Complex Analysis: Assignment-5

(due March 31, 2020)

Note. Please submit solutions to 1, $2(c),(e)$, 4, $5(a)$ 6, $7(b)$ and $8(b),(c),(e)$.

1. Recall that we proved the identity

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \Big(\frac{1}{z-n} + \frac{1}{n} \Big).$$

Recall also, the definition of Bernoulli numbers from the previous assignment

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n}.$$

Finally, for any complex number $Re(s) > 1$, we define $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$.

(a) Prove that

$$\pi z \cot \pi z = 1 - \sum_{n=1}^{\infty} \zeta(2n) z^{2n}.$$

(b) Prove that

$$\zeta(2n) = 2^{2n-1} \frac{B_n}{(2n)!} \pi^{2n},$$

In particular, from your answers in the previous assignment, you should be able to calculate $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$. **Hint.** First observe that

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1}.$$

2. Let $\Omega \subset \mathbb{C}$ be an open set containing the closure $\overline{D}_r(0)$ of the disc of radius r centred at the origin. Suppose $f : \Omega \to \mathbb{C}$ be is a holomorphic function with zeroes $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $D_r(0)$ with multiplicities $m_1, \cdots, m_n$ respectively, and no zero on $\partial D_r(0)$. For any entire function $\varphi : \mathbb{C} \to \mathbb{C}$, show that

$$\frac{1}{2\pi i} \int_{|z|=r} \varphi(z) \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{n} m_j \varphi(\alpha_j).$$

3. Let f be a function that is holomorphic on the annulus $A_{R,\infty}(0)$. The residue of $f(z)$ at infinity is defined to be

$$\mathrm{Res}_{z=\infty} f(z) = -\frac{1}{2\pi i} \int_{|z|=r} f(z)\, dz,$$

where $r > R$. Note that by Cauchy's theorem, the definition is independent of r. The reason for the negative sign is that morally, one would like to define the residue at infinity, in the same way as for a point in $\mathbb{C}$, namely via an integral on a small circle around the point with positive orientation. But a small circle with positive orientation around the point at infinity is a large circle in $\mathbb{C}$ with negative orientation, and hence the negative sign in the above expression.

(a) Prove that
$$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

(b) If f is holomorphic in $\mathbb{C} \setminus \{p_1, \cdots, p_n\}$. Then prove that
$$\text{Res}_{z=\infty} f(z) + \sum_{k=1}^{n} \text{Res}_{z=p_k} f(z) = 0.$$

4. Show that if f is an injective entire function, then it must be linear. That is, $f(z) = az + b$, for some $a, b \in \mathbb{C}$ with $a \neq 0$. **Hint.** First show that $f(z)$ cannot have an essential singularity at infinity.

5. Let f be a non-constant holomorphic map defined in an open set Ω containing the unit disc $\mathbb{D}$ centred at the origin.

   (a) Suppose $|f(z)| = 1$ whenever $|z| = 1$, then show that $f(\Omega)$ contains the unit disc.

   (b) Show that if $|f(z)| \geq 1$ whenever $|z| = 1$, and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, the prove that $f(\Omega)$ contains the unit disc.

6. Show that there is no holomorphic function on $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ such that $f(z) = 1/z$ for all $z \in \partial\mathbb{D}$.

7. In each of the cases below, calculate the total number of solutions (with multiplicity) in the regions indicated.

   (a) $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ in $|z| < 1$.

   (b) $cz^n = e^z,\ |c| > e$ in $|z| < 1$.

8. Compute the following real-variable integrals using the residue theorem.

   (a) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\, dx,\ a \in \mathbb{R}$.

   (b) $\int_0^\infty \frac{x \sin x}{x^2+a^2}\, dx,\ a \in \mathbb{R}$.

   (c) $\int_0^\infty \frac{x^{1/3}}{x^2+1}\, dx$.

   (d) $\int_0^\infty \frac{\log x}{1+x^2}\, dx$.

   (e) $\int_0^\infty \log(1+x^2) \frac{dx}{x^{1+\alpha}}, 0 < \alpha < 2$.