n\}$ is a sequence in $H(\Omega)$ that is Cauchy with respect to the $L^2(\Omega)$ norm, then $f_n \to f$ compactly on Ω.

(c) Hence prove that $H(\Omega)$ with the metric

$$d(f,g) := ||f-g||_{L^2(\Omega)}$$

is a complete metric space.

5. Suppose that f and g are holomorphic in a region containing $\overline{D_1(0)}$. Suppose f has a simple zero at 0 (ie. the order is one), and has no other zero in $\overline{D_1(0)}$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

Show that if ε is sufficiently small, then

(a) f_ε has a unique zero (counted with multiplicity) in $\overline{D_1(0)}$.

(b) Moreover, if that unique zero is p_ε, then $\varepsilon \to p_\varepsilon$ is a continuous function.

6. Find the branch points (including infinity) for the following functions. Also give a branch cut that will make the function a single valued holomorphic function on the complement of the cut.

(a) $\sqrt{z-1}$

(b) $\log(z^2+z+1)$

7. (a) Let $f : \mathbb{D} \to \mathbb{C}$ such that the functions $g = f^2$ and $h = f^3$ are holomorphic on $\mathbb{D}$. Prove that f is holomorphic. Is the statement true if either g is not holomorphic or h is not holomorphic? If so, give a proof. Else give counterexamples.

(b) Either prove or provide a counter-example to the following statement: If f is a continuous function on a connected open subset Ω such that f^2 is holomorphic. Then so is f.

8. (a) Let Q_R be the rectangle with vertices $(-R,0)$, $(R,0)$, (R,R) and $(-R,R)$. Compute the integral

$$\int_{\partial Q_R} \frac{dz}{(1+z^2)^{n+1}},$$

where Q_R has the anti-clockwise orientation.

(b) Use this to prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{(2n)!}{4^n(n!)^2}\pi.$$