Complex Analysis: Assignment-4

(due March 10, 2020)

Note. Please submit solutions to 1, 4, 5, 8

1. In each of the cases below, classify the isolated singularities, and in case of poles, compute the order.
   (a) $\frac{z^2-\pi^2}{\sin^2 z}$.
   (b) $\frac{1-\cos z}{\sin z}$.
   (c) $\frac{1}{e^z-1} - \frac{1}{z-2\pi i}$.
   (d) $\frac{1}{\cos(1/z)}$.

2. If f and g are entire functions such that $|f(z)| < |g(z)|$ for $|z| > 1$, then show that $f(z)/g(z)$ is a rational function.

3. Let $R(z)$ be a rational function such that $|R(z)| = 1$ for $|z| = 1$.
   (a) Show that α is a zero or a pole of order m, if and only if $1/\bar{\alpha}$ is a pole or zero of order m respectively. **Hint.** First show that
   $$M(z) = R(z)\overline{R\Big(\frac{1}{\bar{z}}\Big)}$$
   is a rational function such that $M(z) = 1$ on $|z| = 1$.
   (b) Let $\{\alpha_j\}_{j=1}^N$ be zeroes and poles of $R(z)$ of order m_j in the unit disc $|z| < 1$. Here $m_j > 0$ is α_j is a zero and $m_j < 0$ is it is a pole. Define
   $$B(z) = \Big(\frac{z-\alpha_1}{1-z\bar{\alpha}_1}\Big)^{m_1}\Big(\frac{z-\alpha_2}{1-z\bar{\alpha}_2}\Big)^{m_2}\cdots\Big(\frac{z-\alpha_N}{1-z\bar{\alpha}_N}\Big)^{m_N}.$$
   Show that $R(z) = \lambda B(z)$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$

4. Recall that a function is said to have a removable singularity (resp. pole or essential singularity) at infinity if the function $f(1/z)$ has a removable singularity (resp. pole or essential singularity) at $z = 0$.
   (a) Show that an isolated singularity (including at infinity) of $f(z)$ cannot be a pole for $\exp(f(z))$.
   (b) In particular, if f is a non-constant entire function, then $\exp(f(z))$ has an essential singularity at infinity.

5. Let f and g be entire functions such that $h(z) = f(g(z))$ is a non-constant polynomial. Prove that both $f(z)$ and $g(z)$ are polynomials.

6. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Using the Laurent series, evaluate

$$\int_{|z|=2} \frac{1}{4z - z^2}\, dz,$$

where the circle is given positive orientation.

7. Show that the Laurent series for $(e^z - 1)^{-1}$ at the origin takes the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1},$$

where the numbers B_k are the called the *Bernoulli numbers.* Calculate B_1, B_2, B_3.

8. Find a Laurent series that converges in the annulus $1 < |z| < 2$ to a branch of the function

$$f(z) = \log\left(\frac{z(2-z)}{1-z}\right).$$