Complex Analysis: Practice problems for the midterm

Note. Please submit solutions to 2, 3(a) and (c), 4, 6 and 7.

1. Let f be an entire function such that $Re(f(z)) > 0$ for all $z \in \mathbb{C}$. Prove that f is a constant.
2. Let f be an entire function such that $|f(z)| \leq |z|^n$ for all $z \in \mathbb{C}$. Prove that $f(z) = cz^n$, for some constant c.
3. Let f be an entire function. The prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.
4. Let f and g be holomorphic functions in a neighbourhood fo zero such that both vanish to an order m at 0. Then prove the following L'hospital's rule in this context:
$$\lim_{z \to 0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(0)}{g^{(m)}(0)}.$$
5. Compute $\int_{|z|=1} \tan z \, dz$.
6. Let $f(z)$ be an entire function with finite number of zeroes $\alpha_1, \cdots, \alpha_k$ with orders $n_1, \cdots, n_k$ respectively. If $R > \max_j |\alpha_j|$, prove that
$$\int_{|z|=R} z \frac{f'(z)}{f(z)} \, dz = \sum_j \alpha_j n_j.$$
7. (a) Find the branch points (including infinity) for
$$f(z) = z^{\frac{1}{3}}(1-z)^{\frac{2}{3}},$$
and show that $f(z)$ defines a single valued holomorphic function on $\mathbb{C} \setminus [0, 1]$.
   (b) Is there a contninuous function $g(z)$ on $\mathbb{C} \setminus [0, 1]$ such that $f(z) = e^{g(z)}$?
8. Let
$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$
   (a) For $n \geq 2$, if C_r is any circle that contains all the roots of $p(z)$ in the interior, then show that
$$\int_{C_r} \frac{1}{p(z)} \, dz = 0.$$
   Is the result true if $n = 1$?

(b) More generally, if $P(z)$ and $Q(z)$ are polynomials such that $\deg(Q(z)) \geq \deg(P(z)) + 2$, show that

$$\int_{C_r} \frac{P(z)}{Q(z)}\, dz = 0,$$

for any circle C_r such that all the roots of $Q(z)$ lie in the interior of C_r.

9. Let

$$p(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

(a) Show that

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \frac{1}{z - \alpha_j}.$$

(b) Suppose all the roots lie in the upper half plane i.e $\mathrm{Im}(\alpha_j) > 0$ for all j. Then show that if $\mathrm{Im}(z) \leq 0$, then for all j,

$$\mathrm{Im}\Big(\frac{1}{z - \alpha_j}\Big) > 0.$$

Use this to show that any root α of $p'(z)$ also satisfies $Im(\alpha) > 0$.

(c) More generally, without any assumption on the roots $\alpha_1, \cdots \alpha_n$ show that a root α of $p'(z)$ lies in the convex hull

$$S(\alpha_1, \cdots, \alpha_n) = \{t_1\alpha_1 + \cdots + t_n\alpha_n \mid t_1 + \cdots + t_n = 1\}.$$

10. Prove the following identities using Cauchy's theorem:

(a) $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6}$.

(b) $\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$ **Hint.** Use a contour formed by a sector of the circle $|z| = R$ between $0 \leq \theta \leq 2\pi/3$.