Lectures on Complex Analysis

These are lecture notes based on courses on complex analysis offered at UC Berkeley in Fall 2016 and IISc in the winter semester of 2020.

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Introduction

1.1 Roadmap for the course

Complex analysis is one of the most beautiful branches of mathematics; a subject that lies at the heart of several other subjects, such as topology, algebraic geometry, differential geometry, harmonic analysis, and number theory.

The main objects in calculus are real (scalar or vector) valued functions defined on domains in $\mathbb{R}^n$. The starting point in complex analysis is to consider *complex valued* functions

$$f : \Omega \to \mathbb{C}$$

defined on subsets $\Omega \subset \mathbb{C}$ of complex numbers. Recall that complex numbers can be added, subtracted, multiplied and divided (if non-zero) just like real numbers. Every complex number can be written in the form

$$z = x + iy$$

where x and y are real numbers. So complex numbers can be identified as a set with Euclidean plane $\mathbb{R}^2$. The addition of complex numbers is also equivalent to addition of vectors in $\mathbb{R}^2$. So it might appear as if we are not adding much, and that nothing is lost by simply treating the complex valued function as a two variable vector field. In fact this is true, as we will see later, when talking about limits and continuity.

But there is one key difference between $\mathbb{R}^2$ and $\mathbb{C}$, that of multiplication and division. Indeed things change dramatically when we restrict our attention to *complex differentiable* or *holomorphic* functions, that is, functions for which

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. The important point being that h could be a complex number. Formally this definition is identical to that of a differentiable function in one-variable calculus. But quite surprisingly the mere change of perspective, the fact that h is allowed to take complex values as it goes to zero, produces beautiful new phenomenon that have no counterparts in one-variable calculus, or indeed even multivariable calculus. We now summarize some of these remarkable consequences of holomorphicity.

- **Analyticity.** As we remarked above, complex valued functions can be thought as mapping between sets in $\mathbb{R}^2$. We will prove later in the course that for a holomorphic function, partial derivatives of all orders exist. And moreover, one also has that the Taylor series at ever point converges to the function value, that is holomorphic functions are *analytic*. Recall that this is not true for one-variable functions. For instance if

$$f(x) = \begin{cases} e^{-1/x^2}, \ x > 0 \\ 0, \ x \leq 0, \end{cases}$$

then it is easy to see that at $x = 0$, derivative of any order is zero. So the Taylor series of the function at $x = 0$ is zero, but the function is clearly not zero.

- **Analytic continuation.** Two holomorphic functions defined on an open connected domain are equal in a small open neighbourhood of a point, no matter how small the neighbourhood is, have to be identically equal.

- **Good convergence properties.** If a sequence of holomorphic functions converges uniformly, the limit function is again holomorphic. This is not true for differentiable one-variable functions. For instance, if $f_n : [-1, 1] \to \mathbb{R}$ is defined by

$$f_n(x) = \sqrt{\frac{1}{n} + x^2},$$

  then one can show that $f_n \to |x|$ uniformly, but $|x|$ is not differentiable.

- **Liouville property.** A bounded holomorphic function defined on all of $\mathbb{C}$ is forced to be a constant. As a consequence, one can prove the fundamental theorem of algebra.

Part of the richness of the theory of holomorphic functions comes from the variety in the methods used to study the subject. We next summarize the approaches that we will touch upon in this course.

- **Partial differential equations.** It turns out that real and imaginary parts of holomorphic functions, thought of as real valued two-variable functions, satisfy a system of first-order partial differential functions, called the *Cauchy-Riemann* equations. As a consequence of this, the real and imaginary parts are *harmonic functions*. The theory of harmonic functions is rather well developed, and could be potentially exploited to study holomorphic functions. We will only touch upon the Cauchy-Riemann equations, but will not pursue this approach further. We will instead focus on integral methods.

- **Integral methods.** The viewpoint that we will adopt is centered on a remarkable formula called the Cauchy's integral formula. We will develop a notion of integration of complex valued functions along curves, a generalization of the notion of line integrals in multi-variable calculus. The fundamental fact, which will be the theoretical basis for the rest of the course, is that the complex integral of a holomorphic function around a closed curve is zero. If the real and imaginary parts of the holomorphic function are assumed to have continuous partial derivatives, this result follows from Green's theorem. We will give an independent proof, not because we wish to be clever, but because remarkably this theorem will *imply* that the real and imaginary parts of the holomorphic function indeed have not only continuous partial derivatives but have partial derivatives of all orders, and are in fact *analytic*.

- **Power series methods.** As remarked above, every holomorphic function is represented by a power series. Since power series are algebraic objects, for the most part they can also be manipulated as if they were polynomials. Thus algebraic methods can be used to study holomorphic functions.

- **Geometric mathods.** An elementary but beautiful fact is that holomorphic functions, thought of as mappings (or transformations) between sets in $\mathbb{R}^2$ are *conformal maps*. That is, holomorphic mappings preserve angles between curves, and stretch the distances. We will study some standard examples of conformal maps. Towards the end of the course we will prove the following deep fact, first discovered by Riemann - Any domain in the complex plain which does not have a 'hole' and which is not the entire complex plane, can be mapped conformally to a disc centered at the origin of radius one. Our proof will be due to Koebe.

1.2 Complex numbers

It is known that certain polynomial equations with real coefficients need not have real roots. Complex numbers are obtained from the reals by formally adjoining a number i that solves the equation

$$i^2 = -1.$$

More formally, we define the set of complex numbers by

$$\mathbb{C} := \mathbb{R}[i] = \mathbb{R}[x]/(1+x^2).$$

So a general complex number takes the form $z = x + iy$, where x and y are real numbers, and are called the real and imaginary part of z respectively. We use the notations

$$x = \Re(z) \text{ and } y = \Im(z).$$

Clearly the real numbers can be identified as a subset of the complex numbers in a natural way as numbers with $\Im(z) = 0$. We define addition and subtraction to be component-wise i.e. if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then we define

$$z_1 \pm z_2 = (x_1 + x_2) \pm i(y_1 + y_2).$$

Using this, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ as vector spaces. With this interpretation, a complex number represents a point in the xy-plane; with the x-coordinate given by $\Re(z)$ and the y-coordinate given by $\Im(z)$. This more geometric interpretation will be very useful to us.

But the complex numbers are much more than just 2-dimensional vectors. They also have a multiplicative structure, induced from the multiplicative structure of $\mathbb{R}[x]$. That is we can multiply two complex numbers to obtain another complex number. Indeed, if z_1 and z_2 are as above, we define

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

More simply, we define $i^2 = -1$, and then extend the product to satisfy the distributive property. It is not hard to verify that addition and multiplication satisfy the following properties:

P1 (Additive and Multiplicative identity.) For any complex number z,

$$z + 0 = z,\ z \cdot 1 = z.$$

P2 (Commutativity.) For any $z_1, z_2 \in \mathbb{C}$,

$$z_1 + z_2 = z_2 + z_1,\ z_1 \cdot z_2 = z_2 \cdot z_1.$$

P3 (Associativity.) For any complex numbers z_1, z_2, z_3,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),\ (z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3).$$

P4 (Distribution) For any $z_1, z_2, z_3 \in \mathbb{C}$,

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3.$$

P5 (Additive inverse.) For any $z \in \mathbb{C}$, $-z = (-1) \cdot z$ satisfies

$$z + (-z) = 0.$$

For notational convenience, we sometimes drop the *dot* when multiplying complex numbers. As remarked above, geometrically, addition of complex numbers corresponds to addition of vectors. What is the interpretation for multiplication? This is clearer if we use *polar coordinates.* Recall that any point (x, y) in the plane that is not the origin, can be represented uniquely by a pair (r, θ), where $r > 0$ and $\theta \in (-\pi, \pi]$ via the following transformation law:

$$x = r\cos\theta,\ y = r\sin\theta.$$

Then r is the geometric distance from the origin, and θ is the angle made by the line joining (x, y) to the origin with the positive x-axis. For instance the complex number i corresponds to $(1, \pi/2)$ in polar coordinates. So, any complex number can be represented as

$$z = r(\cos\theta + i\sin\theta).$$

If $w = \rho(\cos\alpha + i\sin\alpha)$ is another complex number, then it follows from the definition of the multiplication formula that

$$\begin{aligned} zw &= r\rho[(\cos\theta\cos\alpha - \sin\theta\sin\alpha) + i(\cos\theta\sin\alpha + \sin\theta\cos\alpha)] \\ &= r\rho(\cos(\theta+\alpha) + i\sin(\theta+\alpha)), \end{aligned}$$

where we used the sum-angle formulas in the last equation. So geometrically multiplication simply corresponds to a dilation (i.e. scaling) and a rotation. For instance multiplication by i corresponds to rotating the vector representing the complex number by $\pi/2$. To form a good number system we will also need to be be able to divide by complex numbers. For any $z_2 \neq 0$, we say that $w = z_1/z_2$ if $z_1 = wz_2$. We call w, the *quotient* obtained by dividing z_1 by z_2. Clearly, if $z_2 = 0$, then by property P1, the quotient cannot be well defined. We next see that the quotient *is* in fact well defined when dividing by non-zero complex numbers.

1.2.1 Conjugate, Absolute value, Argument.

To prove that a quotient always exists on dividing by a non-zero complex number, it is enough to have a formula for $1/z$, where $z \neq 0$. Let $z = x + iy$. Multiplying the numerator and denominator of $1/z$ by $x - iy$ (this is similar to rationalizing irrational denominators) we obtain

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}.$$

Note that in the last equation, the denominator is now a real number, and we already know how to divide by real non-zero numbers. The numerator, is called the *conjugate* of z and is denoted by

$$\bar{z} = x - iy.$$

Geometrically this amounts to reflection of the point representing z about the x-axis. Readers will notice that the denominator is the square of the distance of the point (x, y) from the origin. So we define the absolute value or the *length* of the complex number, denote by $|z|$ as

$$|z| = \sqrt{x^2 + y^2}.$$

This is of course the 'r' in the polar coordinate representation. Some basic properties of these operations are the following.

- $\overline{\overline{z}} = z$.
- $|\overline{z}| = |z|$.
- $|z| = 0 \implies z = 0$.

- $\overline{zw} = \bar{z}\bar{w}$, $|zw| = |z||w|$.
- $\overline{z+w} = \bar{z} + \bar{w}$.

Note that contrary to the conjugate function, the modulus function is not additive. Instead we have an inequality; see Theorem 1.2.1. With these notations in place, we can re-write the above statement as

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2},$$

for any $z \neq 0$. So we in summary we shown that multiplication has another property, that every no-zero number has *multiplicative inverse*. That is we have

P6 (multiplicative inverse.) For every $z \in \mathbb{C}$, $z \neq 0$ there exists a complex number $1/z = \bar{z}/|z|^2$ such that

$$z \cdot \frac{1}{z} = 1.$$

With the two operations of addition and multiplication satisfying these six axioms, the set of complex numbers become what is called as *field* by algebraists. In fact, the complex numbers form an *algebraically closed* field, which means that any polynomial with complex coefficients can be completely factorized using complex roots. Later in the course, somewhat remarkably, we will prove this statement in algebra using our complex analysis techniques. We will in fact give multiple proofs, not just one!

The 'θ' in the polar coordinates also has a name, and is called the *argument* of z, and denote by $\arg(z)$. Using the new notation, it is also easy to see that

$$\Re(z) = \frac{z+\bar{z}}{2},\ \Im(z) = \frac{z-\bar{z}}{2i}.$$

We can now define division by

$$\frac{z}{w} = \frac{z\bar{w}}{|w|^2}.$$

Integer Powers. Given any natural number $n \in \mathbb{N}$, we define z^n to be z multiplied to itself 'n' times. We also define $z^0 := 1$. For negative integers $-n$, we then define z^{-n} to be $1/z^n$ or the multiplicative inverse to z^n.

We end with an important inequality that will be crucial in most of the estimates.

Theorem 1.2.1 (Triangle inequality). *Let $z, w \in \mathbb{C}$. Then we have the following inequalities*

1. $$|z+w| \leq |z| + |w|,$$
   with equality if and only if $z = aw$ where $a \in \mathbb{R}$ i.e. z and w lie on the same line through the origin.

2. $$|z-w| \geq ||w| - |z||,$$
   with equality again if z and w lie on the same line through the origin.

Proof. 1. We first note that if $x, y, u, v \in \mathbb{R}$, then

$$(xu+yv)^2 \leq (x^2+y^2)(u^2+v^2),$$

with equality if and only there exists some $a \in \mathbb{R}$ such that $x = au$ and $y = av$. This follows from the elementary observation that the difference of the two sides in the above inequality equals $(xv-yu)^2$. Next, we have

$$|z+w|^2 = (z+w)\overline{(z+w)} = |z|^2 + |w|^2 + 2\Re(z\bar{w}).$$

So if we let $z = x + iy$ and $w = u + iv$, then

$$2\Re(z\bar{w}) = (x + iy)(u - iv) + (x - iy)(u + iv) = 2(xu + yv) \leq 2\sqrt{x^2 + y^2}\sqrt{u^2 + v^2} = 2|z||w|,$$

which completes the proof of the inequality. We have equality if and only if there exists a $a \in \mathbb{R}$ such that $x = au$ and $y = av$, or equivalently, $z = aw$.

2. For the second inequality, without loss off generality we may assume that $|z| \leq |w|$. Then by the first part we have

$$|w| = |z + w - z| \leq |z| + |w - z|.$$

Again we have equality if and only if there exists a real number b such that $w - z = bz$, or $w = (1 + b)z$.

$\square$

Lecture 2

Lecture-2: Limits, continuity and holomorphicity

2.1 Topology of the complex plane

A consequence of the triangle inequality discussed in the previous lecture is that the function

$$d(z, w) := |z - w|$$

defines a distance function on $\mathbb{C}$, and for simplicity, we denote the corresponding metric space by $(\mathbb{C}, |\cdot|)$. Given a $z_0 \in \mathbb{C}$, the *open* disc of radius R around z_0 is given by

$$D_R(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < R\}.$$

We now review a few standard definitions from topology. The *complement* of a set S, denoted by S^c is the set of all complex numbers NOT in S. Given any set $S \subset \mathbb{C}$, a point $p \in \mathbb{C}$ is a *limit* or an *accumulation* point if for any $r > 0$, the disc $D_r(p)$ has at least one point in common with S other than possibly p itself. A point $p \in S$ is said to be *isolated* if p is not a limit point of S. The *closure* of a set S, denoted by $\bar{S}$ is the union of S with all it's accumulation points. The *interior* of S, denoted by $\mathring{S}$, is the set of all points $p \in S$ such that $D_r(p) \subset S$ for some $r > 0$. The *boundary* of a set S is the set of points $p \in \mathbb{C}$ such that for all $r > 0$, the disc $D_r(p)$ contains at least one point from S and S^c. For instance the boundary of the open disc $D_r(p)$ is the circle of radius r centered at p.

A set S is called *open* is for any point $p \in S$, there exists a disc $D_r(p) \subset S$. That is each point has a neighborhood that is completely contained in the set. A set is called *closed* is it's complement is open. An equivalent definition (why are they equivalent?) is that a set is closed if and only if it completely contains it's boundary. So for any set S, the interior $\mathring{S}$ is the largest open set contained in S and the closure $\bar{S}$ is the smallest closed set containing S. A basic property of open and closed sets is the following.

Proposition 2.1.1.
- *Arbitrary union (possibly infinite) of open sets is again open. Finite intersection of open sets is open.*
- *Arbitrary intersection of closed sets is close. Finite union of closed sets is closed.*

Given a sequence $\{z_n\}$ we say that it *converges* to $p \in \mathbb{C}$ if for all $\varepsilon > 0$, there exists an N such that

$$|z_n - p| < \varepsilon.$$

Proposition 2.1.2. *$z_n \to p$ if and only if $\Re(z_n) \to \Re(p)$ and $\Im(z_n) \to \Im(p)$.*

This is a consequence of the fact that for any $z \in \mathbb{C}$,

$$\max(|\Re(z)|, |\Im(z)|) \le |z| \le \sqrt{2}\max(|\Re(z)|, |\Im(z)|).$$

A disadvantage of the above definition of convergence is that one needs to know the limit a priori, to even decide if a sequence is converging. A convenient alternative is of course the notion of a *Cauchy sequence.* Recall that a sequence z_n is said to be Cauchy if for all $\varepsilon > 0$, there exists an $N > 0$ such that for all $n, m > N$ we have

$$|z_n - z_m| < \varepsilon.$$

It is easy to see (prove it!) that every convergent sequence is Cauchy. Conversely, we have the following fundamental fact.

Theorem 2.1.3. *Every Cauchy sequence in* $\mathbb{C}$ *converges. That is,* $(\mathbb{C}, |\cdot|)$ *is a complete metric space.*

. The theorem follows from the proposition above and the fact that real numbers from a complete metric space. Recall that a set is called *compact* if every open cover has a finite sub-cover. A consequence of completeness is the following useful characterization of compact sets in $\mathbb{C}$.

Theorem 2.1.4. *The following are equivalent for a subset* $K \subset \mathbb{C}$.

1. K *is compact.*
2. K *is closed and bounded.*
3. K *is* sequentially compact. *That is, any infinite sequence* $\{z_n\} \subset K$ *has an accumulation point* $p \in K$.

The last notion we need is that of a connected set. A subset $S \subset \mathbb{C}$ is called connected, if

$$S = (U \cap S) \cup (V \cap S)$$

for some disjoint open sets U and V. If S itself is open, this reduces to saying that S cannot be written as the union of two disjoint open sets. An open, connected subset is called a *region.* We have the following elementary characterization of regions.

2.2 Functions on the complex plane

Let $S \subset \mathbb{C}$ be a subset. A function $f : S \to \mathbb{C}$ is a rule that assigns *unique* complex number, denoted by $f(z)$ to every number $z \in S$. The set S is called the *domain* of the function, and

$$f(S) := \{f(z) \mid z \in S\},$$

is called the *range.* The *pre-image* of a set $T \subset \mathbb{C}$, denoted by $f^{-1}(T)$ is the subset of S defined by

$$f^{-1}(T) = \{z \in S \mid f(z) \in T\}.$$

A function is called *injective* or *one-one* if the pre-image of every point in the range consists of exactly one point, i.e

$$f(z) = f(w) \implies z = w.$$

It is said to *surjective* or *onto* if the range is all of $\mathbb{C}$.

We say that the limit of $f(z)$ as z tends towards p is L, and denote it by

$$\lim_{z\to p} f(z) = L,$$

if the following holds - For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|z - p| < \delta,\ z \in S, \implies |f(z) - L| < \varepsilon.$$

We say that f is *continuous* at $p \in S$ if

$$\lim_{z \to p} f(z) = f(p).$$

f is simply called continuous if it is continuous at all points in its domain. We then have the basic fact.

Theorem 2.2.1. *$f : S \to \mathbb{C}$ is continuous if and only if $Re(f)$ and $Im(f)$ a continuous as real valued functions of two variables.*

So as far as topology, which is the study of continuous functions, is concerned, there is no difference between $\mathbb{C}$ and $\mathbb{R}^2$. With this remark, the following properties follow easily from what is already known about multivariable functions.

Theorem 2.2.2. *Consider a function $f : \Omega \to \mathbb{C}$, where Ω is open.*

1. *It is continuous if and only if $f^{-1}(U)$ is open for any open set $U \subset \mathbb{C}$.*
2. *It is continuous if and only if $f^{-1}(K)$ is closed for every closed set $K \subset \mathbb{C}$.*
3. *It is continuous at $p \in \Omega$ if and only if for any sequence $\{z_n\}$ such that $z_n \to p$, we have*

$$\lim_{z_n \to p} f(z_n) = f(p).$$

4. *If f is continuous, then for any compact subset $K \subset \Omega$, $f(K)$ is compact.*
5. *If f and $g : \Omega \to \mathbb{C}$ are continuous at p then so are $f \pm g$ and fg. If $g(p) \neq 0$, then f/g is also continuous at p.*
6. *if f is continuous at p, and $g : f(\Omega) \to \mathbb{C}$ is continuous at $f(p)$, then the composition $g \circ f$ is also continuous at p.*

Example 2.2.3. *The function $f(z) = z^n$, where n is an integer, is continuous. To see this, note that*

$$z^n - p^n = (z - p)(z^{n-1} + z^{n-2}p \cdots + p^{n-1})$$

A polynomial is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_k \in \mathbb{C}$ for $k = 0, 1, \cdots, n$. Then by the fact that sums of contiuous functions are continuous, it follows that polynomials are continuous at all points. A rational function is a quotient of two polynomials

$$R(z) = \frac{p(z)}{q(z)},$$

wherever q is non-zero. At all such points, by the quotient rule above, a rational function is also continuous.

Example 2.2.4. *The function $f(z) = \bar{z}$ is continuous. Similarly, the function $g(z) = |z|$ is also continuous.*

Example 2.2.5. Arg(z) in not continuous on $\mathbb{C}$. *Recall that if $z = x + iy$, then $\arg(z)$ is defined as the unique angle between $(-\pi, \pi]$ that the line joining the origin to (x, y) makes with the positive x-axis. Now consider any point on the the negative x-axis, say $z = -1$. Then the function is not continuous. The reason being that if $z_n \to -1$ with $\Im(z_n) > 0$, then $\arg(z_n) \to \pi$, while if $\Im(z_n) < 0$, then $\arg(z_n) \to -\pi$.*

2.2.1 Path connected sets in $\mathbb{C}$.

As a special case, we could take S to be an interval in $\mathbb{R}$ thought of as a subset of the x-axis in $\mathbb{C}$. A *path* is defined to be a continuous function $\gamma : I \to \mathbb{R}$, where I is an interval. We then have the following useful characterization of regions in $\mathbb{C}$.

Proposition 2.2.6. *Let $\Omega \subset \mathbb{C}$ be an open subset. Then $\Omega \subset \mathbb{C}$ is a region if and only if Ω is path connected, ie. for any $z_0, z_1 \in \Omega$, there exists a continuous map $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.*

Proof. Suppose Ω is a region. Fix $z_0 \in \Omega$, and let

$$A = \{z \in \Omega \mid \text{there exists a path in } \Omega \text{ connecting } z \text{ to } z_0\}.$$

Note that $z_0 \in A$, and hence A is non-empty. Since Ω is open, and clearly every disc is path connected, A is open. Now we claim that $A^c := \Omega \setminus A$ is also open. To see this, if A^c is non-empty we have some $w \in A^c$. Then since Ω is open, there is disc $D_r(w) \subset \Omega$. Clearly, $D_r(w) \cap A = \Phi$, for if, $z_1 \in D_r(w) \cap A$, then one could simply connect z to z_0, by connecting z to z_1, and z_1 to z_0, contradicting our assumption that $w \in A^c$. This shows that $D_r(w) \subset A^c$, and hence A^c is open. But since Ω is connected and A is non-empty, this forces $A^c = \Phi$. Conversely, suppose Ω is path connected, but is disconnected. Then we can write $\Omega = A \cup A^c$, where both A and A^c are open and non-empty. ☐

2.2.2 Convergence of functions

There are two notions of convergence, that of point-wise, and uniform convergence. We say that

- the sequence of functions $f_n : \Omega \to \mathbb{C}$ converges *point-wise* to $f : \Omega \to \mathbb{C}$, if for every $z \in \Omega$, the sequence $f_n(z) \to f(z)$. Or equivalently, given any $\varepsilon > 0$, and any $z \in \Omega$, there exists an $N > 0$, possibly depending both on ε and z, such that

$$n > N \implies |f_n(z) - f(z)| < \varepsilon.$$

- the sequence of functions is said to converge *uniformly* if given any $\varepsilon > 0$, there exists an $N > 0$ depending only on ε such that for all $z \in \Omega$ and $n > N$ we have that

$$|f_n(z) - f(z)| < \varepsilon.$$

Theorem 2.2.7. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of continuous functions which converge uniformly to $f : \Omega \to \mathbb{C}$, then f itself is continuous.*

2.3 Holomorphic functions

Let $\Omega \subset \mathbb{C}$ be an open subset. We say that a function $f : \Omega \to \mathbb{C}$ is *complex differentiable* at $z = p \in \Omega$, if the limit

$$f'(p) = \frac{d}{dz}\Big|_{z=p} f(z) = \lim_{z \to p} \frac{f(z) - f(p)}{z - p} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

exists and is finite. The limit, denoted by $f'(p)$, is then called the derivative (or the complex derivative if the context is not clear) of f at p. The function is called *holomorphic*, if it is complex differentiable at all points in the domain. We will denote the set of holomorphic functions on Ω by $\mathcal{O}(\Omega)$. Note that formally this definition is identical to the one for real valued functions of one variable. We have also seen that functions of a complex variable can be thought of as vector fields in two variables. In multivariable calculus, there is already a notion of derivatives of such functions. A natural question is to ask for the relation between these two notions. We will return to this question shortly. But first we collect some basic properties of holomorphic functions, the proofs of which are identical to those in for differentiable functions of one real variable.

Theorem 2.3.1. *Let $f, g : \Omega \to \mathbb{C}$ be differentiable at z, and $a, b \in \mathbb{C}$*

1. *If f is constant in a neighbourhood of z, then $f'(z) = 0$.*

2. *(Linearity)* $af + bg$ *is complex differentiable at* z*, and*

$$[af + bg]'(z) = af'(z) + bg'(z)$$

3. *(Product rule)* fg *is complex differentiable at* z *and*

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

4. *(Quotient rule) If* $g'(z) \neq 0$*, then* f/g *is complex differentiable at* z *and*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}.$$

5. *(Chain rule) If* h *is complex differentiable at* $f(z)$*, then* $h \circ f$ *is complex differentiable at* z *and*

$$[h \circ f]'(z) = h'(f(z)) \cdot f'(z).$$

The main theme of the course is that holomorphicity imposes severe restrictions on the functions under consideration. Just to get our feet wet, we start with the following elementary observation.

Proposition 2.3.2. *If a function* f *is complex differentiable at* $z = a$ *then it is automatically continuous at* $z = a$*.*

Proof. We proceed by contradiction. So suppose f is not continuous at a. Then there exists an $\varepsilon > 0$ and a sequence $z_n \to a$ such that $|f(z_n) - f(a)| > \varepsilon$. By holomorphicity, there exists an N such that

$$\left|\frac{f(z_n) - f(a)}{z_n - a} - f'(a)\right| < 1,$$

whenever $n > N$. Or equivalently, that

$$|f(z_n) - f(a) - (z_n - a)f'(a)| < |z_n - a|.$$

By the triangle inequality

$$\begin{aligned}\varepsilon < |f(z_n) - f(a)| &= |f(z_n) - f(a) - (z_n - a)f'(a) + (z_n - a)f'(a)| \\ &< |f(z_n) - f(a) - (z_n - a)f'(a)| + |(z_n - a)f'(a)| \\ &< |z_n - a|(1 + |f'(a)|)\end{aligned}$$

Suppose now N is chosen large enough so that

$$|z_n - a| < \frac{\varepsilon}{2(1 + |f'(a)|)},$$

then the above chain of inequality yields

$$\varepsilon < |f(z_n) - f(a)| < \frac{\varepsilon}{2},$$

which is absurd. □

Example 2.3.3. *As a first example, we compute the derivative of* $f(z) = z^n$*, where* n *is an integer. We first assume that* $n \geq 1$*. Then it is easy to see (try to prove it!) that*

$$z^n - a^n = (z - a)(z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1}).$$

So then

$$f'(a) = \lim_{z\to a} \frac{z^n - a^n}{z-a} = \lim_{z\to a} z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1} = na^{n-1}.$$

For negative integers n we can apply quotient rule to again obtain the same formula. So for integers n we have that

$$\boxed{\frac{d}{dz}\Big|_{z=a} z^n = na^{n-1}}$$

Example 2.3.4. Polynomials and Rational functions. *Recall that polynomials are functions of the type*

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0.$$

Then by the above theorem, such functions are holomorphic. Moreover, by the above calculation the derivative of a degree n polynomial is again a polynomial, but of degree $n-1$. Recall also that rational functions are quotients of two polynomials. By the quotient rule, these are holomorphic at all points where the denominator does not vanish. That is, if

$$R(z) = \frac{p(z)}{q(z)},$$

then $R(z)$ is holomorphic at $z = a$ if and only if $q(a) \neq 0$.

Example 2.3.5. *The function $f(z) = \bar{z}$ is not holomorphic. To see this, consider the difference quotient*

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h}.$$

Then if $h \to 0$ along the real axis i.e. $h \in \mathbb{R}$, this difference quotient is 1. On the other hand if $h \to 0$ along the imaginary axis, i.e. $h = ik$ where $k \in \mathbb{R}$, then this quotient is always -1. So the limit cannot exist.

Example 2.3.6. *The function $f(z) = |z|$ is not holomorphic. By the product rule, it is enough to show that $g(z) = |z|^2$ is not holomorphic. The difference quotient is*

$$\frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h} = \frac{z\bar{h} + \bar{z}h + |h|^2}{h} = z\frac{\bar{h}}{h} + z + \bar{h}.$$

The limit of the last two terms as $h \to 0$ is $\bar{z}$, but, as we saw in the previous example, the limit of the first term does not exist. So $|z|^2$, and hence $|z|$, is not differentiable.

Lecture 3

Power series

A power series centered at $z_0 \in \mathbb{C}$ is an expansion of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n, z \in \mathbb{C}$. If a_n and z are restricted to be real numbers, this is the usual power series that you are already familiar with. A priori it is only a formal expression. But for certain values of z, lying in the so called disc of convergence, this series actually converges, and the power series represents a function of z. Before we discuss this fundamental theorem of power series, let us review some basic facts about complex series, and series of complex valued functions.

3.1 Infinite series of complex numbers: A recap

A series is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n,$$

where $a_n \in \mathbb{C}$ for all n. We say that the series converges to S, and write

$$\sum_{n=0}^{\infty} a_n = S,$$

if the sequence of partial sums

$$S_N = \sum_{n=0}^{N} a_n$$

converges to S. We need the following basic fact.

Proposition 3.1.1. *If $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. We will show that S_N forms a Cauchy sequence if $\sum_{n=0}^{\infty} |a_n|$ converges. Then the theorem will follow from the completeness of $\mathbb{C}$. If we let $T_N = \sum_{n=0}^{N} |a_n|$, then $\{T_N\}$ is a Cauchy sequence by the hypothesis. Then by triangle inequality

$$|S_N - S_M| = \Big| \sum_{n=N+1}^{M} a_n \Big| \leq \sum_{n=N+1}^{M} |a_n| = |T_M - T_N|.$$

Now, given $\varepsilon > 0$, there exists a $K > 0$ such that for all $N, M > K$, $|T_M - T_N| < \varepsilon$. But then $|S_M - S_N| \leq \varepsilon$, and so $\{S_N\}$ is also Cauchy. □

We say that $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges. Next suppose $f_n : \Omega \to \mathbb{C}$ are complex functions, we say that $\sum_{n=0}^{\infty} f_n(z)$ converges *uniformly* if the corresponding sequence of partial sums

$$S_N(z) = \sum_{n=0}^{N} f_n(z)$$

converges uniformly.

Proposition 3.1.2 (Weierstrass' M-Test)**.** *Suppose $f_n : \Omega \to \mathbb{C}$ is a sequence of complex functions, and $\{M_n\}$ is a sequence of positive real numbers such that*

- *$|f_n(z)| \leq M_n$ for all n and all $z \in \Omega$.*
- *$\sum_{n=0}^{\infty} M_n$ converges.*

Then $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly.

Proof. Like before, this time we show that the sequence of partial sums $\{S_N(z)\}$ is uniformly Cauchy. But again by triangle inequality if T_N denotes the N^{th} partial sum of $\sum M_n$, then

$$|S_N(z) - S_M(z)| \leq \sum_{n=N+1}^{M} |f_n(z)| \leq \sum_{n=N+1}^{M} M_n = |T_M - T_N|,$$

for all $z \in \Omega$. Since the right side does no depend on z, given $\varepsilon > 0$, one can make $|S_N(z) - S_M(z)| < \varepsilon$ by choosing $N, M > K$ where K can be chosen independent of z. □

3.2 Convergence of power series

By convergence of the power series, we mean the following. Consider the truncations of the power series at the N^{th} term, also called the N^{th} partial sum -

$$s_N(z) = \sum_{n=0}^{N} a_n (z - z_0)^n.$$

We say that the power series converges (unifomrly) if the sequence of functions $\{s_N(z)\}$ converges (uniformly). We say that the series converges absolutely if the sequence of functions

$$\sum_{n=0}^{N} |a_n||z|^n$$

converges. It is well known, and not difficult to see, that absolute convergence implies convergence. The fundamental fact is the following.

Theorem 3.2.1 (Fundamental theorem of power series)**.** *There exists a $0 \leq R \leq \infty$ such that*

- *If $|z - z_0| < R$, the series*

  $$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

  converges absolutely.

- *For any compact set $K \subset D_R(z_0)$, the absolute convergence is actually uniform.*
- *If $|z - z_0| > R$, then the series diverges.*

Moreover, R can be computed using the Cauchy-Hadamard formula:

$$R = \frac{1}{\limsup |a_n|^{1/n}}.$$

The number R is called the *radius* of convergence, and the domain $D_R = \{z \mid |z - z_0| < R\}$ is called the *disc* of convergence. Recall that for a sequence $\{b_n\}$ of real numbers,

$$L = \limsup b_n$$

if the following two conditions hold

- For all $\varepsilon > 0$, and all $N > 0$, there exists $n > N$ such that
$$b_n > L - \varepsilon.$$
- For all $\varepsilon > 0$, there exists an N such that for all $n > N$,
$$b_n < L + \varepsilon.$$

Remark 3.2.2. It is not difficult to show that if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists, then it is equal to $1/R$. On many occasions the limiting ratio is easier to calculate.

Remark 3.2.3. Suppose $0 < R < \infty$ is the radius of convergence of the above power series. The by the theorem, the series converges on the *open* disc $|z - z_0| < R$. The behavior of the series at points on boundary however is subtle as the examples below indicate.

Example 3.2.4. *The power series*

$$\sum_{n=0}^{\infty} z^n$$

has radius of convergence 1. In fact it easy to see that on $|z| < 1$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

Observe that on the unit disc $|z^n| = |z|^n = 1$, and so by the divergence test, the series cannot converge at any boundary point. On the other hand, the left hand side is defined and holomorphic at all points $z \neq 1$ even though the power series is only defined inside the unit disc. We then say that the holomorphic function $1/1 - z$ is an analytic continuation of the power series to the domain $\mathbb{C} \setminus \{z = 1\}$. We will say more about analytic continuation towards the end of the course. We remark that the following misleading formula often appears in popular culture (most notably in a video on the youtube channel - Numberphile, an otherwise decent math channel), many times accompanied with a quote with the effect that "Oh look - math is magical!":

$$1 - 1 + 1 - 1..... = \frac{1}{2}.$$

The formula as stated is of course junk since the left hand side is clearly a divergent series. But there are ways of interpreting the left hand side. For instance, the left hand side is in fact Cesaro summable, which is a *generalization of usual infinite summation in that a convergent series is also Cesaro summable and the Cesaro sum equals the sum of the series. In this case, the Cesaro sum does turn out to be* $1/2$. *A more fundamental way (at least in my opinion) of interpreting the left hand side, as precisely the analytic continuation of* $\sum_{n=0}^{\infty} z^n$ *to* $z = -1$. *Then as remarked above, this analytic continuation is given by* $1/(1-z)$ *which of course equals* $1/2$ *at* $z = -1$. *Another example of such misleading propogation of math, especially in India, is that Ramanujan proved the "miraculous" identity that*

$$1 + 2 + 3 + \cdots = \frac{-1}{12}.$$

We'll see later in the course that the left hand side should in fact be replaced by the analytic continuation of the series $\sum_{n=1}^{\infty} n^{-s}$ *to* $s = -1$. *The infinite seres is a priori only defined on the region* $Re(s) > 1$, *but can be analytically continued to* $\mathbb{C} \setminus \{1\}$, *and this if of course the famous* $\zeta(s)$ *of Riemann.. We'll then compute that* $\zeta(-1) = -1/12$*!*

Example 3.2.5. *Consider the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n}.$$

Again, it is easy to see that the radius of convergence is 1. At $z = 1$ *this is the usual harmonic series, and is divergent. It turns out in fact, that this series converges for all other points on* $|z| = 1, \ z \neq 1$. *This follows from the following test due to Abel, which we state without proof.*

Lemma 3.2.6 (Abel's test)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n z^n.$$

Suppose

- $a_n \in \mathbb{R}$, $a_n \geq 0$.
- $\{a_n\}$ *is a decreasing sequence such that* $\lim_{n\to 0} a_n = 0$.

Then the power series converges on $|z| = 1$ *except possibly at* $z = 1$.

Clearly the series in the example above satisfies all the hypothesis, and hence is convergent at all points on $|z| = 1$ *except at* $z = 1$.

Example 3.2.7. *Next, consider the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n^2}.$$

Again, the radius of convergence is 1, and again by Abel's test the power series is convergent on $|z| = 1$ *except possibly at* $z = 1$. *But at* $z = 1$, *the series is clearly convergent, for instance by the integral test. So in this example the power series is convergent on the entire boundary.*

Example 3.2.8. *Finally consider the power series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

To find the radius of convergence, we use the ratio test. Denoting $a_n = 1/n!$, *we see that*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So $R = \infty$. Notice that if z is a real number then this is the usual Taylor expansion of the exponential function. Inspired by this, we define the complex exponential function, $exp(z) : \mathbb{C} \to \mathbb{C}$ by

$$exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We will study this function in more detail in the next lecture.

3.3 Holomorphicity of power series

From the previous theorem, since the convergence is uniform on comapct subsets of the disc of convergence, it is clear that a power series represents a continuous function. In fact, much more is true.

Theorem 3.3.1. *Consider the function defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

on the disc of convergence $D_R = \{z \mid |z - z_0| < R\}$ where $0 < R \leq \infty$. Then $f(z)$ is holomorphic on D_R with

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^n.$$

Proof. Without loss of generality we can assume $z_0 = 0$. Firstly, observe that since $\lim_{n\to\infty} n^{1/n} = 1$, the power series $\sum_{n=1}^{\infty} n a_n (z - z_0)^n$ also has radius of convergence R. To prove the theorem, we need to show that for any $p \in D_R(0)$,

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h} = \sum_{n=1}^{\infty} n a_n p^n.$$

Or equivalently, given any $\varepsilon > 0$, we need to find a $\delta > 0$ such that

$$|h| < \delta \implies \Big| \frac{f(p+h) - f(p)}{h} - \sum_{n=1}^{\infty} n a_n p^n \Big| < \varepsilon \tag{3.1}$$

Let us denote by

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \; E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n,$$

the N^{th} partial sum, and the N^{th} error term respectively, so that

$$f(z) = S_N(z) + E_N(z).$$

Then since the partial sums are polynomials, they are holomorphic, and in fact

$$\lim_{N\to\infty} S_N'(z) = \sum_{n=1}^{\infty} n a_n z^n, \tag{3.2}$$

where the convergence is uniform on compact subsets of $D_R(0)$ Now suppose $|p| < r < R$, then for any N, we can break the difference that we need to estimate into three parts -

$$\Big| \frac{f(p+h) - f(p)}{h} - \sum_{n=1}^{\infty} n a_n p^n \Big| = \Big| \frac{S_N(p+h) - S_N(p)}{h} - S_N'(p) \Big|$$

$$+ |S_N'(p) - \sum_{n=1}^{\infty} n a_n p^n| + \Big|\frac{E_N(p+h) - E_N(p)}{h}\Big|$$

Since S_N, being a polynomial, is holomorphic, there exists a $\delta > 0$ so that for $|h| < \delta$, the first term is smaller than $\varepsilon/3$. Similarly, by equation (3.2), the second term can be made smaller than $\varepsilon/3$ by choosing N big enough. So all that remains is to control the error term. Using the factorization $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$, we have

$$\begin{aligned}\Big|\frac{E_N(p+h) - E_N(p)}{h}\Big| &\leq \sum_{n=N+1}^{\infty} |a_n| \Big|\frac{(p+h)^n - p^n}{h}\Big| \\ &\leq \sum_{n=N+1}^{\infty} |a_n| |((p+h)^{n-1} + \cdots + p^{n-1})|\end{aligned}$$

But if $|h| < \delta$ for sufficiently small δ (in particular if $\delta < r - |p|$), then $|p + h| \leq |p| + |h| < r$, and so

$$\Big|\frac{E_N(p+h) - E_N(p)}{h}\Big| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.$$

But this is the tail of the series $\sum n|a_n|r^{n-1}$ which converges for $r < R$, so we can also make this term smaller than $\varepsilon/3$ by choosing N big enough. This shows that we can find δ small enough so that (3.1) is satisfied.

□

Notice that the derivative is again a power series with the same radius of convergence. So applying the above theorem inductively we obtain -

Corollary 3.3.2. *A power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is infinitely complex differentiable in it's disc of convergence. Moreover, the derivatives can be computed by successive term-wise differentiation:*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k}.$$

In particular, the coefficients of the power series are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

We say that a function $f : \Omega \to \mathbb{C}$ is *analytic* if for every $p \in \Omega$, there exists an $r = r(p) > 0$ and a sequence of numbers $\{a_n = a_n(p)\}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - p)^n$$

for every $z \in D_r(p)$. A priori, it is not quite clear that if a function is represented by a power series expansion on a disc of convergence $D_R(p)$ then it is automatically analytic. For instance, it is not clear if there should be a power series expansion around any other point $q \in D_R(p)$. The next proposition answers this question in the afirmative.

Proposition 3.3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with radius of convergence R. Then f is analytic on $D_R(z_0)$. In fact, for every $p \in D_R(z_0)$, and every $z \in D_r(p)$ where $r := R - |p - z_0|$, we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z - p)^n.$$

Proof. We use the binomial expansion. As before, without loss of generality, we assume that $z_0 = 0$. Writing $z = z - p + p$, and applying the binomial theorem, we see that on $|z - p| < R - |p|$ (since $|z| < R$), we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - p + p)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} p^{n-k} (z-p)^k \\ &= \sum_{k=0}^{\infty} b_k (z-p)^k, \end{aligned}$$

where

$$b_k = \sum_{n=k}^{\infty} \binom{n}{k} a_n p^{n-k} = \frac{f^{(k)}(p)}{k!}$$

by Corollary 3.3.2. □

Remark 3.3.4. In some cases the power series on the right in the conclusion of Proposition 3.3.3 might have a larger radius of convergence than $R - |q - p|$. In such cases, the new power will define an analytic continuation of f. For instance, let us consider the power series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

on $|z| < 1$ with $p = 0$. Let $q = -1/2$. Then if $|z + 1/2| < 1 - 1/2 = 1/2$, we have

$$\frac{1}{1-z} = \frac{1}{3/2 - (z + 1/2)} = \frac{2}{3} \cdot \frac{1}{1 - 2(z + 1/2)/3} = \frac{2}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} \left(z + \frac{1}{2}\right)^n.$$

On the other hand, it can be easily seen that the power series on the right has a radius of convergence $R = 3/2$, and hence defines an extension of the original power series in the new region $|z + 1/2| < 3/2$. This was Weierstrass' method of analytically continuing holomorphic functions.

Corollary 3.3.5. *[Principle of analytic continuation for power series] Let f be an analytic function on a connected open set Ω. If there exists a point $p \in \Omega$ such that $f^{(n)}(p) = 0$ for all $n \in \mathbb{N}$, $f \equiv 0$ on all of Ω. In particular, the conclusion is true if there is an open set $U \subset \Omega$ such that $f\big|_U \equiv 0$.*

Proof. Let $S = \{z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \cdots\}$. Then by the continuity of f, S is closed in Ω (ie. all limit points of S *in* Ω, are also contained in S). Also, S is non-empty by the hypothesis. Now suppose $q \in S$. Since the function is analytic, there is an open disc $D_r(q)$ on which

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(q)}{n!} (z - q)^n = 0,$$

since $q \in S$. But then $D_r(q) \subset S$, and so S is open in Ω. But then since Ω is connected, this forces $S = \Omega$, and in particular, $f \equiv 0$ on Ω. □

Appendix: Multiplication and composition of power series

Given two power series

$$\sum_{n=0}^{\infty} a_n z^n, \sum_{n=0} b_n z^n,$$

their product can be defined, at least formally, in the following way-

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n z^n,\end{aligned}$$

where c_n is given by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This product goes by the name of *Cauchy product.* The main theorem, which we state without proof, is the following.

Theorem 3.3.6. *If the radius of convergence of two series centered at z_0 is R_1 and R_2 respectively, then their product power series has a radius of convergence that is at least* $\min(R_1, R_2)$.

Proof of a slightly general version, applicable for any infinite series, can be found on the wiki article `https://en.wikipedia.org/wiki/Cauchy_product#Convergence_and_Mertens.27_theorem`.

Lecture 4

The exponential functions

4.1 The exponential functional

Last lecture, we defined the *complex exponential function* by the power series

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We saw that the power series has infinite radius of convergence, and hence defines a function on the entire complex plain. In fact by the theorem from last lecture, we now know that the exponential function is holomorphic on the entire plane. Such functions, that are holomorphic on the entire complex plane, are called *entire functions*. As we saw in the previous lecture, to find the complex derivative, it is enough to differentiate term-wise:

$$\frac{d}{dz}\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d}{dz}z^n = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

To see the last equality just replace $n-1$ by n in the penultimate term. So we see that

$$\frac{d}{dz}\exp(z) = \exp(z).$$

In fact we'll see later that this property characterizes the exponential function. But first, we collect some important properties of the exponential function.

Theorem 4.1.1. *1.* $\exp(0) = 1$.

2. For any complex numbers z, w we have that

$$\exp(z+w) = \exp(z)\exp(w),$$

and in particular $\exp(-z) = [\exp(z)]^{-1}$.

3. $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$.

4. The restriction of $\exp(z)$ *to* $\mathbb{R}$ *is a positive strictly increasing function. For* $x \in \mathbb{R}$, $e^x = 1$ *if and only if* $x = 0$. *Moreover,* $\lim_{x\to\infty}\exp(x) = \infty$ *and* $\lim_{x\to-\infty}\exp(x) = 0$. *In particular,* $e^x : \mathbb{R} \to \mathbb{R}_+$ *is a bijective function.*

Proof. 1. This is trivial from the definition.

2. This follows from the product formula for power series and the binomial theorem. The left hand side of the equation is
$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}.$$
But then
$$\frac{(z+w)^n}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{k=0}^{n} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}.$$
This is exactly the n^{th} coefficient of the Cauchy product, and the result follows.

3. Suppose $\exp(p) = 0$ for some $p \in \mathbb{C}$. Since $f(z) = \exp(z)$ is an analytic function with an infinite radius of convergence at 0, we also have the following representation formula on all of $\mathbb{C}$:
$$\exp(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n.$$
But since $f^{(n)}(p) = \exp(p) = 0$ for all n, this would mean that $\exp(z) = 0$ for all z, clearly contradicting (1).

4. Since $\exp(z) \neq 0$ and $\exp(0) = 1$, by the intermediate value theorem the function $f(x) = \exp(x)$ defined on $\mathbb{R}$ is strictly positive. Moreover, since $f(0) = 1$ and $f'(x) = e^x > 0$, the function $f(x)$ is strictly increasing, and hence $f(x) > 1$ for all $x > 0$. Then by the mean value theorem $f(x) - 1 > x$ for all $x > 0$. From this it follows that $\lim_{x\to\infty} f(x) = \infty$. Then property (2) implies that $\lim_{x\to-\infty} f(x) = 0$. By the intermediate value theorem $e^x : \mathbb{R} \to \mathbb{R}_+$ is then a surjective map, and hence a bijection.

$\square$

Remark 4.1.2. Due to property (2) and our familiarity with working with exponents from high-school, from now one we adopt the more suggestive notation $\exp(z) = e^z$.

Theorem 4.1.3. *There is a unique holomorphic function $f : \mathbb{C} \to \mathbb{C}$ satisfying*
$$\begin{cases} f'(z) = f(z) \\ f(0) = 1. \end{cases}$$

Proof. The proof uses the fact that if a holomorphic function has complex derivative identically zero, then the function has to be a constant. We will prove this fact later in the course. Assuming this, consider the function
$$g(z) = e^{-z} f(z).$$
Then by the Chain rule, since $f'(z) = f(z)$ we see that
$$g'(z) = e^{-z}(-f(z) + f'(z)) = 0.$$
Hence $g(z)$ is a constant. But by the initial condition we see that $g(0) = 1$. On the other hand by the first property in Theorem 4.1.1, $e^{-z} = 1/\exp(z)$, and so $f(z) = \exp(z)$. $\square$

4.2 Trigonometric functions

We can now analogously define the functions *sine* and *cosine* using power series:
$$\cos z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!}$$

$$\sin z = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!}$$

It is easy to check that the radius of convergence of both the power series is infinity, and hence they define entire functions, just like the exponential function. In fact an easy computation also shows that

$$\frac{d}{dz} \cos z = -\sin z, \ \frac{d}{dz} \sin z = \cos z.$$

The same computation then gives the following generalized Euler identity.

Proposition 4.2.1 (Generalized Euler identity)**.** *For any* $z \in \mathbb{C}$,

$$e^{iz} = \cos z + i \sin z.$$

Proof. This follows trivially from the following observations:

$$i^n = \begin{cases} (-1)^m, \ n = 2m \\ (-1)^m i, \ n = 2m+1. \end{cases}$$

So then by definition

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{z^{2m+1}}{(2m+1)!}.$$

Euler's identity then follows from the observation that the two series on the right are simply the Maclaurin series for sine and cosine respectively. ☐

Remark 4.2.2. Polar coordinates. The Euler identity can be used to give a third representation of complex numbers in terms of the exponential function. Namely, for $z \in \mathbb{C}$, let $r = |z|$ and $\theta = \arg z$. Then we have seen that

$$z = r \cos \theta + ir \sin \theta.$$

So by the Euler identity, we have the representation

$$z = re^{i\theta}.$$

This is sometimes very useful in computations.

Next, we collect some properties of the sine and cosine functions. These can be proved using the generalized Euler identity, and the analogous properties of the exponential function.

Theorem 4.2.3. *The sine and cosine function satisfy the following.*

1. $\sin(0) = 0$, $\cos(0) = 1$, *and for all* $z \in \mathbb{C}$ *we have*

$$\sin(-z) = -\sin z \textit{ and } \cos(-z) = \cos(z).$$

2. *For* $z, w \in \mathbb{C}$,

$$\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w$$
$$\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w.$$

3. *For all* $z \in \mathbb{C}$,

$$\sin^2 z + \cos^2 z = 1.$$

Proof. 1. This follows easily from the definitions.

2. It follows from the definition and property (1) above that
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$
The sum-angle fomrulae then follow from this and property (2) in Theorem 4.1.1.

3. This follows from the sum angle properties and property (1) above. An independent proof can be given by observing that the derivative of $f(z) = \sin^2 z + \cos^2 z$ is identically zero, hence $f(z)$ is a constant function, and hence equal to $f(0) = 1$.

□

One can also define the other trigonmetric functions $\tan z$, $\cot z$, $\sec z$ and $\csc z$ in the usual way.

4.3 Periodicity and the definition of π.

Theorem 4.3.1. *There exists a smallest positive real number which we denote by π such that $e^{2\pi i} = 1$. Moreover, $e^{iz} = 1$ if and only if $z = 2n\pi$ for some $n \in \mathbb{Z}$.*

The above theorem can be taken as the *definition* of the number π. As an immediate consequence of the above theorem and Euler's identity, we have the following.

Corollary 4.3.2. *For all $z \in \mathbb{C}$ and all $n \in \mathbb{Z}$,*
$$e^{z+2n\pi i} = \exp(z), \ \sin(z + 2n\pi) = \sin(z), \ \textit{and} \ \cos(z + 2n\pi) = \cos z.$$
In particular,
$$\sin(2n\pi) = \cos\left(\frac{2n+1}{2}\pi\right) = 0$$
for all n.

Proof of Theorem 4.3.1. We first prove that there exists a real number τ such that $e^{i\tau} = 1$. To see this, note that by property (3) above, $-1 \leq \sin(x), \cos(x) \leq 1$ for all $x \in \mathbb{R}$. Then by the mean value theorem, it is easy to see that for all $x > 0$,
$$\sin x < x, \text{ and } \cos x > 1 - \frac{x^2}{2}.$$
Once again by an application of the mean value theorem, since $(\sin x)' = \cos(x) > 1 - x^2/2$, we see that
$$\sin(x) > x - \frac{x^3}{6}.$$
Finally, by yet another application of the mean value theorem we obtain
$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24},$$
for all $x > 0$. Putting $x = \sqrt{3}$, we see that $\cos\sqrt{3} < 0$, and hence by the intermediate value theorem, there exists a $x_0 \in (0, \sqrt{3})$ such that $\cos(x_0) = 0$. Let $\tau = 4x_0$. Then by the sum, angle formulae, $\cos(\tau) = 1$ and $\sin(\tau) = 0$, and hence by the Euler identity, $e^{i\tau} = 1$.

Next, we argue that *any* period of e^{iz} has to be a real number. For, note that if $p \in \mathbb{C}$ is a period, that is if $e^{iz+ip} = e^{iz}$ for all $z \in \mathbb{C}$, then $e^{ip} = 1$. But then, if $p = a + ib$, then $1 = |e^{ip}| = e^{-b}$, and so by part(4) in Theorem 4.1.1, we see that $b = 0$, or that p has to be real.

Finally, we show that any period is an integral multiple of τ. We first show that τ is the smallest positive period. To see this, note that if $0 < x < x_0 (< \sqrt{3})$, then

$$\sin x > x\Big(1 - \frac{x^2}{6}\Big) > \frac{x}{2} > 0.$$

Since $(\cos x)' = -\sin x$, this shows that $\cos x$ is strictly decreasing in $[0, x_0]$. Then from the identity $\sin^2 x + \cos^2 x = 1$, since $\sin x > 0$, we see that $\sin x$ is strictly increasing in $[0, x_0]$. In particular, $0 < \sin x < 1$, and $e^{ix} \neq \pm 1$ or $\pm i$. Hence $e^{4ix} \neq 1$, and τ is indeed the smallest positive period. Now, if p is any other period of e^{iz}, then we can write $p/\tau = n + c$, where $n \in \mathbb{Z}$ and $c \in [0, 1)$. Then $1 = e^{ip} = e^{i\tau c}$. By our discussion above, if $c > 0$, then $\tau c \geq \tau$, which is a contradiction. Hence $c = 0$, and p is an integral multiple of τ. The proof of the theorem is complete with $\pi := \tau/2$. $\square$

4.4 The logarithm function and complex powers

For functions of one real variable, the logarithm is the inverse function of the exponential function. Indeed by Theorem 4.1.1, $e^x : \mathbb{R} \to \mathbb{R}_+$ is a bijective map, and so we can define

$$\ln x : \mathbb{R}_+ \to \mathbb{R}$$

to be the inverse function. We would like to generalize this to the complex plane. In particular, we would like to have a definition for logarithm that makes it a holomorphic function. An immediate difficulty is that while on real line, the exponential function is strictly increasing, and hence one-one, on the complex plane, we have already seen that the exponential function is not one-one. For instance $e^0 = e^{2\pi i n} = 1$ for all $n \in \mathbb{Z}$. So to define an inverse function, one has to make a choice of the pre-image. For instance, we can choose to $\log 1 = 0$ or indeed any of $2\pi i n$, for $n = 1, 2, 3, \cdots$.

In fact, writing in polar coordinates $z = re^{i\theta}$, $f(z)$ satisfies $e^{f(z)} = f(\exp(z)) = z$ on any open connected set if and only if

$$f(z) = \ln r + i\theta + 2\pi i n,\ n = 0, 1, -1, 2, -2, \cdots,$$

where $\ln r$ is the logarithm on positive real numbers defined above. That is, we can at best define logarithm as *multivalued* function. A choice of n corresponds to defining a *single valued* logarithm, the corresponding function is called a *branch* of the logarithm. For instance, choosing $n = 0$, and defining

$$\log z = \log r + i\theta,$$

picks out what is called as the *principal branch* of the logarithm. This might seem like a good definition until we realize that the logarithm so defined is not even continuous. To see this, suppose $z \to -1$ from the 2nd quadrant. Then $\log z$ will tend to $i\pi$. But on the other hand, as $z \to -1$ from the third quadrant, $\log z$ will tend to $-\pi$. This is not only a minor irritant that can be fixed by some trick, but as we will see later in the course, is a fundamental issue. In fact, we will see that there is actually no way to define a holomorphic logarithm on which is defined on all of $\mathbb{C} \setminus \{0\}$. The best we can do is to define it outside of a ray. In fact we have the following.

Theorem 4.4.1. *The function*

$$\log z := \log |z| + i \arg z$$

defines a holomorphic function on $\mathbb{C} \setminus \mathrm{Re}(z) \leq 0$ satisfying $e^{\log z} = z$. Moreover, applying chain rule, we see that

$$\frac{d}{dz} \log z = \frac{1}{z}.$$

We will see a proof of this later in the course. The logarithm function then has the property that $\log 1 = 0$ and

$$\log zw = \log z + \log w,$$

assuming it is defined at all those points.

4.4.1 Logarithm as power series.

Since $e^0 = 1$, and definition of logarithm must satisfy $\log 1 = 0$. Let's see if we can have a definition of logarithm using power series centered at $z_0 = 1$. By the chain rule, if have a holomorphic function $\log z$ near $z_0 = 1$, then

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

Iteratively, we must have

$$\frac{d^n}{dz^n}\Big|_{z=1}\log z = (-1)^{n-1}(n-1)!.$$

If $\log z$ has a power series expansion around $z_0 = 1$, then the coefficients must be given by

$$a_n = \frac{1}{n!}\frac{d^n}{dz^n}\Big|_{z=1}\log z = (-1)^{n-1}\frac{1}{n}.$$

Turning this around, we consider the power series

$$\mathrm{L}(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n.$$

We then have the following

Proposition 4.4.2. *The holomorphic function* $\mathrm{L}(z) : D_1(1) \to \mathbb{C}$ *satisfies*

$$\mathrm{L}(z) = \log z,$$

where $\log z$ *is the principal branch of the logarithm on* $\mathbb{C} \setminus Re(z) \leq 0$ *defined above.*

Proof. We already know that $d\log z/dz = 1/z$ for the principal branch of the logarithm. Also, $\mathrm{L}(1) = \log 1 = 0$. So, similar to the above proof, all we need to show (modulo the theorem on identically zero derivatives to be covered later) is that $\mathrm{L}'(z) = 1/z$. Since $F(z)$ is a power series, by term-wise differentiation,

$$\mathrm{L}'(z) = \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^{n-1} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n.$$

From the geometric series expansion, we know that for $|w| < 1$,

$$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}.$$

Putting $w = 1 - z$ (we can do this since $|z-1| < 1$) in the above expansion

$$\frac{1}{z} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n = \mathrm{L}'(z).$$

□

4.4.2 Complex powers

Once logarithm is defined, one can also define the powers of complex to other complex numbers by the simple formula

$$z^w = e^{w\log z}.$$

Example 4.4.3. The n^{th} roots. *If* n *is an integer, then* $e^{n\log z} = (e^{\log z})^n = z^n$, *and hence the new definition of complex powers agrees with our usual definition of integer powers of complex numbers.*

Lecture 5

Cauchy Riemann equations

5.1 A review of multivariable calculus

Let $\Omega \subset \mathbb{R}^2$ be an open set, and $p = (a, b) \in \Omega$. Then a function $f : \Omega \to \mathbb{R}^2$ is said to be *(totally) differentiable* at p, if there exists a linear map $Df_p : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{|h| \to 0} \frac{f(p+h) - f(p) - Df_p(h)}{|h|} = 0.$$

The linear map Df_p is called the *(total) derviative* of f at p. If f is differentiable at p, then f is also continuous, and moreover the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at p. In fact, if

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are the standard basis vectors for $\mathbb{R}^2$, then

$$\frac{\partial f}{\partial x}(p) = Df_p(\vec{e_1}), \text{ and } \frac{\partial f}{\partial y}(p) = Df_p(\vec{e_2}).$$

In particular if $f = (u, v)$, the matrix for the linear map Df_p in terms of the standard basis, called the *Jacobian matrix*, is given by

$$\mathbf{J_f}(p) := \begin{pmatrix} \frac{\partial u}{\partial x}(p) & \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

The determinant of the Jacobian matrix is called simply the *Jacobian*, and we will denote it by $J_f(p)$.

Remark 5.1.1. Note that the mere existence of partial derivatives is not sufficient for the function to be differentiable. On the other hand, if the partial derivatives exist and are continuous, then the function is indeed differentiable.

5.2 Cauchy-Riemann equations

Recall that a function $f : \Omega \to \mathbb{C}$ is holomorphic if

$$f'(z) = \frac{d}{dz} f(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite at all points $z \in \Omega$. Putting $z = x + iy$ and decomposing f into it's real and imaginary parts, we can write

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where $u, v : \Omega \to \mathbb{R}$ are real valued functions of two real variables.

Question 5.2.1. *What restrictions does holomorphicity of f put on the functions u and v?*

So suppose f is holomorphic. In the limit above, suppose h goes to zero along the the real axis ie. $h = h + i0$ is a real number. Then one can re-write the difference quotient as

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).\end{aligned}$$

Notice that since the limit on the let exists, by definition of the limits of complex values functions, the individual limits of the real and imaginary parts also exist, and hence the two limits on the right exist. In other words, if f is holomorphic, then the partials of u and v with respect to x exist. On the other hand, if h goes to zero along the imaginary axis ie. $h = ik$, where $k \in \mathbb{R}$ goes to zero, then

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{ik} + i \lim_{k\to 0} \frac{v(x,y+k) - v(x,y)}{ik} \\ &= \lim_{k\to 0} \frac{v(x,y+k) - v(x,y)}{k} - i \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{k} \\ &= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y).\end{aligned}$$

Again, the fact that f is holomorphic implies that the partials of u and v with respect to y also exist. But of course, if f is holomorphic, then these two limits must coincide. Setting the real and imaginary parts equal to each other we obtain the so called *Cauchy-Riemann equations*. In fact we have the following fundamental characterisation of holomorphicity. To state it, we first define "complex" multiplication of vectors in $\mathbb{R}^2$ as the map $J : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}.$$

This is of course, simply multiplication by i, once we identify $\mathbb{R}^2$ with $\mathbb{C}$. It is easy to check that this is a linear isomorphism satisfying $J^2 = -id$, and that the matrix with respect to the standard basis is

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 5.2.2. *Let $\Omega \subset \mathbb{C}$ be an open set and $f = u + iv : \Omega \to \mathbb{C}$ a function. Then the following are equivalent.*

1. *$f = u + iv$ is complex differentiable at a point $z \in \Omega$.*
2. *Df_p exists and u and v satisfy the Cauchy-Riemann (CR) equations:*

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) &= -\frac{\partial u}{\partial y}(x,y) \end{cases} \tag{CR}$$

3. *Df_p exists and is $\mathbb{C}$-linear in the sense that for any vector $\vec{v} \in \mathbb{R}^2$,*

$$Df_p(J \cdot \vec{v}) = J \cdot Df_p(\vec{v}).$$

Moreover, if f is complex differentiable at $p = a + ib$, then

$$f'(p) = \frac{\partial f}{\partial x}(a,b) = i\frac{\partial f}{\partial y}(a,b). \tag{5.1}$$

Proof.

- **Proof of** $(1) \implies (2)$**.** Firstly, the remarks above prove that if f is complex differentiable at $p = a + ib$, then the partial derivatives of u and v exist at (a, b) and satisfy the Cauchy-Riemann equations. Moreover, $f'(p)$ can be computed using the formula (5.1) above. All it is remains to be shown is that f is in fact differentiable as a vector valued function of two variables. From the definition of differentiability, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

  $$\Big|\frac{f(p+h) - f(p) - f'(p)h}{h}\Big| < \varepsilon,$$

  whenever $|h| < \delta$. By the Cauchy Riemann equations, the Jacobian matrix at p is given by

  $$\mathbf{J_f}(p) = \begin{pmatrix} u_x(a,b) & u_y(a,b) \\ -u_y(a,b) & u_x(a,b) \end{pmatrix}.$$

  If Df_p is the linear transformation associated to the Jacobian, then an easy computation shows that for any complex number $h = \lambda + i\mu$,

  $$Df_p(h) = (\lambda u_x(a,b) + \mu u_y(a,b)) + i(-\lambda u_y(a,b) + \mu u_x(a,b)) = f'(p)h.$$

  Here the left hand side is a vector in $\mathbb{R}^2$, while the right hand side is a complex number, and the identification is the usual one. So give $\varepsilon > 0$, for the $\delta > 0$ chosen above, we have that

  $$\Big|\frac{f(p+h) - f(p) - Df_p(h)}{|h|}\Big| = \Big|\frac{f(p+h) - f(p) - f'(p)h}{h}\Big| < \varepsilon,$$

  whenever $|h| < \delta$. Hence f is differentiable at p with derivative given by Df_p.

- **Proof of** $(2) \iff (3)$**.** Let $\vec{e_1}$ and $\vec{e_2}$ be the standard basis vectors for $\mathbb{R}^2$ as above. Note that $J(\vec{e_1}) = \vec{e_2}$ and $J(\vec{e_2}) = -\vec{e_1}$. Then

  $$\begin{aligned} Df_p(J(\vec{e_1})) &= Df_p(\vec{e_2}) = \begin{bmatrix} \frac{\partial u}{\partial y}(p) \\ \frac{\partial v}{\partial y}(p) \end{bmatrix} \\ J(Df_p(\vec{e_1})) &= J\Big(\begin{bmatrix} \frac{\partial u}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) \end{bmatrix}\Big) = \begin{bmatrix} -\frac{\partial v}{\partial x}(p) \\ \frac{\partial u}{\partial x}(p) \end{bmatrix}. \end{aligned}$$

  So the Cauchy-Riemann equations are satisfied if and only if $Df_p(J(\vec{v})) = J(Df_p(\vec{v}))$.

- **Proof of** $(2) \implies (1)$**.** Let $p = a + ib \in \Omega$ be a point, and let

  $$\begin{cases} A &= \frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \\ B &= \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b). \end{cases}$$

  By the definition of differentiability,

  $$\begin{aligned} u(a+h, b+k) &= u(a,b) + hA - kB + \varepsilon_1(h,k) \\ v(a+h, b+k) &= v(a,b) + hB + kA + \varepsilon_2(h,k), \end{aligned}$$

  where $\varepsilon_1(h,k)/\sqrt{h^2+k^2} \to 0$ and $\varepsilon_2(h,k)/\sqrt{h^2+k^2} \to 0$ as $(h,k) \to 0$. So

  $$\frac{f(a+ib+(h+ik)) - f(a+ib)}{h+ik} = \frac{h(A+iB) - k(B-iA)}{h+ik} + \frac{\varepsilon_1(h,k) + \varepsilon_2(h,k)}{h+ik}.$$

For the first term, multiplying and dividing by the conjugate $h - ik$ and simplifying, we see that

$$\frac{(hA - kB) + i(hB + kA)}{h + ik} = A + iB.$$

For the second term, using triangle inequality, we see that

$$\left|\frac{\varepsilon_1(h,k) + \varepsilon_2(h,k)}{h + ik}\right| \leq \left|\frac{|\varepsilon_1(h,k)|}{\sqrt{h^2+k^2}}\right| + \left|\frac{|\varepsilon_2(h,k)|}{\sqrt{h^2+k^2}}\right| \to 0$$

as $(h,k) \to 0$. And so we see that

$$\lim_{h+ik \to 0} \frac{f(a + ib + (h + ik)) - f(a + ib)}{h + ik}$$

exists, and is in fact equal to $A + iB$. This proves that f is holomorphic, with

$$f'(p) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b) = i\frac{\partial u}{\partial y}(a,b) - \frac{\partial v}{\partial y}(a,b).$$

□

As a consequence, we have the following useful observation.

Corollary 5.2.3. *Let $f = u + iv : \Omega \to \mathbb{R}$ be holomorphic at $p \in \Omega$. Then, $J_f(p) = |f'(z)|^2$. In particular, the Jacobian of is always positive, and hence any holomorphic map is orientation preserving.*

Proof. By the Cauchy Riemann equations and (5.1),

$$J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2.$$

□

The Cauchy-Riemann equations help us compute complex derivatives, or rule out the possibilities of some functions being holomorphic.

Example 5.2.4. *The function $f(z) = \bar{z}$ cannot be holomorphic. To see this, note that $f(z) = x - iy$. Applying the above theorem with $u(x,y) = x$ and $v(x,y) = -y$, we see that $u_x = 1$ while $v_y = -1$, and so the above equations are not satisfied. For this function $\bar{f}(z) = z$ is holomorphic. Functions such as these whose conjugates are holomorphic, are called anti-holomorphic functions. It is easy to prove that the only holomorphic and anti-holomorphic functions are the constants.*

Example 5.2.5. *We have already seen that $f(z) = |z|$ is not holomorphic. Using the above theorem it is easy to see that in fact, any function, defined on an open connected set, that takes only real values cannot be holomorphic, unless it is a constant function. This is because from the above theorem (since v=0) we get that $u_x = u_y = 0$. That is, the gradient of u is zero in an open connected set in $\mathbb{R}^2$. But then, by a standard fact from multivariable calculus, u (and hence f) has to be a constant.*

Example 5.2.6. *The function $f(x + iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin. On the other hand, one can show that the function is not holomorphic at $z = 0$. The problem is of course that Df_0 does not exist.*

We now discuss some important consequences of the Cauchy-Riemann equations.

Corollary 5.2.7. *Let $\Omega \subset \mathbb{C}$ be a connected, open subset, and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $f'(z) \equiv 0$, then f is a constant.*

Proof. If $f'(z) \equiv 0$, then by the Cauchy Riemann equations, and (5.1), $Df_p \equiv 0$. By the mean value theorem (applied to line segments in $\mathbb{R}^2$), we see that f has to be a constant. □

Corollary 5.2.8. *Let Ω be an open subset of $\mathbb{C}$ containing the origin. There does not exist a holomorphic function $f(z)$ on Ω such that $\exp(f(z)) = z$ for all $z \in \Omega$.*

Proof. Since Ω is an open set containing the origin, there exists an $\delta > 0$ such that $D_\delta(0) \subset \Omega$. We proceed by contradiction. Let $f = u + iv$ be such a function. Then necessarily

$$f'(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

By the Cauchy Riemann equations we then have that

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2} := P, \ \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} := Q,$$

that is $\nabla u = \vec{F} = (P, Q)$. Then by the fundamental theorem of line integrals

$$\int_{\partial D_\delta(0)} \nabla u \cdot d\vec{r} = 0.$$

On the other hand

$$\begin{aligned}\int_{\partial D_\delta(0)} \vec{F} \cdot d\vec{r} &= \int_{\partial D_\delta(0)} P\,dx + Q\,dy \\ &= 2\pi,\end{aligned}$$

which is a constradiction. □

5.2.1 New notation

A more compact way to write down the Cauchy-Riemann equations is to introduce the

$$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$$

operators. These are the analogs of the partial derivative operators $\partial/\partial x, \partial/\partial y$ in the complex setting. To see how to define these operators, note that any point in the plane (x, y) can be described using the $(z, \bar{z})$ variables via the formulas

$$x = \frac{z+\bar{z}}{2}, \ y = \frac{z-\bar{z}}{2i}.$$

Formally, using chain rule from multivariable calculus (any sensible definition of these operators should satisfy chain rule after all!), and treating z and $\bar{z}$ as independent variables, we see that

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} \\ &= \frac{1}{2}\Big(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\Big)\end{aligned}$$

Similarly we see that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\Big).$$

Motivated by these formulae, we define the two operators as

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\Big)$$
$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\Big)$$

Then for a holomorphic function $f(z)$ the Cauchy-Riemann equations can be re-written as

$$\frac{\partial f}{\partial \bar{z}} = 0,\ \frac{\partial f}{\partial z} = f'(z).$$

So holomorphicity implies that the function is independent of the $\bar{z}$ variable. Note that for any complex valued two variable function, we then have

$$\frac{\partial f}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}.$$

Remark 5.2.9. (Chain rule in the new notation). Let f and g be function from domains in $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose f is differentiable at $p = (a, b)$, g is defined in a neighborhood of $f(p)$ and is differentiable at $f(p)$, then $g \circ f$ is differentiable at p. Then the chain rule from multivariable calculus is equivalent to

$$\frac{\partial g \circ f}{\partial z}(p) = \frac{\partial g}{\partial w}(f(p)) \cdot \frac{\partial f}{\partial z}(p) + \frac{\partial g}{\partial \bar{w}}(f(p)) \cdot \frac{\partial \bar{f}}{\partial z}(p)$$
$$\frac{\partial g \circ f}{\partial \bar{z}}(p) = \frac{\partial g}{\partial w}(f(p)) \cdot \frac{\partial f}{\partial \bar{z}}(p) + \frac{\partial g}{\partial \bar{w}}(f(p)) \cdot \frac{\partial \bar{f}}{\partial \bar{z}}(p).$$

Heuristically, this would be the "obvious" chain rule one might write down, if we consider $(z, \bar{z})$ as two independent variables of f (and resp. $(w, \bar{w})$ two independent variables of g). The proof of course requires an interpretation of the chain rule in terms of multiplication of Jacobian matrices, and the expression above of the Jacobian matrix in terms of the holomorphic and anti-holomorphic derivatives. In particular, if f and g are both holomorphic, then

$$\frac{\partial g \circ f}{\partial z}(p) = g'(f(p)) \cdot f'(p),\ \frac{\partial g \circ f}{\partial \bar{z}}(p) = 0.$$

Lecture 6

Harmonic functions

6.1 Harmonic functions in the plane and holomorphic functions

For a function of two variables $u(x, y)$ whose first two partials exist and are continuous, the *Laplacian* is defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

The function is said to be *harmonic* if $\Delta u = 0$ at all points. Harmonic functions show up almost everywhere in physics, and most prominently in electrostatics. For instance the electric potential in a charge free region is harmonic function! In two dimensions, the study of harmonic functions is equivalent to the study of holomorphic functions via the following theorem.

Proposition 6.1.1. *Let $f = u + iv : \Omega \to \mathbb{C}$ be a holomorphic function, Suppose u and v have continuous second partial derivatives, then u and v are both harmonic functions.*

Proof. The proof is an easy consequence of the Cauchy-Riemann equations. By CR equations, $u_x = v_y$, and $u_y = -v_x$. differentiating the first with respect to 'x' and the second equation with respect to 'y', and recalling that since v has continuous second partials, the mixed partials commute, we see that

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

$\square$

Conversely we have the following.

Proposition 6.1.2. *Let u be a harmonic function on the unit disc $D_R(z_0)$. Then there exists a harmonic function v on $D_R(z_0)$ such that $f := u + iv$ is a holomorphic function on $D_R(z_0)$. Moreover, if v' is another such function, then $v - \tilde{v}$ is constant on the disc.*

Proof. Without loss of generality we may assume that $z_0 = 0$. The uniqueness follows from the Cauchy-Riemann equations since

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial \tilde{v}}{\partial y} + i\frac{\partial \tilde{v}}{\partial x}.$$

In particular, $\nabla(v - \tilde{v}) \equiv 0$, and hence $v - \tilde{v}$ is a constant. Let $\vec{F} = (P, Q)$ be the vector field obtained by

$$P := -\frac{\partial u}{\partial y},\ Q = \frac{\partial u}{\partial x}.$$

Since u is harmonic, we automatically have that $\mathrm{curl}\vec{F} = 0$.

Claim. There exists a function v on $D_R(0)$ with continuous second partials such that $\nabla v = \vec{F}$.

Assuming this the proposition follows since

$$\Delta v = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0,$$

by Clairaut's theorem on commuting mixed partials.

Proof of the Claim. We let

$$v(x,y) = \int_{l_{(x,y)}} \vec{F} \cdot d\vec{r} = \int_0^1 \Big(P(tx,ty)x + Q(tx,ty)y \Big)\, dt.$$

Then

$$\begin{aligned} \frac{\partial v}{\partial x} &= \int_0^1 \Big(tx\frac{\partial P}{\partial x}(tx,ty) + P(tx,ty) + ty\frac{\partial Q}{\partial x} \Big)\, dt \\ &= \int_0^1 t\frac{d}{dt}P(tx,ty)\, dt + \int_0^1 P(tx,ty)\, dt \\ &= tP(tx,ty)\Big|_{t=0}^1 - \int_0^1 P(tx,ty)\, dt + \int_0^1 P(tx,ty)\, dt \\ &= P(x,y). \end{aligned}$$

Similarly we can also prove that

$$\frac{\partial v}{\partial y} = Q(x,y).$$

□

□

6.2 Mean value property

6.3 The Poisson kernel

Part II

Integration theory

Lecture 7

Complex Integration

7.1 Curves in the complex plane

A *parametrized curve* (or simply a curve) in a domain $\Omega \subset \mathbb{C}$ is a continuous function $z(t) : [a, b] \to \Omega$. Writing

$$z(t) = x(t) + iy(t),$$

we say that $z(t)$ is differentiable with derivative $z'(t)$, if $x'(t)$ and $y'(t)$ exist for all $t \in (a, b)$, and then we set

$$z'(t) = x'(t) + iy'(t).$$

We call it *regular* if $z'(t)$ exists and $z'(t) \neq 0$ for all $t \in (a, b)$. Geometrically, the vector

$$x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$$

gives the tangent vector to the curve at the point $(x(t), y(t))$, and so the complex number $z'(t)$ encodes the information of the tangent vector. Moreover

$$|z'(t)| = \sqrt{x'(t)^2 + y'(t)^2},$$

measures the *speed* at which the curve is traversed.

For a parametrised curve as above, the points $z(a)$ and $z(b)$ are called the *initial* and *final* points of the curve respectively, and together they are referred to as the end points of the curve. The curve is said to be *closed* if $z(a) = z(b)$, and is called *simple* if $z(t)$ is injective on the interval (a, b).

7.1.1 Orientation

The choice of a parametrization fixes the orientation of the curve. Given a parametrization $z(t) : [a, b] \to \mathbb{C}$ of the curve C, we say that $w(s) : [c, d] \to \mathbb{C}$ is an *orientation preserving* re-parametrization or that $z(t)$ and $w(s)$ are *equivalent* parametrizations, if there exists a strictly increasing function $\alpha : [c, d] \to [a, b]$ with $\alpha(c) = a$ and $\alpha(d) = b$, such that

$$w(s) = z(\alpha(s)).$$

Example 7.1.1. *Consider the curve defined by* $z : [0, 2\pi] \to \mathbb{C}$ *where*

$$z(t) = R(\cos t + i \sin t).$$

The image of the curve is of course a circle of radius R. The tangent vector is given by

$$z'(t) = R(-\sin t + i\cos t),$$

and so the speed is $|z'(t)| = 1$. A parametrization is given by $w : [0, \pi] \to \mathbb{C}$,

$$w(t) = R(\cos 2t + i\sin 2t),$$

which traverses the same circle, but with double the speed. On the other hand the parametrization

$$z(t) = R(\cos t - i\sin t)$$

for $t \in (0, 2\pi)$ also describes the same circle with the same speed, but traversed in a clock-wise direction. A circle is said to be positively oriented if the parametrization traverses the circle in the anti-clockwise direction, while negatively oriented otherwise.

We will need to consider slightly more general curves. A curve $z(t) : [a, b]$ is said to be *piecewise regular* if there is a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that $z(t)$ restricted to each (t_{i-1}, t_i) is a regular curve.

7.1.2 Notation and conventions.

If the image of restriction of the curve $z(t)$ to the interval (t_{i-1}, t_i) is denoted by C_i, and the image of the full curve is denoted by C, we then write

$$C = \sum_{i=1}^{n} C_n.$$

We denote by $-C$, the curve C traced in the opposite direction. For instance, if $z(t) : [a, b] \to \mathbb{C}$ is a parametrization for C, a parametrization for $-C$ is given by $z^-(s) : [a, b] \to \mathbb{C}$ where

$$z^-(s) = z(a + b - s).$$

For a positive integer $a > 0$, we denote by aC to be the curve C traversed 'a' times. A circle $C_R(p)$ or $|z - p| = R$, unless otherwise specified, will always mean a circle of radius R centred at p traversed once in the anti-clockwise direction.

7.2 Complex line integrals

For a continuous function $f = u + iv : [a, b] \to \mathbb{C}$ of one *real* variable, we can extend the definition of integration by defining

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Now, suppose we are given a smooth curve as above, and a function $f : \Omega \to \mathbb{C}$, we then define the *complex integral along the curve* by

$$\int_C f(z)\,dz := \int_a^b f(z(t))z'(t)\,dt,$$

where C denotes the image of the curve. Note that the multiplication above is the complex multiplication. It is convenient to think of dz as a complex differential, representing an infinitesimal complex change, and given by

$$dz = dx + idy.$$

So if $f = u + iv$, then

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx),$$

where the integrals on the right are now the usual line integrals from multivariable calculus.

Remark 7.2.1. Recall that the differential forms dx and dy are defined to be duals of the vector fields $\partial/\partial x$ and $\partial/\partial y$ in the xy-plane. That is,

$$dx\Big(\frac{\partial}{\partial x}\Big) = dy\Big(\frac{\partial}{\partial x}\Big) = 1,\ dx\Big(\frac{\partial}{\partial y}\Big) = dy\Big(\frac{\partial}{\partial x}\Big) = 0.$$

Then it is easy to compute that $dz = dx + idy$ and $d\bar{z} = dx - idy$ are dual *complex valued* differential forms to the complex valued vector fields $\partial/\partial z$ and $\partial/\partial\bar{z}$ vector fields defined in the previous lecture.

To make sure the integral is well defined, we need to show that it is independent of orientation preserving parametrizations.

Lemma 7.2.2. *Let C be a curve with parametrization $z(t)$. Let $w(s) = z(\alpha(s))$ be another* orientation preserving *parametrization, where $\alpha : [a, b] \to [c, d]$. Then*

$$\int_C f(w)\,dw = \int_C f(z)\,dz.$$

Proof. By the chain rule, since $w'(s) = z'(\alpha(s))\alpha'(s)$, we see that

$$\int_C f(w)\,dw = \int_a^b f(w(s))w'(s)\,ds = \int_a^b f(z(\alpha(s))z'(\alpha(s))\alpha'(s)\,ds.$$

Putting $t = \alpha(s)$, we see that $\alpha'(s)ds = dt$, and hence

$$\int_C f(w)\,dw = \int_a^b f(z(t))z'(t)\,dt = \int_C f(z)\,dz.$$

This shows that the definition of integration is independent of the parametrization chosen. $\square$

We can now extend the definition of complex integrals to piecewise smooth curves by linearity. That is, if $C = C_1 + \cdots + C_n$ is a piecewise smooth curve with C_j smooth curve for all $j = 1, \cdots, n$, then we define

$$\int_C f(z)\,dz := \sum_{j=1}^n \int_{C_j} f(z)\,dz.$$

Similar to our definition of $\int_C f(z)\,dz$ we can define the integral with respect to $d\bar{z}$ by

$$\int_C f(z,\bar{z})\,d\bar{z} := \overline{\int_C \overline{f(z)}\,dz}.$$

We can also define the integral with respect to arc-length. For a complex or real valued function $f(z)$ and a curve $z = z(t) : [a, b] \to \mathbb{C}$ we also define

$$\int_C f(z)\,|dz| := \int_a^b f(z)|z'(t)|dt.$$

We then define the *length* of the curve by

$$\mathrm{len}(C) = \int_C |dz|.$$

We next state, without proof, some basic properties of the complex line integral. The proofs follow from the definition of the complex integral and corresponding properties of the Riemann integral.

Proposition 7.2.3. *Let C be a parametric piecewise regular curve in an open set $\Omega \subset \mathbb{C}$.*

1. *For any complex numbers a, b and any complex valued functions f and g we have that*
$$\int_C [af + bg](z)\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz.$$

2. *For integers $a_j \in \mathbb{Z}$ and piecewise smooth curves C_j, if we denote by $C = a_1C_1 + \cdots + a_nC_n$, we have*
$$\int_C f(z)\,dz = \sum_{j=1}^{n} a_j \int_{C_j} f(z)\,dz.$$
In particular,
$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

3. *(Triangle inequality)*
$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz| \leq \sup_{z\in C}|f(z)| \cdot \text{length}(C),$$
with all inequalities replaced with equality if and only if f is a constant real number.

4. *If $f_n \xrightarrow{u.c} f$, then*
$$\lim_{n\to\infty}\int_C f_n(z)\,dz = \int_C f(z)\,dz.$$

7.3 A fundamental computation

We now compute the integrals
$$\int_{C_R} z^n\,dz,$$
where $C_R = \{z \mid |z| = R\}$ is the circle of radius R centered at the origin positively oriented and traversed once. It is not an overstatement to claim that this integral, although elementary, plays a fundamental role in complex analysis. A parametrization of the circle is given by $z(t) = Re^{i\theta},\ \theta \in [0, 2\pi]$. Then $dz = iRe^{i\theta}d\theta$, and so
$$\begin{aligned}\int_{C_R} z^n,\,dz &= \int_0^{2\pi} R^n e^{in\theta}(iRe^{i\theta})\,d\theta\\ &= iR^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta\end{aligned}$$
Now if $n+1 \neq 0$, then
$$\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = \frac{1}{(n+1)i}e^{i(n+1)\theta}\Big|_{\theta=0}^{\theta=2\pi} = 0,$$
since $e^{i\theta}$ is periodic with period 2π. On the other hand, if $n = -1$, then
$$\int_{C_R} z^n,\,dz = iR^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$
So summarizing, we have the following:
$$\frac{1}{2\pi i}\int_{|z|=R} z^n\,dz = \begin{cases} 0, & n \neq -1\\ 1, & n = -1.\end{cases}$$
In particular, the integral is independent of the radius R. More generally, we have the following.

Proposition 7.3.1. *Let D be any disc in $\mathbb{C}$, and let p be a point not lying on the boundary circle $C = \partial D$. If C is traversed only once with positive orientation, then*

$$n(C;p) := \frac{1}{2\pi i}\int_C \frac{1}{z-p}\,dz = \begin{cases} 1, & p \in D \\ 0, & p \notin \overline{D}. \end{cases}.$$

Moreover, if $n \neq -1$ and $n \in \mathbb{Z}$, then

$$\int_C (z-p)^n\,dz = 0.$$

The number $n(C,p)$ is called the *index* or the *winding number* of the circle C around p.

Proof. Without loss of generality, we can assume that $z_0 = 0$. Now suppose $p \in D$. Then for any $z \in C$, $|z| > |p|$. Then by the geometric series expansion, for $z \in C$ we have

$$\frac{1}{z-p} = \frac{1}{z}\cdot\frac{1}{1-p/z} = \frac{1}{z} + \sum_{n=2}^{\infty}\frac{p^{n-1}}{z^n}.$$

Integrating both sides (this can be done since convergence is uniform), and using the above computation, we see that $n(C,p) = 1$. On the other hand, if $p \notin \overline{D}$, then $|z| < |p|$ for all $z \in C$, and hence

$$\frac{1}{z-p} = -\frac{1}{p}\cdot\frac{1}{1-z/p} = -\frac{1}{p}\sum_{n=0}^{\infty}\frac{z^n}{p^n}.$$

Again integrating both sides, we see that $n(C,p) = 0$, since there are only positive powers of z on the right. For the second part, if $n > 0$, then the integrand is a polynomial and hence the integral is zero by the above computations. If $n = -m < 0$, then we can write

$$(z-p)^n = \left(\frac{1}{z-p}\right)^m.$$

Once again using the geometric series expansions above, in both cases, there will no terms with exponent -1. And hence by the computation above, the integral will be zero. $\square$

Remark 7.3.2. Later in the course, we will define the index $n(\gamma, p)$ of a general curve γ around a point p by a similar formula, and we shall prove (rather indirectly) that the index of *any* closed curve is always an integer. Assuming this, we can provide a more conceptual explanation of the above result. It is clear that the $n(C,p)$, as a function of p defined on the open set $\mathbb{C}\setminus C$ is a continuous function. But then being integer valued, it must be locally constant. From our elementary observation, $n(C,0) = 1$, and hence $n(C,p) = 1$ for all $p \in D$. On the other hand, clearly as $|p| \to \infty$, $n(C,p)$ approaches 0. Again by virtue of being locally constant, this implies that $n(C,p) = 0$ for all $p \in \mathbb{C}\setminus\overline{D}$.

7.4 Primitives

We then have the following theorem, which is a generalization of the fundamental theorem for line integrals from multivariable calculus.

Proposition 7.4.1 (Fundamental theorem for complex integrals). *If C is any curve joining the point p to q, then*

$$\int_C F'(z)\,dz = F(q) - F(p).$$

Proof. Let $z(t) : [0,1] \to \mathbb{C}$ be a parametrization for C such that $p = z(0)$ and $q = z(1)$. Then

$$\int_C F'(z) = \int_0^1 F'(z(t))z'(t)\,dt.$$

But by Chain rule, if we let $g(t) = F(z(t))$, then $g'(t) = F'(z(t))z'(t)$, and so

$$\int_C F'(z)\,dz = \int_0^1 \frac{dg}{dt}\,dt = g(1) - g(0),$$

where we use the usual one variable fundamental theorem of calculus. But $g(0) = F(z(0)) = F(p)$ and $g(1) = F(q)$, and this completes the proof. □

Recall that an open set is called *connected* is any two points can be joined by a continuous curve lying completely inside the open set. An important and immediate consequence of the fundamental theorem is the following.

Corollary 7.4.2. *Let $\Omega \subset \mathbb{C}$ be an open connected subset, and $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f'(z) = 0$ for all $z \in \Omega$ if and only if $f(z)$ is a constant.*

For a domain $\Omega \subset \mathbb{C}$, and $F, f : \Omega \to \mathbb{C}$ complex valued functions, we say that $F(z)$ is a *primitive* of $f(z)$ if

$$F'(z) = f(z)$$

for all $z \in \Omega$. Then another direct corollary of the above theorem is the following.

Corollary 7.4.3. *Suppose $f : \Omega \to \mathbb{C}$ has a primitive on F, then*

$$\int_C f(z)\,dz = 0$$

for every closed curve $C \subset \Omega$.

Using the above corollary, we can explain the results of the calculation above. Recall that for any integer $n \neq -1$, we have that

$$\Big(\frac{1}{n+1}\Big)\frac{d}{dz}z^{n+1} = z^n.$$

Or in other words, for $n \neq -1$, z^n has a primitive at least on $\mathbb{C} \setminus \{0\}$. Hence the integral on any closed loop not passing through $z = 0$, is always zero. In particular, the integrals around $|z| = R$ are zero. That leaves the case when $n = -1$. We have already seen (as a consequence of the chain rule) that if a holomorphic logarithm $\log z$ can be defined, then it is a natural primitive for $1/z$. But we saw some lectures back, we saw that going around a circle centered at the origin, makes the logarithm function discontinuous, leave alone non-holomorphic. In fact, combining the corollary with the calculations above, we have managed to *prove* the following.

Proposition 7.4.4. *Let $\Omega \subset \mathbb{C} \setminus \{0\}$ be an open set containing at least one circle $|z| = r$. Then there is no holomorphic function $F : \Omega \to \mathbb{C}$ such that*

$$e^{F(z)} = z.$$

In particular, there cannot be a holomorphic logarithm defined on all of $\mathbb{C}^*$, or indeed on any punctured neighbourhood $D_r(0) \setminus \{0\}$ of 0.

In fact, the theorem can also be used to explain the fact that $n(\partial D, p) = 0$ when $p \notin \overline{D}$. Again without loss of generality, we assume that $D = D_R(0)$. Recall that $\log w$ can indeed be defined via a power series in the region $|w - 1| < 1$, namely

$$\log w = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(w-1)^n,$$

and it satisfies $(\log w)' = w^{-1}$. Now putting $w = z/p - 1$, $f(z) = \log(z/p - 1)$ is a holomorphic function in the region $|z| < |p|$ with

$$f'(z) = \frac{1}{z-p}.$$

Since C is completely contained in the region $|z| < |p|$, by Corollary 7.4.3, $n(C, p) = 0$.

7.4.1 A differential forms interpretation

Recall that (real) differential one forms on an open subset $\Omega \subset \mathbb{R}^2$ are formal $C^\infty(\Omega, \mathbb{R})$-linear combinations of the symbols dx and dy. The space of one forms is usually denoted by $\mathcal{A}^1(\Omega)$. But one can just as well consider the set $\mathcal{A}^1_{\mathbb{C}}(\Omega)$ of complex valued differential forms where instead of smooth real valued functions, one considers $C^\infty(\Omega, \mathbb{C})$-linear combinations. Instead of using dx and dy as bases, it is more convenient to use

$$dz = dx + idy, \ d\bar{z} = dx - idy$$

as bases elements, so that a general element of $\mathcal{A}^1_{\mathbb{C}}(\Omega)$ can then be written as

$$\alpha = f(z, \bar{z})dz + g(z, \bar{z})d\bar{z}.$$

The form α is real if and only if $\bar{f} = g$. A form is said to be of type $(1, 0)$ if $b = 0$. Recall that one can integrate (real) differential one-forms on curves. Indeed if $\alpha = adx + bdy$, and $\gamma = z(t) = x(t) + iy(t) : [0, 1] \to \mathbb{C}$ is a C^1-curve, then one defines

$$\int_\gamma \alpha := \int_0^1 \Big(a(z(t))x'(t) + b(z(t))y'(t) \Big)\, dt.$$

One can extend this definition to integrating complex one forms by simply using linearity. That is, if $\eta = \alpha + i\beta$ is a complex valued one form then we simply define

$$\int_\gamma \eta = \int_\gamma \alpha + i \int_\gamma \beta.$$

One can now easily check that if $\eta = f(z)\, dz$ is a form on type $(1, 0)$, then

$$\int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t))\gamma'(t)\, dt,$$

and so our definition of complex integration is simply integration of complex valued forms of type $(1, 0)$.

Lecture 8

Cauchy's theorem : Local versions

In the previous lecture, we saw that if f has a primitive in an open set, then

$$\int_\gamma f\,dz = 0$$

for all closed curves γ in the domain. This was a simple application of the fundamental theorem of calculus. It is somewhat remarkable, that in many situations the converse also holds true. In the next few lectures we will explore this theme, and prove theorems that will form the basis of all that we will accomplish in the rest of the course. The simplest version is the following:

Theorem 8.0.1 (Cauchy's theorem on a disc)**.** *Let D be a disc in the complex plane. If $f : D \to \mathbb{C}$ is holomorphic, then*

$$\int_\gamma f\,dz = 0$$

for all closed curves γ contained in D.

We will prove Cauchy's theorem on a disc by showing that all holomorphic functions in the disc have a primitive. The key technical result we need is *Goursat's theorem.*

Theorem 8.0.2 (Goursat)**.** *Let $\Omega \subset \mathbb{C}$ be an open subset, and $T \subset \Omega$ be a triangle contained inside Ω. If $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_T f(z)\,dz = 0.$$

From this we immediately obtain a Goursat theorem for rectangles. We leave the proof as an exercise.

Corollary 8.0.3. *Let $\Omega \subset \mathbb{C}$ be an open subset, and $R \subset \Omega$ be a triangle contained inside Ω. If $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_R f(z)\,dz = 0.$$

Remark 8.0.4 (Relation to Green's theorem.)**.** If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field, such that P and Q have continuous first partials, then for any close curve

$$\int_R (Q_x - P_y)\,dx\,dy = \int_{\partial R} P\,dx + Q\,dy.$$

Now, suppose that $f = u + iv$ and that u and v have continuous partials. Then

$$\int_\gamma f\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx).$$

Consider the vector fields $(P_1, Q_1) = (u, -v)$ and $(P_2, Q_2) = (v, u)$. Then for each $j = 1, 2$, by the Cauchy Riemann equations

$$\frac{\partial Q_j}{\partial x} = \frac{\partial P_j}{\partial y}.$$

Then by Green's theorem, the line integral is zero and we recover Cauchy's theorem (even for general domains). The important point is our assumption that u and v have continuous partials, while in Cauchy's theorem we only assume holomorphicity which *a priori* only guarantees the existence of the partial derivatives. Later in the course, we will in fact show that holomorphicity does imply that the first partials are continuous partials (actually we will prove that partial derivatives of *all* orders exists and are continuous), but that will be a *consequence* of Cauchy's theorem. And hence we need a proof that avoids Green's theorem. In fact the methods we use also yield a proof of Green's theorem.

8.1 Proof of Cauchy's theorem assuming Goursat's theorem

Cauchy's theorem follows immediately from the theorem below, and the fundamental theorem for complex integrals (cf. Proposition 7.4.1).

Theorem 8.1.1. *Let $f : D \to \mathbb{C}$ be a holomorphic function. Then $f(z)$ has a primitive on D.*

Proof. We first observe that By translation, we can assume without loss of generality that the disc D is centered at the origin. For any points $z, w \in \mathbb{C}$, we denote by $l_{z,w}$ the straight line segment from z to w. For any $z = (x, y) \in D$, let γ_z denote the path from the origin to z consisting of a horizontal segment from 0 to $(x, 0)$ followed by a vertical segment from $(x, 0)$ to (x, y). We then define

$$F(z) = \int_{\gamma_z} f(w)\,dw,$$

and claim that $F(z)$ is holomorphic with $F'(z) = f(z)$. To see this, first note that if $z_1 = (x, 0)$, then for any small $h \in \mathbb{R}$,

$$\int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw = \int_{l_{z_1,z_1+h}} f(w)\,dw + \int_{l_{z_1+h,z+h}} f(w)\,dw - \int_{l_{z_1,z}} f(w)\,dw.$$

On the other hand, by Theorem 8.0.2,

$$\int_{l_{z_1,z_1+h}} f(w)\,dw + \int_{l_{z_1+h,z+h}} f(w)\,dw + \int_{l_{z+h,z}} f(w)\,dw + \int_{l_{z,z_1}} f(w)\,dw = 0,$$

and so we have the key identity:

$$F(z+h) - F(z) = \int_{l_{z,z+h}} f(w)\,dw. \tag{8.1}$$

Intuitively, if $|h| << 1$, then $f(w) \approx f(z)$ on $l_{z,z+h}$, and so the integral is approximately $f(z)h$. To make this rigorous, we write

$$\int_{l_{z,z+h}} f(w)\,dw = \int_{l_{z,z+h}} f(z)\,dw + \int_{l_{z,z+h}} (f(w) - f(z))\,dw$$

$$= f(z)h + \int_{l_{z,z+h}} (f(w) - f(z))\, dw,$$

and so

$$\frac{F(z+h) - F(z)}{h} = f(z) + \frac{1}{h}\int_{l_{z,z+h}} (f(w) - f(z))\, dw.$$

Dividing by h and using triangle inequality,

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{l_{z,z+h}} (f(w) - f(z))\, dw\right| \leq \frac{1}{|h|}\int_{l_{z,z+h}} |f(w) - f(z)|\, |dw|.$$

By the continuity of f, given any $\varepsilon > 0$ there exists a δ such that if $|h| < \delta$, then for all $w \in l_{z,z+h}$

$$|f(w) - f(z)| \leq \varepsilon.$$

Using this in the above estimate, and remembering that $\mathrm{length}(l_{z,z+h}) = |h|$ we see that

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \leq \varepsilon,$$

so long as $|h| \leq \delta$. This shows that

$$\frac{\partial F}{\partial x}(z) := \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

Next, consider a path σ_z consisting of a vertical line segment from 0 to iy followed by a horizontal segment from iy to z. By Theorem 8.0.2,

$$F(z) = \int_{\sigma_z} f(w)\, dw.$$

By an argument similar to the one above, we can prove that $\partial F/\partial y$ exists, and that

$$\frac{\partial F}{\partial y}(z) = \lim_{k\to 0} \frac{f(z+ik) - f(z)}{k} = if(z).$$

The analog of the key identity is that

$$F(z+ik) - F(z) = \int_{l_{z,z+ik}} f(w)\, dw,$$

which is approximately $if(z)k$. (integrating in the vertical direction incurs an i). In any case, this shows that the partials of F exist *and* are continuous, and hence F is a (totally) differentiable map from D to $\mathbb{R}^2$. On the other hand, since

$$\frac{\partial F}{\partial y}(z) = i\frac{\partial F}{\partial x}(z),$$

the partials also satisfy the Cauchy-Riemann equations. Hence F is complex differentiable at z, and moreover, $F'(z) = f(z)$. □

8.2 Proof of Goursat's theorem

The main idea of the proof is to localize the integral. We start by choosing a sequence of "nested" triangles T_n

$$\mathrm{int}(T_{n+1}) \subset \mathrm{int}(T_n),$$

with $T_0 = T$. Note that when we say triangle we mean the one-dimensional object, and we denote by $\mathrm{int}(T)$ the region bounded by the triangle. Suppose we have already constructed the triangle T_{n-1}. The first step in the construction of T_n is to bisect each side of T_{n-1}. This results in four new triangles which we label $T_{n-1}^{(1)}, T_{n-1}^{(2)}, T_{n-1}^{(3)}$ and $T_{n-1}^{(4)}$. We also give them the an orientation consistent with the original triangle (see figure) so that the integrals over the common boundaries cancel, and we have that

$$\int_{T_{n-1}} f(z)\,dz = \sum_{j=1}^{4} \int_{T_{n-1}^{(j)}} f(z)\,dz.$$

By triangle inequality,

$$\Big|\int_{T_{n-1}} f(z)\,dz\Big| \leq \sum_{j=1}^{4} \Big|\int_{T_{n-1}^{(j)}} f(z)\,dz\Big|,$$

and so for at least one triangle $T_{n-1}^{(j)}$,

$$\Big|\int_{T_{n-1}} f(z)\,dz\Big| \leq 4\Big|\int_{T_{n-1}^{(j)}} f(z)\,dz\Big|.$$

Choose any one such triangle, and label it T_n. Inductively, we have

$$\Big|\int_{T} f(z)\,dz\Big| \leq 4^n\Big|\int_{T_n} f(z)\,dz\Big|. \tag{8.2}$$

Recall that the diameter of a subset of $\mathbb{C}$ is the maximum distance between any two points in that subset. We then have the following elementary observation.

Lemma 8.2.1. *If d_n and p_n denote the diameter and perimeter of the triangle $T^{(n)}$ respectively, then*

$$d_n = 2^{-n}d_0, \text{ and } p_n = 2^{-n}p_0.$$

Moreover, there exists a unique $p \in \cap_{n=0}^{\infty}\mathrm{int}(T_n)$.

Proof. The diameter of a triangle is the length of the longest side. Since $T_{n-1}^{(1)}$, $T_{n-1}^{(2)}$, $T_{n-1}^{(3)}$ and $T_{n-1}^{(4)}$ are all congruent triangles with side lengths that are half of the respective side lengths of $T^{(n-1)}$, clearly $d_n = 2^{-1}d_{n-1}$ and $p_n = 2^{-1}p_{n-1}$, and we obtain the result by induction. Now, let $x_n \in T_n$ be any point. Then since the sequence $\{x_n\}$ is contained in T_0, a compact subset, there exists a limit point $p \in T = T_0$, which will trivially lie inside all the triangles. So the intersection is non-empty. On the other hand since $\lim_{n\to\infty} d_n = 0$ the intersection can contain at most one point. □

Continuing with the proof of Goursat's theorem, since f is holomorphic at $z = p$, we can write

$$f(z) = f(p) + f'(p)(z-p) + \psi(z)(z-p),$$

where $\psi(z) \to 0$ as $z \to p$. Now the constant function $f(p)$ and the linear function $f'(p)(z-p)$ both have primitives; $f(p)z$ and $f'(p)(z-p)^2/2$ respectively. So by the fundamental theorem, their line integrals on T_n are zero. So

$$\int_{T_n} f(z)\,dz = \int_{T_n} \psi(z)(z-p)\,dz.$$

Now since $d_n \to 0$ as $n \to \infty$, given any $\varepsilon > 0$ there exists a n such that

$$|\psi(z)| < \varepsilon$$

on T_n. But then

$$\Big|\int_{T_n} f(z)\,dz\Big| \le \varepsilon \sup_{T_n} |z-p| p_n = \varepsilon d_n p_n \le 4^{-n} \varepsilon d_0 p_0.$$

Then by the inequality in (8.2)

$$\Big|\int_T f(z)\,dz\Big| \le \varepsilon d_0 p_0,$$

which can be made arbitrarily small. Hence

$$\int_T f(z)\,dz = 0,$$

and this proves the theorem in the case that f is holomorphic everywhere.

Remark 8.2.2. The reader should attempt to understand why a similar argument would fail if f is merely assumed to be smooth (as opposed to holomorphic) as a function of two variables.

8.3 Extensions to punctured domains

For applications, we need a slightly stronger version of Cauchy's theorem, which in turn relies on a slightly stronger version of Goursat's theorem. For any open set $\Omega \subset \mathbb{C}$, and any $p \in \Omega$ we denote $\Omega_p^* := \Omega \setminus \{p\}$.

Theorem 8.3.1. *Let D be a disc, and $p \in D$. Let $f : D_p^* \to \mathbb{C}$ be a holomorphic function such that* $\lim_{z\to p}(z-p)f(z) = 0$. *Then for any closed curve $\gamma \subset D_p^*$,*

$$\int_\gamma f(z)\,dz = 0.$$

As before, the kye technical input is a version of Goursat's theorem. This time we will state the version only for rectangles.

Theorem 8.3.2. *We only need to prove the theorem in the case that $p \in \mathrm{int}(R)$. Let Ω be any open subset of $\mathbb{C}$. For some $p \in \Omega$, let $f : \Omega_p^* \to \mathbb{C}$ be a holomorphic function satisfying* $\lim_{z\to p}(z-p)f(z) = 0$. *Then for any rectangle $R \subset \Omega$ with $p \notin R$*

$$\int_R f(z)\,dz = 0.$$

Proof. Without loss of generality, we may assume that p lies in the interior of the region bounded by R. Let $\varepsilon > 0$, and let $\delta > 0$ such that

$$|f(z)| \le \frac{\varepsilon}{|z-p|},$$

whenever $|z-p| < \delta$. Let R_0 be a small square of side length δ with p at it's centre. Note that $|z-p| > \delta/2$ for all $z \in \partial R$. By extending the sides of R_0, divide R into nine rectangles $R_0, \cdots, R_8$. Clearly

$$\int_{R_j} f(z)\,dz = 0$$

for $j = 1, 2, \cdots, 8$ by Theorem 8.0.2. Since the integrals over the common boundaries cancel out if we choose the correct (ie. anti-clockwise) orientations

$$\int_R f(z)\,dz = \int_{R_0} f(z)\,dz.$$

Next, we estimate the integral over R_0,

$$\Big|\int_{\partial R_0} f(z)\,dz\Big| \leq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z-p|} < \frac{2\varepsilon}{\delta}\mathrm{len}(\partial R_0) = 8\varepsilon.$$

Since ε is arbitrary, this shows that $\int_{\partial R} f(z)\,dz = 0$.

□

Proof of Theorem 8.3.1. The idea of the proof is to again show that $f(z)$ has a primitive on D_p^*, and we proceed as in the proof of Theorem 8.1.1. Let $p = (a, b)$. Pick a point $z_0 = (x_0, y_0)$ such that $x_0 \neq a$ and $y_0 \neq b$. Let $z + (x, y) \in D_p^*$. If $x \neq a$, we let $\gamma_{z_0,z}$ be the path consisting of the line segment (x_0, y_0) to (x, y_0), followed by a vertical line from (x, y_0) to (x, y). On the other hand, if $x = a$, then we let $\gamma_{z_0,z}$ consist of three segments - A vertical segment from (x_0, y_0) to (x_0, y_η) followed by a horizontal segment from (x_0, y_η) to (x, y_η) followed by another vertical segment from (x, y_η) to (x, y). We then define $F : D_p^* \to \mathbb{C}$ by

$$F(z) = \int_{\gamma_{z_0,z}} f(w)\,dw.$$

First, we claim that $\partial F/\partial x = f(z)$ for every $z \in D_p^*$. Note that the key step in the proof of Theorem 8.1.1 is to obtain the identity (8.1). We obtain the same identity in this new situation. Let $h \in \mathbb{R}$ be a small number. If $x \neq a$, then the same argument as before will imply that $\partial F/\partial x$ exists at (x, y) and equals $f(z)$. So suppose $x = a$. Let R be the rectangle with vertices z_0, $(a + h, y_0)$, $(a + h, y_\eta)$ and (x_0, y_η) and let R' be the rectangle with vertices (a, y_η), $(a+h, y_\eta)$, $z+h = (a+h, y)$ and $z = (a, y)$. Note that p lies either in the interior or the exterior of R (but not on the boundary), and so by Theorem 8.0.2 or Theorem 8.3.2,

$$\int_{\partial R} f(w)\,dw = 0.$$

On the other hand, p lies in the exterior of R', and so by Theorem (8.0.2),

$$\int_{\partial R'} f(w)\,dw = 0.$$

It is easy to check that

$$F(z+h) - F(z) = \int_{l_{z,z+h}} f(w)\,dw.$$

Now the argument proceeds exactly as in the proof of Theorem 8.1.1. Next, as before, one proves that $\partial F/\partial y$ exists everywhere on D_p^* and equals $if(z)$ using a suitable modification of the path σ.

□

Lecture 9

Cauchy's integral formula

9.1 Cauchy integral formula and Analyticity of holomorphic functions

The main consequence of Cauchy's theorem for the punctured disc is the following beautiful formula - a fundamental mean value property or a localization property. It is not an exaggeration to say that all of complex analysis is contained in this formula, if only a little bit of cleverness is added to the mix.

Theorem 9.1.1 (Cauchy integral formula (CIF))**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $\overline{D_r(z_0)} \subset \Omega$, then for any $z \in D_r(z_0)$,*

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof. For a fixed $z \in D_R(z_0)$, we define $h_z : D^* := D_R(z_0) \setminus \{z\} \to \mathbb{C}$ by

$$h_z(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}.$$

Clearly h_z is holomorphic in D^*. Also $\lim_{\zeta \to z}(\zeta - z)h_z(\zeta) = 0$, since f is holomorphic, and hence continuous at z. Then by Cauchy's theorem

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} h_z(\zeta) &= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta - f(z) \Big(\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{1}{\zeta - z} \, d\zeta \Big) \\ &= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta - f(z), \end{aligned}$$

since the quantity in the bracket is simply $n(C, z)$, where $C = \partial D_r(z_0)$, which is equal to 1 because $z \in D_r(z_0)$. □

An immediate consequence of the Cauchy integral formula is that holomorphic functions are analytic

Theorem 9.1.2. *Let $\Omega \subset \mathbb{C}$ open, and $f : \Omega \to \mathbb{C}$ a holomorphic function. Then f is analytic. Moreover, if $D_R(z_0)$ is* any *disc whose closure is contained in Ω, then for all $z \in D_R(z_0)$, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, where*

$$a_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta. \tag{9.1}$$

In particular, holomorphic functions are infinitely complex differentiable.

Proof. By CIF, if D is a of radius R centered at a with boundary circle C_R, then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=R}\frac{f(\zeta)}{\zeta-z}\,dz.$$

Writing $\zeta - z = (\zeta - z_0) - (z - z_0)$, we see that

$$\frac{1}{\zeta-z} = \frac{1}{\zeta-z_0}\Big(1-\frac{z-z_0}{\zeta-z_0}\Big)^{-1}.$$

For $z \in D$ and $\zeta \in C_R$, $|z - z_0| < R = |\zeta - z_0|$, or equivalently $|(z - z_0)/(\zeta - z_0)| < 1$, and hence using the geometric series

$$\frac{1}{\zeta-z} = \frac{1}{\zeta-z_0}\sum_{n=0}^{\infty}\Big(\frac{z-z_0}{\zeta-z_0}\Big)^{n}.$$

Since power series converge uniformly, we can also integrate term-wise (see Appendix), we get that

$$\begin{aligned}2\pi i f(z) &= \int_{C_R}\frac{f(\zeta)}{\zeta-z_0}\sum_{n=0}^{\infty}\Big(\frac{z-z_0}{\zeta-z_0}\Big)^{n}d\zeta\\ &= \sum_{n=0}^{\infty}\int_{C_R}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\cdot(z-z_0)^n\\ &= 2\pi i\sum_{n=0}^{\infty}a_n(z-z_0)^n,\end{aligned}$$

where a_n is given by the formula (9.1), and this completes the proof of the theorem. Infinite complex differentiability follows from analyticity by Corollary 1.1 from Lecture-3. □

As an immediate corollary, we have the following versions of the principle of analytic continuation for holomorphic functions.

Corollary 9.1.3 (Principle of analytic continuation)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, and suppose Ω is connected.*

1. *If there exists a $p \in \Omega$ such that $f^{(n)}(p) = 0$ for $n = 1, 2, \cdots$, then f is a constant function.*
2. *If there exists a n open subset U such that $f\Big|_U \equiv 0$, then $f\Big|_\Omega \equiv 0$.*

This follows immediately from Theorem 9.1.2 above and Corollary 3.3.5 from Lecture-4. Another important consequence of the analyticity is the following criteria for holomorphicity.

Corollary 9.1.4 (Morera)**.** *Any continuous function on an open set Ω that satisfies*

$$\int_{\partial R} f(z)\,dz = 0$$

for all rectangular regions $R \subset \Omega$, is holomorphic.

Proof. Let $p \in \Omega$ and $r > 0$ such that $\overline{D_r(p)} \subset \Omega$. The given condition is equivalent to f having a primitive $F_p(z)$ on $D_r(p)$, essentially by the proof of Theorem 2.1 in Lecture-7. Note that continuity of f is crucial for this. But then by Theorem 9.1.2, $F_p'(z) = f(z)$ is also holomorphic on $D_r(p)$. In particular, f is complex differentiable at p. Since this is true for all $p \in \Omega$, $f \in \mathcal{O}(\Omega)$. □

9.2 Cauchy integral formula for derivatives

Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then by Theorem 9.1.2, the n^{th} complex derivative $f^{(n)}(z)$ exists for all $z \in \Omega$. Moreover, by equation (9.1), for any $z \in \Omega$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta,$$

where $r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$. More generally, just as for the Cauchy integral formula, one can obtain a similar formula for $f^n(z_0)$ where the integral is over a circle centred at possibly a point other than z_0.

Theorem 9.2.1 (Cauchy integral formula for derivatives)**.** *If D is a disc with boundary C whose closure is contained in Ω, then for any $z \in D$, we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta. \tag{9.2}$$

Remark 9.2.2. Essentially what this theorem says is that one can differentiate the Cauchy integral formula, and take the derivative inside the integral.

Proof. Since $f(z)$ is analytic in the neighborhood of any point $z \in \Omega$, it is automatically holomorphic in Ω. To prove the formula (9.2) we use induction. For $n = 0$, this is simply the Cauchy integral formula.

$$\begin{aligned}
\frac{f^{(n-1)}(z) - f^{(n-1)}(z_0)}{z - z_0} &= \frac{(n-1)!}{2\pi i (z - z_0)} \int_C f(\zeta) \Big(\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z_0)^n} \Big) \, d\zeta \\
&= \frac{(n-1)!}{2\pi i} \int_C f(\zeta) \cdot \frac{(\zeta - z_0)^{n-1} + \cdots + (\zeta - z)^{n-1}}{(\zeta - z)^n (\zeta - z_0)^n} \, d\zeta \\
&= \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^n} \sum_{k=0}^{n-1} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} \, d\zeta.
\end{aligned}$$

Suppose $D = D_R(p)$, and suppose $|f(\zeta)| < M$ on C. By choosing z sufficiently close to z_0, we can ensure that for all $\zeta \in C$,

$$|\zeta - z| > (R - |z_0 - p|)/2 := \lambda(z_0).$$

Also, trivially, $|\zeta - z|, |\zeta - z_0| < 2R$ and $|\zeta - z_0| > \lambda(z_0)$. Then for any $k < n$,

$$\Big| \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} - \frac{1}{\zeta - z_0} \Big| = \frac{|z - z_0| (|\zeta - z|^k + \cdots |\zeta - z_0|^k)}{|\zeta - z|^{k+1} |\zeta - z_0|} < n|z - z_0| \Big(\frac{2R}{\lambda(z_0)} \Big)^{n+2}.$$

Hence, given any $\varepsilon > 0$, there exists a $\delta = \delta(n, R, M, z_0) > 0$ such that for any $|z - z_0| < \delta$, and any $\zeta \in C$, we have

$$\Big| \frac{f(\zeta)}{(\zeta - z_0)^n} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} - \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \Big| < \varepsilon.$$

uniformly in the sense that the rate of convergence is independent of ζ. So if $|z - z_0| < \delta$, then for each k,

$$\Big| \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^n} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} \, d\zeta - \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \Big| \le (n-1)! R \varepsilon,$$

and hence

$$\lim_{z \to z_0} \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^n} \frac{(\zeta - z_0)^k}{(\zeta - z)^{k+1}} \, d\zeta = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.$$

Summing up over k, we see that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta.$$

□

An extremely useful consequence is the following estimate on the derivatives of a holomorphic function.

Corollary 9.2.3. *[Cauchy's estimate] Let f be a holomorphic function in an open set containing the closure of a disc $D_R(z_0)$. If we denote the boundary of the disc by C, then for any $z \in D_R(z_0)$,*

$$|f^{(n)}(z)| \leq \frac{n!R}{(R - |z - z_0|)^{n+1}}||f||_C,$$

where $||f||_C := \sup_{\zeta\in C}|f(\zeta)|$. In particular,

$$f^{(n)}(z_0) \leq \frac{n!}{R^n}||f||_C,$$

Proof. By CIF for derivatives, we have that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Applying the triangle inequality, and remembering that on C, $|\zeta - z| \geq R - |z - z_0|$,

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi}\int_C \left|\frac{|f(\zeta)|}{|\zeta - z|^{n+1}}\right| |d\zeta| \\ &\leq \frac{n!}{2\pi(R - |z - z_0|)^{n+1}} \sup_C |f(\zeta)| \cdot \mathrm{len}(C) \\ &= \frac{n!R}{(R - |z - z_0|)^{n+1}}||f||_C. \end{aligned}$$

□

9.3 Liouville Theorem

Recall that an *entire* function is a function that is holomorphic on the entire complex plane $\mathbb{C}$. We then have the following surprising fact.

Theorem 9.3.1 (Liouville)**.** *There are no bounded non-constant entire functions.*

Proof of Theorem 24.3.4. Let $f(z)$ be a bounded entire function. We show that it then has to be a constant. Suppose $|f(\zeta)| < M$ for all $\zeta \in \mathbb{C}$, and let $z \in \mathbb{C}$ be an arbitrary point. Since f is entire, it is holomorphic on any disc $D_R(z)$. Denoting the boundary by C_R, by the estimate above,

$$|f'(z)| \leq \frac{||f||_{C_R}}{R} < \frac{M}{R}.$$

Letting $R \to \infty$ the right hand side approaches zero, and hence $f'(z) = 0$ for all $z \in \mathbb{C}$. Since $\mathbb{C}$ is connected, this forces $f(z)$ to be a constant, completing the proof of Liouville's theorem. □

Remark 9.3.2. More generally, we can show that an entire function $f(z)$ satisfying

$$|f(z)| \leq M(1 + |z|^\alpha),$$

for some constants $M, \alpha > 0$ and all $z \in \mathbb{C}$, has to be a polynomial of degree at most $\lfloor \alpha \rfloor$, where $\lfloor \cdot \rfloor$ is the usual floor function. We leave this as an exercise.

Lecture 10

Further applications of Cauchy integral formula

In this lecture, we give three further applications of the Cauchy integral formula.

10.1 The fundamental Theorem of algebra

Recall that in the previous lecture we proved Liouville's theorem, namely that there are no bounded, non-constant, entire functions. As a simple and beautiful consequence of this, we can prove the fundamental theorem of algebra, namely that any polynomial can be completely factored into linear factors over complex numbers.

Theorem 10.1.1. *Any non-constant polynomial $p(z)$ has a complex root, that is there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$. As a consequence, any polynomial is completely factorable, that is we can find $\alpha_1, \cdots, \alpha_n$ (some of them might be equal to each other), and $c \in \mathbb{C}$ such that*

$$p(z) = c(z - \alpha_1) \cdots (z - \alpha_n),$$

where $n = \deg(p(z))$.

Proof. For concreteness, let

$$p(z) = a_n z^n + \cdots + a_1 z + a_0,$$

where $a_n \neq 0$. Then we claim that there exists an R such that

$$\frac{|a_n|}{2}|z|^n \leq |p(z)| \leq 2|a_n||z|^n,$$

whenever $|z| > R$. To see this, by the triangle inequality,

$$|p(z)| \leq |a_n||z|^n + \cdots |a_1||z| = |a_0| = |a_n||z|^n \Big(1 + \frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_0|}{|z|^n}\Big) \leq \frac{3}{2}|a_n||z|^n$$

if $|z| > R$ and R is sufficiently big. For the other inequality, we use the other side of the triangle inequality. That is,

$$|p(z)| \geq |a_n||z|^n \Big|1 - \Big|\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\Big|\Big|.$$

But
$$\Big|\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\Big|\leq\frac{|a_{n-1}|}{|z|}+\cdots+\frac{|a_0|}{|z|^n}<\frac{1}{2},$$
if $|z|>R$ when R is as above. So when $|z|>R$,
$$|p(z)|\geq\frac{|a_n||z|^n}{2}.$$
Now suppose, that $p(z)$ has no root in $\mathbb{C}$. Then
$$f(z)=\frac{1}{p(z)}$$
will be an entire function. Moreover, on $|z|>R$, by the lower bound above,
$$|p(z)|>|a_n||z|^n/2>|a_n|R^n/2,$$
that is $|f(z)|\leq M$ for some M on $|z|>R$. On the other hand, we claim that there exists an $\varepsilon>0$ such that $|p(z)|>\varepsilon$ on $|z|\leq R$. If not, then there is a sequence of points $z_k\in\overline{D_R(0)}$ such that $|p(z_k)|\to 0$. Since $\overline{D_R(0)}$ is compact, there exists a subsequence, which we continue to denote by z_k, such that $z_k\to a\in\overline{D_R(0)}$. But then by continuity, $p(a)=0$, contradicting our assumption that there is no root. This proves the claim. The upshot is that on $|z|\leq R$, $|f(z)|\leq 1/\varepsilon$. This shows that $f(z)$ is a bounded, entire function, and hence by Liouville, must be a constant, which in turn implies that $p(z)$ must be a constant. This proves the first part of the theorem.

The second part follows from induction. If $p(z)$ is a non-constant polynomial, let α_1 be a root, which is guaranteed to exist by the first part. Then by the remainder theorem, $p(z)$ is divisible by $(z-\alpha_1)$. For a proof of the remainder theorem, see remark below. Then we define a new polynomial
$$p_1(z)=\frac{p(z)}{z-\alpha_1}.$$
The crucial observation is that $\deg(p_1)$ is *strictly* smaller than $\deg(p)$. In finitely many steps we should reach a linear polynomial which is obviously factorable. □

Remark 10.1.2. We used the remainder theorem for the proof of the second part of the theorem which basically says that for a polynomial $p(z)$,
$$p(z)-p(a)=(z-a)q(z)$$
for some polynomial $q(z)$ of a strictly smaller degree. To see this, let
$$p(z)=a_nz^n+\cdots+a_1z+a_0,$$
and so
$$p(z)-p(a)=a_n(z^n-a^n)+\cdots+a_1(z-a).$$
Each term is of the form
$$a_k(z^k-a^k)=a_k(z-a)(z^{k-1}+z^{k-2}a+\cdots+za^{k-2}+a^{k-1}),$$
and this completes the proof. It is also easy to see that highest degree term in $q(z)$ is a_nz^{n-1}, and hence in particular, the degree of $q(z)$ is strictly smaller.

10.2 Zeroes of a holomorphic function

A complex number a is called a *zero* of a holomorphic function $f : \Omega \to \mathbb{C}$ if $f(a) = 0$. A basic fact is that zeroes of holomorphic functions are *isolated*. This follows from the following theorem.

Theorem 10.2.1. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function that is not identically zero, and let $a \in \Omega$ be a zero of f. Then there exists a disc D around a, a non vanishing holomorphic function $g : D \to \mathbb{C}$ (that is, $g(z) \neq 0$ for all $z \in D$), and a unique positive integer n such that*

$$f(z) = (z-a)^n g(z).$$

Moreover, we have that

$$n = \min\{\nu \in \mathbb{N} \mid f^{(\nu)}(a) \neq 0\}.$$

The positive integer n is called the *order* or *multiplicity* of the zero at a.

Proof. By the principle of analytic continuation, if f is not identically zero, there exists an n such that $f^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$ but $f^{(n)}(a) \neq 0$. Let $D_\varepsilon(a)$ be a disc such that $\overline{D_\varepsilon(a)} \subset \Omega$. Then f has a power series expansion in the disc centered at $z = a$ with the first n terms vanishing. So we write

$$f(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots$$

with $a_n \neq 0$. The the theorem is proved with

$$g(z) = a_n + a_{n+1}(z-a) + \cdots .$$

□

As an immediate corollary to the theorem we have the following.

Corollary 10.2.2. *Let $f : \Omega \to \mathbb{C}$ holomorphic.*

1. *The set of zeroes of f is isolated. That is, for every zero a, there exists a small disc $D_\varepsilon(a)$ centered at a such that $f(z) \neq 0$ for all $z \in D_\varepsilon(a) \setminus \{a\}$.*
2. *(strong principle of analytic continuation) If Ω is connected, and $f, g : \Omega \to \mathbb{C}$ are holomorphic functions such that for some sequence of points $p_n \in \Omega$, $f(p_n) = g(p_n)$. Then either $f \equiv g$ or $\{p_n\}$ does not have a limit point in Ω.*

Proof. By the theorem, if $a \in \Omega$ is a root, then there exists a disc D around a, and integer m, and a holomorphic function $g : D \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = (z-a)^m g(z).$$

Since $g(a) \neq 0$, by continuity, there is a small disc $D_\varepsilon(a) \subset D$ such that $g(z) \neq 0$ for any $z \in D_\varepsilon(a)$. But then on this disc $(z-a)$ is also not zero anywhere except at a, and hence for any $z \in D_\varepsilon(a) \setminus \{a\}$, $f(z) \neq 0$ exactly what we wished to prove. Part (2) is a trivial consequence of the Lemma. □

Example 10.2.3. *Even though the zeroes are isolated, they could converge to the boundary. For instance, consider the holomorphic function*

$$f(z) = \sin\left(\frac{\pi}{z}\right)$$

on $\mathbb{C}^*$. Clearly $z = 1/n$ is a sequence of zeroes. They are isolated, but converge to $z = 0$ which is not in the domain.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then by Corollary 10.2.2 any disc D such that $\overline{D} \subset \Omega$ contains only finitely many zeroes of the function in the interior. The next proposition allows one to calculate the number of zeroes counted with multiplicity.

Corollary 10.2.4. *Let $f : \Omega \to \mathbb{C}$ be holomorphic, and let $D \subset \Omega$ be a disc such that $\overline{D} \subset \Omega$, and $C := \partial D$ contains no zeroes of f. Let $p_1. \cdots, p_k$ are the zeroes of f in D with multiplicity $n_1. \cdots, n_k$.*

1. *There exists a no-where vanishing holomorphic function $g : D \to \mathbb{C}$ such that*

$$f(z) = (z - p_1)^{n_1} \cdots (z - p_k)^{n_k} g(z).$$

2. *The total number of roots (counted with multiplicity) is given by*

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{k} n_j.$$

Proof. 1. For each $j \in \{1, \cdots, k\}$, there exist a radius $r_j > 0$ and a nowhere vanishing holomorphic function $g_j : D_{r_j}(p_j) \to \mathbb{C}$ such that for all $z \in D_{r_j}(p_j)$,

$$f(z) = (z - p_j)^{n_j} g_j(z).$$

We can choose r_j small enough so that the discs have mutually disjoint closures. Let $U := D \setminus \left(\cup_{j=1}^{k} \overline{D_{r_j/2}(p_j)} \right)$. Define the function

$$g(z) := \begin{cases} \frac{g_j(z)}{\Pi_{i \neq j}(z-p_i)^{n_i}}, & z \in D_{r_j}(p_j) \text{ where } j = 1, \cdots, k \\ \frac{f(z)}{\Pi_j (z-p_j)^{n_j}}, & z \in U. \end{cases}$$

Clearly $g(z)$ satisfies all the required properties.

2. With $g(z)$ as above, for any $z \neq p_j$, a simple computation shows that

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{k} \frac{n_j}{z - p_j} + \frac{g'(z)}{g(z)}.$$

Since $g(z)$ is nowhere vanishing, $g'(z)/g(z)$ is holomorphic on D, and hence by Cauchy's theorem it's integral on C vanishes. It then follows that

$$\int_C \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{k} n_j \int_C \frac{dz}{z - p_j} = \sum_j n_j.$$

□

10.3 Sequences of holomorphic functions

Recall that if a sequence of differentiable functions $f_n : I \to \mathbb{R}$ converges uniformly to $f : I \to \mathbb{R}$, for some interval $I \subset \mathbb{R}$, it does not necessarily imply that f is also differentiable. For instance, consider $f_n : [-1, 1] \to \mathbb{R}$ defined by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

Then it is not difficult to see that $f_n \to |x|$ uniformly, but of course $|x|$ is not differentiable at $x = 0$. It turns out that holomorphic function behave much better under uniform convergence. We say that a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ converges to $f : \Omega \to \mathbb{C}$ *compactly* if the convergence is uniform over compact subsets $K \subset \Omega$. That is, for all $\varepsilon > 0$ and $K \subset \Omega$ compact, there exists a $N = N(\varepsilon, K)$ such that

$$|f_n(x) - f(x)| < \varepsilon,$$

for all $n > N$ and all $x \in K$.

Theorem 10.3.1. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge compactly to $f : \Omega \to \mathbb{C}$, then $f(z)$ is holomorphic. Moreover*

$$f_n^{(k)} \to f^{(k)}$$

compactly on Ω for all $k \in \mathbb{N}$.

Proof. Fix a $a \in \Omega$, and let D be a disc around a such that it's closure is also in Ω. Then for any triangle $T \subset D$, by Goursat's theorem

$$\int_T f_n(z)\, dz = 0.$$

Since $\overline{D}$ is compact, $f_n \to f$ converges uniformly on D, and hence

$$\int_T f(z)\, dz = \lim_{n\to\infty} \int_T f_n(z)\, dz = 0.$$

But this is true for all triangles in D, and hence by Morera's theorem f is holomorphic in D and in particular at a. Since a is arbitrary, this shows that f is holomorphic on Ω.

We prove that $f_n' \to f'$ compactly. For $k > 1$, the result will follow from induction. There is no loss of generality in assuming that $\overline{\Omega} \subset \mathbb{C}$ is compact. First we define

$$\Omega_r = \{z \in \Omega \mid \overline{D_r(z)} \subset \Omega\}.$$

Geometrically, this is the set of all points in Ω that are at least a distance r away from the boundary of Ω. Given any compact set K, there exists a $r > 0$ such that $K \subset \Omega_r$, and hence it suffices to show that $f_n' \to f'$ uniformly on Ω_r. The key point is the following estimate.

Claim. Let $F : \Omega \to \mathbb{C}$ be holomorphic. Then for any $r > 0$,

$$\sup_{z\in\Omega_r} |F'(\zeta)| \leq \frac{2}{r} \sup_{\zeta\in\Omega_{r/2}} |F(\zeta)|.$$

First notice that if $z \in \Omega_r$ then $\overline{D_{r/2}(z)} \subset \Omega_r$. To see this, let $w \in \overline{D_{r/2}(z)}$ i.e. $|z - w| \leq r/2$, and we need to show that $w \in \Omega_{r/2}$ or equivalently that $\overline{D_{r/2}(w)} \subset \Omega$. But for any $w' \in \overline{D_{r/2}(w)}$, $|w' - w| \leq r/2$ and hence by the triangle inequality $|w' - z| \leq r$, that is $w' \in \overline{D_r(z)}$ which is contained in Ω by definition, since $z \in \Omega_r$. This shows that $\overline{D_{r/2}(w)} \subset \Omega$ and hence that $\overline{D_{r/2}(z)} \subset \Omega_{r/2}$. Now by Cauchy's integral formula, if we denote the boundary of $D_r(z)$ by $C_r(z)$, then for any $z \in \Omega_{2r}$,

$$F'(z) = \frac{1}{2\pi i} \int_{C_{r/2}(z)} \frac{F(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

By the above observation, if $z \in \Omega_r$ then $C_{r/2}(z) \subset \Omega_{r/2}$, and hence by triangle inequality, for all $z \in \Omega_r$, since $|\zeta - z| = r/2$ for $\zeta \in C_{r/2}(z)$ we have

$$|F'(z)| \leq \frac{1}{2\pi r^2} \sup_{\zeta\in C_{r/2}(z)} |F(\zeta)| \mathrm{len}(C_{r/2}(z))$$

$$\leq \frac{2}{r} \sup_{\zeta \in \Omega_r} |F(\zeta)|$$

This proves the claim. Now given any $\varepsilon > 0$, since $\overline{\Omega_{r/2}} \subset \Omega$ is compact (remember we are assuming without any loss of generality that Ω is bounded), there exists an $N = N(r, \varepsilon)$ such that for all $n > N$ and all $\zeta \in \Omega_{r/2}$,

$$|f_n(\zeta) - f(\zeta)| < \frac{\varepsilon r}{2}.$$

But then by the claim, for $n > N$ and for all $z \in \Omega_r$, we have the estimate

$$|f(z) - f_n(z)| \leq \frac{2}{r} \cdot \frac{\varepsilon r}{2} = \varepsilon,$$

proving that $f_n \to f$ uniformly on Ω_r, and this completes the proof of the theorem.

□

Recall that Weiestrass' theorem states that any continuous function on a compact interval is the uniform limit of polynomials. On the other, by the above theorem, a continuous non-holomorphic function cannot be the uniform limit of polynomials. Instead we have the following, which we state without a proof.

Theorem 10.3.2 (Runge's thoerem). *Let $K \subset \mathbb{C}$ and let f be a function that is holomorphic in a neighbourhood of K.*

1. *There exists a sequence of rational functions $R_n(z)$ such that $R_n \xrightarrow{u.c} f$ on K, and such that the singularities of the rational functions all lie in K^c.*
2. *If K^c is connected, then one there exists a sequence of polynomials $p_n(z)$ such that $p_n \xrightarrow{u.c} f$.*

Lecture 11

Cauchy's theorem : Homology version

11.1 Index of a curve

For a piecewise smooth (not necessarily close) curve, we defined the *index* or *winding number* around a point $p \notin \gamma$ by

$$n(\gamma, p) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z-p}\, dz.$$

We have already seen that if γ is a circle traversed n number of times, then

$$n(\gamma, p) = \begin{cases} n, & p \text{ is inside the disc bounded by } \gamma \\ 0, & \text{otherwise} \end{cases}$$

We now argue that this number is a measure of the change in the argument along the curve. For simplicity, lets suppose that $p = 0$, and that the curve γ joins $u = re^{i\theta}$ to $v = re^{i\varphi}$ (note that γ need not be a circular arc), where $\theta, \varphi \in (-\pi, \pi)$. In particular, the curve lies in $\mathbb{C} \setminus \{z < 0\}$. On this domain, $1/z$ has a primitive, which we take to be the principal branch of the logarithm. Then by the fundamental theorem

$$n(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z}\, dz = \log v - \log u = \frac{\varphi - \theta}{2\pi},$$

and so up to a factor of 2π, the index measures the change in the argument. The following theorem contains the basic properties of the index.

Theorem 11.1.1. *Let γ be any closed curve. Then*

1. *$n(\gamma, p)$ is an integer for any $p \in \mathbb{C} \setminus \gamma$.*
2. *$n(\gamma, p)$ is a continuous function on $\mathbb{C} \setminus \gamma$, and hence is locally constant.*
3. *If γ is any curve lying in the interior of a disc D, then $n(\gamma, p) = 0$ for all $p \in \mathbb{C} \setminus \overline{D}$.*

Proof. 1. If we could take holomorphic logarithms freely, and argument as above would suffice. Instead we will give a computational proof. Let $\gamma : [0, 1] \to \mathbb{C}$ be parametrization. Then

$$n(\gamma, p) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t) - p}\, dt.$$

Consider a function

$$g(s) = \frac{1}{2\pi i} \int_0^s \frac{\gamma'(t)}{\gamma(t) - p}\, dt.$$

This is a continuous function on $[0, 1]$, and is differentiable wherever $\gamma'(t)$ is continuous (and hence at all but finitely many points) with derivative

$$g'(s) = \frac{1}{2\pi i} \frac{\gamma'(t)}{\gamma(t) - p}.$$

Then, letting

$$G(s) = \exp(-2\pi i g(s))(\gamma(s) - p),$$

at all but finitely many points, $G'(s) = 0$. This shows that $G(s)$ is locally constant, which together with continuity, forces it to be a constant. In particular, $G(1) = G(0)$, from which it follows (since $g(0) = 1$ and $\gamma(0) = \gamma(1)$) that

$$e^{2\pi i g(1)} = \frac{\gamma(0) - p}{\gamma(1) - p} = 1.$$

So $g(1) = n(\gamma, p)$ has to be an integer.

2. Continuity is easy to check since $p \notin \gamma$. Since the index is integer valued it has to be then locally constant.

3. If $|p| >> 1$, then clearly $n(\gamma, p)$ can be made really small. But then since the index is locally constant, and $\mathbb{C} \setminus \overline{D}$ is connected, it ought to be zero for all $p \in \mathbb{C} \setminus \overline{D}$.

□

Remark 11.1.2. There is a deeper reason that the index is always an integer, and a full explanation requires some knowledge of covering space theory. If $a \in \mathbb{C}$ does not lie on $\gamma : [0, 1] \to \mathbb{C}$, we can think of γ as a curve in $\mathbb{C}_a^* := \mathbb{C} \setminus \{a\}$. Then it follows from standard covering space theory that γ has a "lift" to a curve $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ such that $e^{2\pi i \tilde{\gamma}(t)} = \gamma(t) - a$. The relevant jargon is that $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering space map. Now additionally if γ is closed, then $\gamma(1) = \gamma(0)$, and hence $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer. On the other hand, by chain rule, $\gamma'(t) = 2\pi i e^{2\pi i \tilde{\gamma}(t)} \tilde{\gamma}'(t)$, and hence

$$\tilde{\gamma}'(t) = \frac{1}{2\pi i} \cdot \frac{\gamma'(t)}{\gamma(t) - a}.$$

Integrating both sides we see that

$$\mathbb{Z} \ni \tilde{\gamma}(1) - \tilde{\gamma}(0) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t) - a}\, dt = \int_\gamma \frac{dz}{z - a} := n(\gamma, a),$$

and hence $n(\gamma, a)$ is an integer.

11.2 The homology version of Cauchy's theorem

A *chain* is a formal linear combination of curves $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$, where $a_i \in \mathbb{Z}$ and each γ_i is a regular curve. We interpret $a\gamma$ as γ traversed a times if $a > 0$ and γ traversed in the the reverse direction $-a$ times if $a < 0$. Chains can be then added in an obvious way. The union of the set theoretic images of γ_j is called the *support* of γ and denoted by $\mathrm{Supp}(\gamma)$. A chain is called a *cycle* if each of it's components γ_j is a closed curve. We can extend the definition of the index $n(\gamma, p)$ for a cycle γ and a point p that does not lie on any of the components of γ. Then it has the same properties as in Theorem 11.1.1. Additionally, we have the linearity property that

$$n(\gamma_1 + \gamma_2, p) = n(\gamma_1, p) + n(\gamma_2, p).$$

We say that a chain γ is *homologous* to zero in Ω, and write $\gamma \sim_\Omega 0$ (or $\gamma \sim 0(mod\Omega)$), if for any point $a \in \Omega^c$, $n(\gamma, p) = 0$. We also say that γ_1 is homologous to γ_2 in Ω and write $\gamma_1 \sim_\Omega \gamma_2$ if $\gamma_1 - \gamma_2 \sim_\Omega 0$, or equivalently if $n(\gamma_1, p) = n(\gamma_2, p)$ for all $p \in \Omega^c$. Note that if $\Omega \subset \Omega'$ then $\gamma \sim_\Omega 0$ implies that $\gamma \sim_{\Omega'} 0$, but the converse need not be true as can be seen in the example below.

Example 11.2.1. *Consider the disc $D_3(0)$ and the annulus $A_{1,2}(0) := \{z \in \mathbb{C} \mid 1 < |z| < 3\}$. By Cauchy's theorem for discs, any curve γ in $D_3(0)$ is homologous to zero. On the other hand the curve $\gamma(t) = 2e^{2\pi i t},\ t \in [0,1]$ is NOT homologous to zero in $A_{1,2}(0)$. This is because $n(\gamma, 0) = 1 \neq 0$.*

Now we are ready to state the most general form of Cauchy's theorem.

Theorem 11.2.2 (Generalised Cauchy's theorem)**.** *If $f : \Omega \to \mathbb{C}$ is holomorphic and $\gamma \sim_\Omega 0$, then*

$$\int_\gamma f(z)\, dz = 0.$$

More generally, if γ_1, γ_2 are curves in Ω such that $\gamma_1 \sim_\Omega \gamma_2$, then

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

In other words, Cauchy's theorem says that if the integrals of the holomorphic functions $1/(z-a)$ is zero on a closed curve in Ω, then the integral of *any* holomorphic function on γ is zero.

Proof. This beautiful proof is taken from [1]. We first assume that Ω is bounded. For a small $\delta > 0$, we cover the plane with a net consisting of squares with sides parallel to the axes, and length δ. Since Ω is bounded, and if $\delta > 0$ is chosen sufficiently small, there exists a finite set of number of cubes $Q_1, \cdots, Q_N$ such that

1. The collection of squares $\{Q_1, \cdots, Q_j\}$ are all the squares in the net which lie completely inside Ω. That is, if Q is a square in the net, then $Q = Q_j$ for some j if and only if $Q \subset \Omega$.
2. $\gamma \subset \Omega_\delta := \Big(\cup_{j=1}^N Q_j\Big)^\circ$.

Consider the cycle

$$\Gamma_\delta = \sum_j \partial Q_j,$$

where the boundary of each Q_j is oriented in the anti-clockwise direction. Then Γ_δ is *equivalent* to $\partial\Omega_\delta$ in the sense that for any function

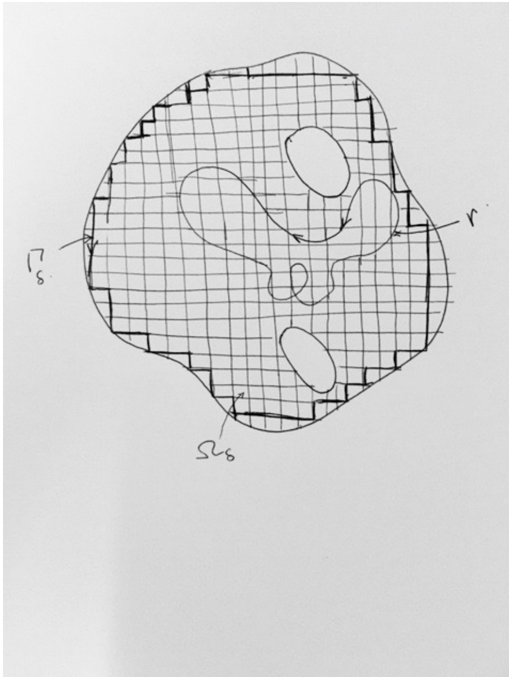
$$\int_{\Gamma_\delta} f(z)\, dz = \int_{\partial\Omega_\delta} f(z)\, dz,$$

since the integrals over the common boundaries cancel.

Now, let γ be a cycle homologous to zero in Ω. Let $\zeta \in \Omega \setminus \Omega_\delta$, and let Q be a square in the net such that $\zeta \in Q$. By definition of Ω_δ, $Q \neq Q_j$ for any j. Again, by our choice of the squares that make up Ω_δ, there exists a point $\zeta_0 \in Q \cap \Omega^c$. Since $\zeta_0 \notin \Omega$ and $\gamma \sim_\Omega 0$, we have that $n(\gamma, \zeta_0) = 0$. But then by continuity, since ζ_0 and ζ can be joined by a straight in Q and hence not intersecting γ, we have $n(\gamma, \zeta) = 0$. In particular, $n(\gamma, \zeta) = 0$ for all $\zeta \in \partial\Omega_\delta$.

Suppose now that $f(z)$ is holomorphic on Ω. For any $z \in \overset{\circ}{Q}_{j_0}$, we have

$$\frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} f(z),\ j = j_0, \\ 0,\ \text{otherwise}, \end{cases}$$



and hence

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{11.1}$$

Since j_0 was arbitrary, (11.1) holds for all $z\in\cup_j Q_j^\circ$. But since both sides are continuous in z, clearly (11.1) must hold on all of Ω_δ. As a consequence,

$$\int_\gamma f(z)\,dz=\int_\gamma\left(\frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\zeta)}{\zeta-z}\,d\zeta\right)dz.$$

By Fubini's theorem (which can be applied since the integrand is continuous in both z and ζ),

$$\int_\gamma f(z)\,dz=\int_{\Gamma_\delta}f(\zeta)\Big(\frac{1}{2\pi i}\int_\gamma\frac{dz}{\zeta-z}\Big)\,d\zeta=\int_{\Gamma_\delta}f(\zeta)n(\gamma,\zeta)\,d\zeta=0.$$

Finally, if Ω is not bounded, consider a large disc $D_R(0)$ which contains $\operatorname{supp}(\gamma)$ in the interior, and let $\Omega'=\Omega\cap D_R(0)$. Then since $\gamma\sim_{D_R(0)}0$, one can easily see that $\gamma\sim_{\Omega'}0$, and the previous argument then applies to Ω' completing the proof. $\square$

Using the generalised Cauchy theorem, we can prove the following generalisation of the CIF.

Theorem 11.2.3 (Generalised Cauchy integral formula (GCIF))**.** *Let $f\in\mathcal{O}(\Omega)$, and $\gamma\subset\Omega$ a cycle. If $\gamma\sim_\Omega 0$, then for any $z\in\Omega\setminus\operatorname{Supp}(\gamma)$,*

$$n(\gamma,z)f(z)=\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Proof. Fix a $z\in\Omega\setminus\operatorname{Supp}(\gamma)$, and let $\varepsilon_0>0$ such that $D_{\varepsilon_0}(z)\cap\operatorname{Supp}(\gamma)=\phi$. For $\varepsilon\in(0,\varepsilon_0)$, let C_ε be the circle centred at z with radius ε.

Claim. For every $\varepsilon\in(0,\varepsilon_0)$, $\gamma\sim_{\Omega_z^*}n(\gamma,z)C_\varepsilon$, where as usual $\Omega_z^*=\Omega\setminus\{z\}$.

Proof of the Claim. We need to prove that for any $\zeta \in \mathbb{C} \setminus \Omega_z^*$,

$$n(\gamma, \zeta) = n(\gamma, z) n(C_\varepsilon, \zeta). \tag{11.2}$$

First suppose $\zeta \in \mathbb{C} \setminus \Omega$. Then in particular, ζ lies outside $D_\varepsilon(z)$ and hence $n(C_\varepsilon, \zeta) = 0$. On the other hand since $\gamma \sim_\Omega 0$, we also have $n(\gamma, \zeta) = 0$, and hence (11.2) is verified. The only other possibility is that $\zeta = z$. But then $n(C_\varepsilon, \zeta) = 1$, and hence again (11.2) is verified.

□

Now, applying Cauchy's theorem to the holomorphic function $f(\zeta)/(\zeta - z)$ on Ω_z^*, we see that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{n(\gamma, z)}{2\pi i} \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, d\zeta. \tag{11.3}$$

The integral on the right is $n(\gamma, z) f(z)$ by the CIF for discs, and we are done. But it is in fact possible to avoid the CIF altogether, and in the process provide a second proof for the CIF on discs. The argument is as follows. Given any $\eta > 0$, by choosing $\varepsilon << 1$, we can make sure that

$$|f(\zeta) - f(z)| < \eta$$

for all $\zeta \in C_\varepsilon$. Then

$$\left| \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, d\zeta - f(z) \right| = \left| \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta \right| < \eta.$$

Hence from (11.3),

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta = n(\gamma, z) \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, d\zeta = n(\gamma, z) f(z).$$

□

11.3 Simply connected domains

There are many equivalent ways of defining simply connected domains. Following Ahlfors, we take a slightly non-standard route. Consider the sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

We denote it's north pole by $N = (0, 0, 1)$, and south pole by $S = (0, 0, -1)$. Then consider the stereographic projection $\Phi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ defined by

$$\Phi(x, y, z) = \frac{x + iy}{1 - z}.$$

Then Φ_N is a bijection, and is in fact a homeomorphism. We can thus identify $\mathbb{S}^2$ as a one-point compactification of $\mathbb{C}$, and call it the extended complex plane, and think of N as the "point at infinity".

We then say that a domain $\Omega \subset \mathbb{C}$ is *simply connected* if it is connected, *and* $\mathbb{S}^2 \setminus \Omega$ is also connected. We note that this is not a commonly used definition since it does not work in higher dimensions. In the next lecture, we will provide equivalent characterisations, one of which will be what is the modern "textbook" definition.

Example 11.3.1. *A disc, $\mathbb{C}$ itself, and half planes are simply connected. A parallel strip, say $\{z \mid a \leq Im(\zeta) \leq b\}$ is also simply connected. This shows the importance of taking the complement of Ω in the extended plane, rather than $\mathbb{C}$ itself. Similarly $\mathbb{C} \setminus \{\zeta \leq 0\}$ is also simply connected. In this case the complement of the set in the extended plane is the complement of half a great circle in $\mathbb{S}^2$. On the other hand the complement of a line L passing through the origin in $\mathbb{C}$ in the extended plane is an entire great circle, and hence $\mathbb{C} \setminus L$ is not simply connected. Similarly, $\mathbb{C}^*$ is not simply connected, since $\mathbb{S}^2 \setminus \mathbb{C}^* = \{N, S\}$*

Lecture 12

Cauchy's theorem : Multiply connected domains

12.1 Characterisations of simply connected sets

Recall that a connected subset $\Omega \subset \mathbb{C}$ is called simply connected if the complement in the extended complex plane is also connected. We now provide several equivalent characterisations of being simply connected. First we need to introduce two important notions.

Recall that a curve $\gamma : [a, b] \to \Omega$ is a *simple, closed* curve if γ is injective on (a, b) and $\gamma(a) = \gamma(b)$. Such curves are called *Jordan curves*, and their name stems from the following historically significant theorem.

Theorem 12.1.1 (Jordan curve theorem)**.** *Let γ be a Jordan curve and C be it's image. Then it's complement $\mathbb{C} \setminus C$ consists of exactly two open connected subsets. One of these components is bounded while the other is unbounded.*

The bounded component is called the *interior* and the unbounded component is called the *exterior*, denoted respectively by $int(\gamma)$ and $ext(\gamma)$. While intuitively obvious, the proof is extremely non-trivial. So much so that the the theorem is notorious for numerous incorrect proofs from well known mathematicians. In fact it will not be an exaggeration to say that attempts to prove this theorem led to the modern development of algebraic topology.

We also need to introduce the notion of homotopy. We say that a *closed* piecewise regular curve $\gamma : [0, 1] \to \Omega$ is *contractible*, or *null homotopic* in Ω if there exists a point $p \in \Omega$ and a continuous function $H : [0, 1] \times [0, 1] \to \Omega$ such that

$$\begin{cases} H(0, t) = \gamma(t),\ H(1, t) = p, \forall t \in [0, 1] \\ H(s, 0) = H(s, 1),\ \forall s \in [0, 1]. \end{cases}$$

We use the notation $\gamma_s(t) := H(s, t)$.

Example 12.1.2.

1. *Let D be a disc centred at p. Then for every curve γ, consider the homotopy*

$$H(s, t) = (1 - s)(\gamma(t) - p) + p.$$

   Then $H(1, t) = p$ and hence γ is null homotopic. More generally any convex domain has the property that every closed curve is null homotopic. To see this, let γ be an aribitrary closed curve. Then

2. *On the other hand, consider the curve $\gamma(t) = e^{it}$ in the annulus $A_{0,2}(0) := \{z \mid 0 < |z| < 2\}$. Then γ is not null-homotopic, as can for instance be seen using the theorem below.*

We then have the following fundamental result.

Theorem 12.1.3. *Let $\Omega \subset \mathbb{C}$ be a connected set. Then the following are equivalent.*

1. *Ω is simply connected.*
2. *For every Jordan curve γ in Ω, $int(\gamma) \subset \Omega$.*
3. *For all cycles γ in Ω, $\gamma \sim_\Omega 0$.*
4. *(Cauchy theorem for simply connected domains) For all holomorphic functions $f \in \mathcal{O}$ and all cycles γ in Ω,*
$$\int_\gamma f(z)\, dz = 0.$$
5. *Every closed piecewise regular curve is null homotopic.*

We need to use the following crucial lemma which we state without proof.

Lemma 12.1.4. *Let γ be a piecewise regular curve which is null homotopic. Then one can choose the homotopy H such that each γ_s is piecewise regular.*

Proof. The implication $(2) \iff (3)$ is a consequence of the generalised Cauchy theorem. W

- $(1) \implies (2)$. We can assume that $\Omega \neq \mathbb{C}$ for else the implication is trivial. Suppose $a \in int(\gamma) \cap \Omega^c$. Then since Ω is simply connected, $\mathbb{S}^2 \setminus \Omega$ is connected, and hence there exists a $p \in \Omega^c \cap ext(\gamma)$, and a path σ lying in Ω^c and connecting a to p. But since $int(\gamma)$ is connected this is a contradiction.
- $(1) \implies (3)$. Let γ be a cycle in Ω and $p \notin \Omega$. Since $\mathbb{S}^2 \setminus \Omega$ is connected, there is a sequence of points p_n such that $|p_n| \to \infty$ and there is a path σ_n from p to p_n. Since $\lim_{n\to\infty} n(\gamma, p_n) = 0$ and index is locally constant, this implies that $n(\gamma, p) = 0$.
- $(3) \implies (1)$. Suppose Ω is not simply connected. Then $\mathbb{S}^2 \setminus \Omega = A \cup B$, where B is the component at infinity, and A is a compact (possibly disconnected) set. Let
$$\delta := \inf\{|z - w| \mid z, \in A,\ w \in B\}.$$
Then $\delta > 0$. Now we cover the entire plane with a net $\mathcal{N}$ of squares of a fixed side length $\delta/4$ (any side length strictly smaller than $\delta/\sqrt{2}$ will do). We choose the net so that a certain square, say Q_1 has the point $a \in A$ at it's centre. Let $Q_1, \cdots, Q_N$ be the squares whose interiors have a non-empty intersection with A, and let
$$\Gamma = \partial\Big(\cup_{j=1}^N Q_j\Big)$$
oriented in an anticlockwise direction. Note that
$$n(\partial Q_j, a) = \begin{cases} 1,\ j = 1 \\ 0, j > 1, \end{cases}$$
and hence $n(\partial\Gamma, a) = 1$, since the integrals over the common boundaries vanish. But then since Γ clearly does not meet B, we have found a cycle in Ω such that $n(\Gamma, a) \neq 0$ but $a \in \Omega^c$. This is a contradiction.
- $(2) \implies (1)$. Suppose Ω is not simply connected, then the Γ produced above gives a Jordan curve whose interior is not completely contained inside Ω.

- $(4) \implies (2)$. It is enough to prove that every closed smooth curve that index zero. Let γ be one such curve, and let $p \in \Omega^c$. There exists a homotopy $H : [0,1] \times [0,1] \to \Omega$ contracting γ to a point $a \in \Omega$. For $s \in [0,1]$, consider the function
$$f(s) := \frac{1}{2\pi i} \int_0^1 \frac{\gamma_s'(t)}{\gamma_s(t) - p}\, dt.$$
Note that the definition makes sense because of the Lemma above. Now, since $p \in \Omega^c$, clearly $f(s)$ is a continuous function. Moreover, $f(1) = 0$, and hence $f(0) = n(\gamma, p) = 0$.
- $(2) \implies (4)$. Since this implication will not play any further role in the course, we simply direct the reader to the argument on page 252 of *Complex Analysis* by Theodore Gamelin.

□

An important consequence of this is the following.

Theorem 12.1.5. *Let Ω be a simply connected domain and $f \in \mathcal{O}$. Then f has a primitive on Ω.*

Proof. The proof is along the lines of the proof for existence of primitives on disc that was used in the proof of Cauchy's theorem. So we fix a $p \in \Omega$, and define
$$F(z) = \int_{\gamma_z} f(w)\, dw,$$
where the integral is along some path γ_z joining p to z. If we choose another path $\tilde{\gamma}_z$ joining the two points, then $\gamma_z - \tilde{\gamma}_z$ will form a cycle. Since the domain is simply connected, $\gamma_z \sim_\Omega \tilde{\gamma}_z$, and hence the integral of f along both would be the same. Hence our definition is actually independent of the path. By openness of Ω, for any h small, the straight line joining z to $z + h$ will lie entirely in Ω, and we call this path as l. Then $\gamma_{z+h} - l$ and γ_z are both piecewise smooth paths joining p to z, so once again by simple connectedness of Ω and Theorem 12.1.3
$$\int_{\gamma_{z+h}} f(w)\, dw - \int_l f(w)\, dw = \int_{\gamma_z} f(w)\, dw,$$
or equivalently
$$F(z+h) - F(z) = \int_l f(w)\, dw.$$
Then the same argument as in the proof of Theorem 2.1 in Lecture-7 implies that $F(z)$ is holomorphic with $F'(z) = f(z)$.

□

12.2 Cauchy's theorem for multiply connected domain

A connected domain $\Omega \subset \mathbb{C}$ is said to be n-*connected* if it's complement in the extended complex plane has n-connected components. So for instance, a simply connected set is 1-connected, and an n-connected set has $n - 1$ number of "holes". Our convention will be to label the components as $A_1, \cdots, A_n$, where A_n is component containing the north pole (or the point at "infinity"). Using the argument in the proof of the implication $(2) \implies (1)$ in Theorem 12.1.3 we obtain the following.

Theorem 12.2.1. *For every $i = 1, \cdots, n-1$, there exists a cycle γ_i such that*
$$n(\gamma_i, p) = \begin{cases} 1, & p \in A_i \\ 0, & p \in \Omega^c \setminus A_i. \end{cases} \tag{12.1}$$
Moreover we have the following observations.

1. *The set* $\{\gamma_1, \cdots, \gamma_n\}$ *is a* linearly independent *set, in the sense that* $\sum_{i=1}^n c_i\gamma_i \sim_\Omega 0$ *if and only if* $c_i = 0$ *for all* i.
2. *The set* $\{\gamma_1, \cdots, \gamma_n\}$ *is a* spanning set, *in the sense that if* γ *is any other cycle in* Ω, *then*

$$\gamma \sim_\Omega c_1\gamma_1 + \cdots c_{n-1}\gamma_{n-1},$$

   where $c_i = n(\gamma, p_i)$ *for any* $p_i \in A_i$.
3. *For any* $f \in \mathcal{O}(\Omega)$, *we have*

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n-1} n(\gamma,p_i)\int_{\gamma_i} f(z)\,dz,$$

   where $(p_1, p_2, \cdots, p_{n-1})$ *is any collection of points in* $A_1 \times \cdots A_{n-1}$.
4. *If* $\{\sigma_1, \cdots, \sigma_m\}$ *is a linearly independent spanning set, then* $m = n$.

We call the collection $\{\gamma_1. \cdots, \gamma_n\}$ the *homology basis* for Ω. In general there will be multiple choices of homology bases, but by an elementary theorem in linear algebra, all of these must be n in number. In fact, if

Example 12.2.2. *Consider the domain* $\Omega := D_2(0) \setminus \{-1, i, 1\}$. *Then clearly the domain is* 4*-connected. In fact if we label the points by* $p_1 = -1, p_2 = i, p_3 = 1$, *then* $A_i = \{p_i\}$ *for* $i = 1, 2, 3$ *and* $A_4 = \mathbb{S}^2 \setminus D_2(0)$. *Let* γ_i *be given by*

$$\gamma_i(t) = p_i + \frac{1}{2}e^{it}.$$

Clearly $n(\gamma_i, p_i) = 1$. *On the other hand if* $p \in \Omega^c\setminus\{p_i\}$, *then* p *lies outside* $D_{1/2}(p_i)$ *and hence* $n(\gamma_i, p) = 0$. *Hence by the above theorem,* $\{\gamma_1, \gamma_2, \gamma_3\}$ *forms a homology basis for* Ω.

As a consequence of the above result, we have the following. We say that a Jordan curve γ is *positively oriented* if while the curve is traversed, $int(\gamma)$ remains to the left.

Corollary 12.2.3. *Let* $\Gamma, \Gamma_1, \cdots, \Gamma_n$ *be positively oriented, pairwise non intersecting, piecewise smooth Jordan curves such that* $\Gamma_j \subset int(\Gamma)$ *for all* $j = 1, \cdots, n$. *Let* $\Omega = int(\Gamma) \cap \Big(\cap_{j=1}^n ext(\Gamma_j)\Big)$. *Let* f *be a function that is holomorphic in a neighbourhood of* Ω. *Then*

$$\int_\Gamma f(z)\,dz = \sum_j \int_{\Gamma_j} f(z)\,dz.$$

Proof. Firstly, one can construct Jordan curves $C, C_1, \cdots, C_n$ such that $int(\Gamma) \subset int(C)$ and $int(C) \subset int(\Gamma)$ for all i, and such that f is holomorphic in $\Omega' := int(C) \cap \Big(\cap_{j=1}^n ext(C_j)\Big)$ which of course contains Ω. Let $p_i \in int(C_i)$. It is then easy to check that $\Gamma_1, \cdots, \Gamma_n$ forms a homology basis for Ω' and the result then follows from Theorem 12.2.1. Note that constructing the Jordan curves C_i is non-trivial. If Γ_i is smooth, then one can construct C_i by perturbing slightly in the direction of the inner normal (and similarly C be perturbing Γ a little bit along the *outer* normal). But since our curves are only piecewise smooth, extra care must be take to "round off"the "corners". ☐

12.3 A real variable integral

We will now apply Cauchy's theorem to compute a real variable integral. Later in the course, once we prove a further generalization of Cauchy's theorem, namely the residue theorem, we will conduct a more systematic study of the applications of complex integration to real variable integration. For now, let us compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx.$$

This is an improper integral which is convergent, so by definition

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1-\cos x}{x^2}\, dx,$$

Consider now the function

$$f(z) = \frac{e^{iz}}{1+z^2}.$$

This is a holomorphic function on $\mathbb{C} \setminus \{i\}$. Moreover, on the real line the real part of this function is precisely the function that we are looking to integrate. Now consider a contour $\Gamma_R := \{z \in \mathbb{C} \mid |z| = R,\ Im(z) > 0\} \cup \{(x, 0) \mid x \in (-R, R)\}$ oriented in the anti-clockwise direction. Let $C_\varepsilon; = \{|z - i| = \varepsilon\}$ be a small circle around i. Then for $R >> \varepsilon$, we clearly have that $\Gamma_R \sim_{\mathbb{C}_i^*} C_\varepsilon$, and hence by Theorem 12.2.1,

$$\int_{\Gamma_R} f(z)\, dz = \int_{C_\varepsilon} f(z)\, dz.$$

Now note that

$$f(z) = \frac{g(z)}{z-i},$$

where $g(z) = e^{iz}/(z + i)$ is holomorphic in a neighbourhood of the disc $D_\varepsilon(i)$, and hence by the Cauchy integral formula (applied to $g(z)$),

$$\int_{C_\varepsilon} f(z)\, dz = \int_{C_\varepsilon} \frac{g(z)}{z-i}\, dz = 2\pi i g(i) = \frac{\pi}{e}.$$

Letting S_R be the semi-circle $\{z \in \mathbb{C} \mid |z| = R,\ Im(z) > 0\}$, we see that

$$\int_{\Gamma_R} f(z)\, dz = \int_{\gamma_R} f(z)\, dz + \int_{-R}^{R} \frac{e^{ix}}{1+x^2} = \int_{\gamma_R} f(z)\, dz + \int_{-R}^{R} \frac{\cos x}{1+x^2},$$

where we have used the fact that $\sin x$ is an odd function. On γ_R we claim that

$$\Big|\frac{1-e^{iz}}{z^2}\Big| \leq \frac{2}{R^2}.$$

To see this, for $z \in \gamma_R$, we can write $z = x + iy$ with $y > 0$. So $|e^{iz}| = e^{-y} < 1$, and hence by triangle inequality $|1 - e^{iz}| < 2$ which proves the claim since $|z| = R$ on γ_R. Using this we can estimate that

$$\Big|\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\, dz\Big| \leq \frac{2}{R}\mathrm{len}(\gamma_R) = \frac{2\pi}{R} \to 0$$

as $R \to \infty$. So the contribution on γ_R goes to zero as R goes to infinity, and hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} = \frac{\pi}{e}.$$

Lecture 13

Logarithm and roots revisited

13.1 Criteria for existence of holomorphic logarithms

The purpose of this lecture is twofold - first, to characterize domains on which a holomorphic logarithm can be defined, and second, to show that the only obstruction to defining a holomorphic logarithm is in defining a continuous logarithm. From henceforth, we let $\mathcal{O}^*(\Omega)$ be the set of nowhere vanishing holomorphic functions on Ω. For instance, $e^z \in \mathcal{O}^*(\mathbb{C})$.

To set the stage, let us revisit the difficulties we had in defining a holomorphic logarithm. For $z = re^{i\theta}$, consider the function

$$\log z = \log |z| + i\theta, \tag{13.1}$$

where, for instance, we can let $\theta \in [\theta_0, \theta_0 + 2\pi)$. If $\arg p = \theta_0$, and we traverse a circle of radius $|p|$ centred at 0 and return to the point p, the argument goes from θ_0 to $\theta_0 + 2\pi$, and hence the $\log z$ does not return to the original value. In other words $\log z$ as defined is not continuous. On the other hand, if we return to p along a small circle not containing 0 in the interior, then the argument does return to θ_0, and $\log z$ does not jump in value. The difference between the situations if of course that the first curve goes around 0 while the second does not. Thus logarithm is an example of a *multivalued function*, and zero in this case is called a *branch* point.

In general, we can consider any holomorphic function $f : \Omega \to \mathbb{C}^*$. Then, a holomorphic function $g : \Omega \to \mathbb{C}$ (if it exists) is called a *branch* of the logarithm of f, and denoted by $\log f(z)$, if

$$e^{g(z)} = f(z)$$

for all $z \in \Omega$. A natural question to ask is the following.

Question 13.1.1. *Given a holomorphic function $f : \Omega \to \mathbb{C}^*$, when can we define a holomorphic branch of* $\log f(z)$.

From the point of view of the Cauchy theory, the multivalued behaviour of the logarithm function is esentially because $1/z$, which would be the derivative of a holomorphic logarithm function, does not integrate out to zero around curves that contain the origin in the interior. Keeping this in mind, we have the following basic theorem.

Theorem 13.1.1 (Fundamental theorem on existence of holomorphic logarithms)**.** *Let Ω be a connected domain, and $f \in \mathcal{O}^*(\Omega)$ such that*

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for all closed loops $\gamma \subset \Omega$.

- *Then there exists a holomorphic function* $g : \Omega \to \mathbb{C}$*, denoted by* $g(z) = \log f(z)$*, such that*

$$e^{g(z)} = f(z).$$

- $g' = f'/f$*, and hence for any fixed* $p \in \Omega$*,*

$$g(z) = g(p) + \int_p^z \frac{f'(w)}{f(w)}\, dw.$$

Moreover, if $\tilde{g}$ *is another function satisfying* $e^{\tilde{g}(z)} = f(z)$*, then there exists an* $n \in \mathbb{Z}$ *such that*

$$\tilde{g}(z) - g(z) = 2\pi n, \ \forall z \in \Omega.$$

Remark 13.1.2. Note that different choices of $g(p)$ corresponding to the countable number of solutions to $e^z = f(p)$ give different formulae for $g(z)$, all of which differ by integral multiples of $2\pi i$. Conversely, if g_1 and g_2 are two logarithms, then they have to differ by a multiple of $2\pi i$. The various logarithm functions are called *branches*.

Theorem 13.1.1 above combined with Cauchy's theorem for simply connected domains gives the following.

Corollary 13.1.3. *Let* Ω *be a simply connected domain.*

1. *Then for any* $f \in \mathcal{O}^*(\Omega)$*, there exists a holomorphic* $\log f(z)$ *with* $(\log f)' = f'/f$*, and hence*

$$\log f(z) = \log f(p) + \int_p^z \frac{f'(w)}{f(w)}\, dw,$$

w here $\log f(p)$ *is* any *solution to* $e^z = f(p)$*.*

2. *In particular, if* $0 \notin \Omega$*, then there is a holomorphic branch of* $\log z$ *on* Ω *with* $(\log z)' = 1/z$*. Moreover, for any* $p \in \Omega$*,*

$$\log z = \log p + \int_p^z \frac{1}{z}\, dz,$$

where we integrate along any path from p *to* z*, and* $\log p$ *is any solution to* $e^z = p$*.*

Proof of Theorem 13.1.1. Fix a point $p \in \Omega$, and let $g(p)$ be a solution to $e^{g(p)} = f(p)$. Since $f(p) \neq 0$, such a solution always exists. We then define $g(z)$ by

$$g(z) = g(p) + \int_p^z \frac{f'(w)}{f(w)}\, dw,$$

where we integrate over any curve joining p and z. By the hypothesis, this is independent of the path chosen. Then, by the same argument used before (as in the proof of Theorem), $g(z)$ is holomorphic with

$$g'(z) = \frac{f'(z)}{f(z)}.$$

Now, consider the function $F(z) = e^{-g(z)} f(z)$. Then

$$F'(z) = -e^{g(z)} g'(z) f(z) + e^{-g(z)} f'(z) = 0.$$

Since Ω is connected, this implies that $F(z)$ is a constant, and hence $F(z) = F(p) = 1$. This completes the proof. $\square$

Recall that the branch of $\log z$ defined by (13.1) is not even a continuous function over $\mathbb{C}^*$. This is not a coincidence. Our next theorem, says that continuity in fact, is the only obstruction to define a holomorphic logarithm.

Theorem 13.1.2. *Let $\Omega \subset \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous. If $e^{g(z)}$ is holomorphic, then so is $g(z)$.*

In other words if $f(z)$ is holomorphic, and we can define a continuous $\log f(z)$, then $\log f(z)$ is automatically holomorphic.

Proof. Let $f(z) = e^{g(z)}$, which by the hypothesis, is holomorphic, and fix a $p \in \Omega$. There is a $\delta > 0$ such that $\overline{D_\delta(p)} \subset \Omega$ and a holomorphic function $g_p(z)$ on $D_\delta(p)$ such that $e^{g_p(z)} = f(z)$. Then for all $z \in D_\delta(p)$,

$$\frac{g(z) - g_p(z)}{2\pi i} \in \mathbb{Z}.$$

Now, since $g(z)$ is continuous, $\frac{g(z)-g_p(z)}{2\pi i}$ is a continuous function on $D_\delta(p)$ which only takes integer values, and hence has to be a constant. That is,

$$g(z) = g_p(z) + 2\pi i n,$$

for some fixed $n \in \mathbb{Z}$. But then $g(z)$ has to be holomorphic in $D_\delta(p)$, since $g_p(z)$ is holomorphic, and hence is in complex differentiable at p. Since p was arbitrary, this completes the proof of the theorem.

□

13.2 Some examples

Somewhat vaguely, for a (multi-valued) function $g(z)$, the point $z = a$ is defined to be a *branch point* if $g(z)$ is discontinuous while traversing an arbitrarily small circle around the point. We define infinity to be a branch point if $z = 0$ is the branch point of $g(1/z)$. An alternate way is to consider a curve "enclosing" infinity. This is a large curve that contains all the other branch points in it's interior. Then infinity is a branch point if along this large curve, the function $g(z)$ is discontinuous. A *branch cut* is a union of curves such that $g(z)$ defines a single valued holomorphic function on the complement. Branch cuts should usually connect branch points to prevent the possibility of going around branch points and making the function value jump.

13.2.1 Principal branch of the logarithm

Since $\mathbb{C} \setminus \{\mathrm{Re}(z) \leq 0\}$ is simply connected, this immediately implies that there is a holomorphic logarithm on that domain. The *principal branch* of the logarithm is defined to be the one with

$$\log 1 = 0.$$

By Theorem 13.1.1, for any $z \in \mathbb{C} \setminus \{\mathrm{Re}(z) \leq 0\}$, we then have

$$\log z = \int_1^z \frac{1}{w}\, dw,$$

where we integrate over any piecewise smooth path from 1 to z. Suppose $z = re^{i\theta}$, then one such path is $C = C_1 + C_2$ where C_1 is parametrized by $z_1(t) : [0, 1] \to \mathbb{C}$ with $z_1(t) = tr + (1 - t)$, and C_2 is given by $z_2(t) : [0, \theta] \to \mathbb{C}$ where $z_2(t) = re^{e^{it}}$ So C is simply the path going first from 1 to r along the x-axis, and then the circular arc to z. Then

$$\int_{C_1} \frac{1}{w}\, dw = \int_0^1 \frac{r-1}{t(r-1)+1}\, dt = \log(t(r-1)+1)\Big|_{t=0}^{t=1} = \log r,$$

where the log is the usual logarithm defined on real numbers. On the other hand,

$$\int_{C_2} \frac{1}{w}\,dw = i\int_0^\theta dt = i\theta.$$

So the principal branch of the logarithm is given by

$$\log z = \log r + i\theta,$$

where $\theta \in (-\pi,\pi)$. We end with the following remark.

Remark 13.2.1. Unlike the real logarithm, in the complex case, in general

$$\log z_1 z_2 \neq \log z_1 + \log z_2.$$

For example, let $z_1 = e^{3\pi i/4}$,$z_2 = e^{\pi i/2}$ and $\log z$ be the principal branch. Then $\log z_1 = 3\pi i/4$ and $\log z_2 = \pi i/2$. But $z_1 z_2 = e^{5\pi i/4} = e^{-3\pi i/4}$ (remember the range of arg is $(-\pi,\pi]$, and so $\log z_1 z_2 = -3\pi i/4 \neq \log z_1 + \log z_2$. Similarly, even though $e^{\log z} = z$ for all z, $\log e^z \neq z$ generally, again due to the periods of e^z.

13.2.2 Branch cut for $\log(z^2-1)$.

The points where $z^2-1=0$, namely $z=\pm 1$ are certainly branch points. Any branch cuts should include these two points. To see that infinity is also a branch point, note that the logarithm should be defined as a primitive of

$$\frac{2z}{z^2-1} = \frac{1}{z+1} + \frac{1}{z-1}.$$

It is clear that as we integrate along a curve of radius $R>1$ around the origin, both terms will make a contribution with the same sign, and hence the integral will not be zero. In other words if we define log as a primitive of $\frac{2z}{z^2-1}$, it will have a jump if we traverse a large curve. Hence ∞ is a also branch point. Possible branch cuts, that will prevent going around $z=\pm1$ or $z=\infty$ are

$$(-\infty,1] \text{ or } (-\infty,1]\cup[1,\infty) \text{ or } [-1,\infty).$$

Of course there are infintely many choices of branch cuts. Each of the above branch cuts renders the domain simply connected, and hence a holomorphic branch does exist.

A convenient way to write down a formula for the branch is by using "**double polar coordinates**". That is, we let

$$z = -1 + r_1 e^{i\theta_1} = 1 + r_2 e^{i\theta_2}.$$

If we restrict the "phases" θ_1,θ_2 to be in the usual range $(-\pi,\pi)$, then using the principal branch of the log we obtain

$$\log(z+1) = \log r_1 + i\theta_1$$
$$\log(z-1) = \log r_2 + i\theta_2.$$

Adding up these two we see that

$$g(z) = \log r_1 + \log r_2 + i(\theta_1+\theta_2) \tag{13.2}$$

does define a branch of $\log(z^2-1)$ since it is easy to see that $e^{g(z)} = z^2-1$.

Claim. $g(z)$ defines a holomorphic branch of $\log(z^2-1)$ on $\mathbb{C}\setminus(-\infty,1]$.

By Theorem 13.1.2 it is enough to check that it defines a continuous branch. But this is obvious since θ_1 is continuous everywhere except $z \leq -1$ and θ_2 is continuous everywhere except $z \leq 1$, and since these points are removed in the branch cut, $g(z)$ is continuous everywhere else.

If we instead, restrict $\theta_1 \in (-\pi, \pi]$ and $\theta_2 \in (0, 2\pi)$, then formula (13.2) defines a holomorphic branch on the complement of the branch cut $(-\infty, 1] \cup [1, \infty)$. The reader should work these out carefully.

13.2.3 Branch cuts for $\log\left(\frac{z+1}{z-1}\right)$.

A holomorphic definition would have primitive

$$\frac{d}{dz}\log\left(\frac{z+1}{z-1}\right) = \frac{1}{z+1} - \frac{1}{z-1}.$$

Clearly $z = \pm 1$ are branch points. To analyze branching at infinity, consider a large disc $D_R(0)$ with $R > 2$. Then both the terms contribute an integral of $2\pi i$ but with opposite signs, and hence the integral vanishes. In other words the argument does not change as we traverse this big circle. Hence infinity is NOT a branch point. Hence we can then choose the branch cut to be $[-1, 1]$, even though $\mathbb{C} \setminus [-1, 1]$ is not simply connected.

Again, lets analyze this using the double polar coordinates $z = -1 + r_1 e^{i\theta_1} = 1 + r_2 e^{i\theta_2}$, This time let $\theta_1, \theta_2 \in [-\pi, \pi)$, and define

$$g(z) = \log r_1 - \log r_2 + i(\theta_1 - \theta_2).$$

Clearly this defines a branch of $\log\left(\frac{z+1}{z-1}\right)$, and we need to check if it is a continuous branch. The function so defined is surely continuous (and hence holomorphic by Theorem 13.1.2) everywhere on $\mathbb{C} \setminus [-1, 1]$ except possibly the real axis to the left of $z = -1$. As you approach this part of the real axis from the top, both $\theta_1, \theta_2 \to \pi$. On the other hand when you approach from the bottom, both $\theta_1, \theta_2 \to -\pi$, and so their difference cancels out. So the resulting function defines a continuous, and hence a holomorphic, $\log\left(\frac{z+1}{z-1}\right)$. Again if we change the domain for either θ_1 or θ_2 to $(0, 2\pi)$, we are forced ton consider other branch cuts. Once again, the reader should work both these cases out carefully.

13.3 n^{th}-roots of holomorphic functions

Given a logarithm function, and a $w \in \mathbb{C}$, one can define a holomorphic complex power, by

$$z^w = e^{w \log z}. \tag{13.3}$$

When $w = 1/n$ is the reciprocal of a natural number, we call $z^{1/n}$ the n^{th} root of z.

Example 13.3.1. Roots of unity. *Consider the polynomial $z^n - 1$. Clearly $\zeta_n = e^{2\pi i/n}$, is a root. Moreover, ζ_n^k, for $k = 0, 1, \cdots, n-1$ is also a root, and since the degree of the polynomial is n, these are all the possible roots. We call ζ_n the primitive n^{th} root of unity.*

We then have the following analogs of Corollary 13.1.3 and Theorem 13.1.2.

Theorem 13.3.1. 1. *If Ω is simply connected and $f(z)$ is holomorphic and zero free, then there exists a holomorphic function $g(z)$ such that $g(z)^n = f(z)$. Moreover, if $g_1(z)$ is any other such function, then $g_1(z) = \zeta_n^k g(z)$ where $\zeta_n = e^{2\pi i/n}$ is the primitive n^{th} root of unity and $k = 0, 1, \cdots, n-1$.*

2. *If $g : \Omega \to \mathbb{C}$ is continuous such that $g(z)^n$ is holomorphic for some positive integer n, then $g(z)$ itself is holomorphic.*

Proof. 1. For the first part, by Corollary 13.1.3, there exists a holomorphic $\log f(z)$. We then simply take

$$g(z) = e^{\frac{\log f(z)}{n}}.$$

It is also clear that if $g_1(z)$ is another such function, then $(g_1(z)/g(z))^n = 1$, and hence there exists

2. We proceed as in the proof of Theorem 13.1.2. Let $f(z) = g(z)^n$. Then for any $p \in \Omega$ if $r > 0$ such that $\overline{D_r(p)} \subset \Omega$, by the first part, there exists a holomorphic function $g_p(z)$ on $D_r(p)$ such that $g_p(z)^n = f(z)$. But then on the disc, $(g/g_p)^n = 1$ and hence by continuity, there exists a fixed integer $0 \leq k \leq n-1$ (independent of z) such that $g(z) = g_p(z)e^{2\pi ik/n}$, which in turn implies that $g(z)$ is holomorphic.

□

13.3.1 Principal, and other branches of the square root

We can define the principal branch of the square root so that $\sqrt{1} = 1$. Doing a similar computation as above, we can then see that if $z = re^{i\theta}$ with $\theta \in (-\pi, \pi)$, then

$$\sqrt{z} = re^{i\theta/2}.$$

On the other hand, if we want $\sqrt{1} = -1$, then

$$\sqrt{z} = re^{i\pi + i\theta/2}.$$

13.3.2 Branch cuts for $\sqrt{z^2-1}$

Any of the branch cuts for $\log(z^2-1)$ will allow us to define $\sqrt{z^2-1}$ on their complement by equation (13.3). Each of those branch cuts extend out to infinity. But it turns out we can define a holomorphic branch of $\sqrt{z^2-1}$ on the complement of *finite* cut. This is possible because ∞ is not a branch point (even though it is a branch point of $\log(z^2-1)$).

To see this, we again make use of double polar coordinates. Let $z = -1 + r_1e^{i\theta_1} = 1 + r_2e^{i\theta_2}$ as before, where we let $\theta_j \in (-\pi, \pi)$, and we define

$$g(z) = \sqrt{r_1r_2}e^{i\left(\frac{\theta_1+\theta_2}{2}\right)}.$$

Clearly this defines a branch of $\sqrt{z^2-1}$ since

$$g(z)^2 = r_1r_2e^{i(\theta_1+\theta_2)} = (z+1)(z-1) = z^2-1.$$

All we need to now do, is to find a branch cut such that $g(z)$ is continuous in the complement.

Clearly θ_1 is continuous everywhere except $(-\infty, -1]$ and θ_2 is continuous everywhere except $(-\infty, 1]$. So $g(z)$ is continuous everywhere except possibly for $(-\infty, 1]$. Let us analyze the two intervals $(-\infty, -1)$ and $[-1, 1]$. If z approaches $[-1, 1]$ from above, $\theta_1 \to 0$ but $\theta_2 \to \pi$, and so

$$g(z) \to \sqrt{r_1r_2}e^{i\pi/2} = i\sqrt{r_1r_2}.$$

But if it approaches from below, then $\theta_1 \to 0$ while $\theta_2 \to -\pi$. So

$$g(z) \to \sqrt{r_1r_2}e^{-\pi i/2} = -i\sqrt{r_1r_2},$$

and so $g(z)$ is discontinuous on $[-1, 1]$. Next we analyse continuity along $(-\infty, -1)$. As z approaches $(-\infty, -1)$ from above, $\theta_1 \to \pi$ and $\theta_2 \to \pi$, hence

$$g(z) \to e^{2i\pi/2}\sqrt{r_1r_2} = -\sqrt{r_1r_2}.$$

On the other hand when z approaches $(-\infty, -1)$ from below $\theta_1, \theta_2 \to -\pi$, and so

$$g(z) \to \sqrt{r_1 r_2} e^{-i\pi} = -\sqrt{r_1 r_2},$$

and hence $g(z)$ defines a continuous branch on $(-\infty, -1)$. The upshot is that $g(z)$ defines a continuous (and hence holomorphic) branch of $\sqrt{z^2 - 1}$ on $\mathbb{C} \setminus [-1, 1]$.

Note that with above formula, we can also compute the value of $\sqrt{z^2 - 1}$. For instance we demonstrate how to calculate the value of $\sqrt{i^2 - 1}$ for our particular branch. Of course the answer has to be either $\pm i\sqrt{2}$, but the question is which one of these values? We can write (draw a picture to see what is happening geometrically)

$$i = -1 + \sqrt{2} e^{i\pi/4} = 1 + \sqrt{2} e^{3\pi i/4},$$

so that $r_1 = r_2 = \sqrt{2}$ and $\theta_1 = \pi/4$ and $\theta_2 = 3\pi/4$. Since $\theta_1 + \theta_2 = \pi$ by the formula above

$$g(i) = \sqrt{\sqrt{2}\sqrt{2}}\, e^{i\pi/2} = i\sqrt{2}.$$

Lecture 14

Isolated Singularities

A punctured domain is an open set with a point removed. For $p \in \Omega$, we use the notation

$$\Omega_p^* = \Omega \setminus \{p\},$$

or simply Ω^* for $\Omega \setminus \{0\}$, if $0 \in \Omega$ or when there is no confusion about the point removed. The aim of this lecture is to study functions that are holomorphic on punctured domains. The puncture, that is the point p in the above case, is called an isolated singularity. These come in three types -

- Removable singularities
- Poles
- Essential singularities

14.1 Removable singularities

A holomorphic function $f \in \mathcal{O}(\Omega_p^*)$ is said to have a *removable singularity* at p if there exists a holomorphic function $\tilde{f} \in \mathcal{O}(\Omega)$ such that

$$\tilde{f}\Big|_{\Omega_p^*} = f.$$

Theorem 14.1.1. *Let $f \in \mathcal{O}(\Omega_p^*)$. Then the following are equivalent.*

1. *f has a removable singularity at p.*
2. *f can be extended to a continuous function on Ω.*
3. *f is bounded in a neighborhood of p.*
4. $\lim_{z\to p}(z-p)f(z) = 0$.

Proof. The implications $(1) \implies (2) \implies (3) \implies (4)$ are trivial. To complete the proof, we need to show that $(4) \implies (1)$.

For convenience, suppose $p = 0$. So suppose $f(z)$ satisfies

$$\lim_{z\to 0} zf(z) = 0,$$

and define a new function

$$g(z) = \begin{cases} z^2 f(z), \ z \neq 0 \\ 0, \ z = 0 \end{cases}$$

Claim. $g(z)$ is holomorphic on Ω and moreover, $g'(0) = 0$.

Clearly $g(z)$ is holomorphic on Ω^*. So we only need to prove holomorphicity at $z = 0$. Let us compute the difference quotient. Since $g(0) = 0$, $g'(0)$ if it exists is equal to

$$\lim_{h \to 0} \frac{g(h)}{h} = \lim_{h \to 0} h f(h) = 0$$

by hypothesis. This proves the claim. By analyticity, $g(z)$ has power series expansion in a neighborhood of $z = 0$. That is, there is a small disc $D_\varepsilon(0)$ such that for all $z \in D_\varepsilon(0)$,

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n.$$

Now, since $g(0) = g'(0) = 0$,

$$g(z) = z^2 \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-2}.$$

By comparing to the definition of $g(z)$ we see that for $z \in D_\varepsilon(0) \setminus \{0\}$,

$$f(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-2}.$$

and so we define

$$\tilde{f}(z) = \begin{cases} \sum_{n=2}^{\infty} \frac{g^{(n)}(0)}{n!} z^{n-2}, \ z \in D_\varepsilon(0) \\ f(z), \ z \in \Omega^*. \end{cases}$$

This is a well defined function, since on the intersection $\Omega^* \cap D_\varepsilon(0)$, $f(z)$ is equal to the infinite series. $\tilde{f}$ is clearly holomorphic on Ω^* since it equals $f(z)$ in this region. Moreover, since it is a power series in a neighborhood of $z = 0$, it is also holomorphic at $z = 0$. Hence $\tilde{f}$ satisfies all the properties in (1), and this completes the proof. □

Remark 14.1.1. Recall that in lecture-7 we proved that Goursat's theorem was valid for functions that are holomorphic at all but one point in a domain, so long as they are bounded near that point. In view of the above theorem, such a result is not surprising, since the function does extend to a holomorphic function on the entire domain, to which Goursat's theorem applies.

Example 14.1.2. *Consider the holomorphic function $Si : \mathbb{C}^* \to \mathbb{C}$ defined by*

$$Si(z) = \frac{\sin z}{z}.$$

Then clearly

$$\lim_{z \to 0} z \cdot \mathrm{Si}(z) = \lim_{z \to 0} \sin z = 0.$$

Hence by the theorem, $\mathrm{Si}(z)$ has a removable singularity at $z = 0$ and hence can be extended to an entire function. It is instructive to look at the power series of $\sin z$. Recall that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...,$$

and so dividing by z, we see that for $z \neq 0$,

$$\mathrm{Si}(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

The power series on the right clearly defines an entire function (an is in particular also defined at $z = 0$), and hence $\mathrm{Si}(z)$ defines an entire function.

14.2 Poles

Let $f \in \mathcal{O}(\Omega_p^*)$. We say that p is a *pole* if

$$\lim_{z \to p} |f(z)| = \infty.$$

Theorem 14.2.1. *Let $f \in \mathcal{O}(\Omega_p^*)$. Then the following are equivalent.*

1. *f has a pole at p.*
2. *There exists a small disc $D_\varepsilon(p)$ and a holomorphic function $h : D_\varepsilon(p) \to \mathbb{C}$ such that $h(p) = 0$ and $h(z) \neq 0$ for any other $z \in D_\varepsilon(p)$, and*
$$f(z) = \frac{1}{h(z)}$$
for all $z \in D_\varepsilon(p) \setminus \{p\}$.
3. *There exists a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g(p) \neq 0$, and an integer $m \geq 1$ such that for all $z \in \Omega_p^*$,*
$$f(z) = \frac{g(z)}{(z-p)^m}.$$
4. *There exists a $M > 1$ and integer $m \geq 1$ such that on some disc $D_\varepsilon(p)$ around p, we have the estimates*
$$\frac{1}{M|z-p|^m} \leq |f(z)| \leq \frac{M}{|z-p|^m}.$$

Note that the integer m in (3) and (4) above has to be the same, and is called the *order* of the pole at p, and written as $\nu_f(p)$.

Proof. Again for convenience, lets assume $p = 0$, and we denote $\Omega_p^* = \Omega^*$. Suppose $|f(z)| \to \infty$ as $z \to 0$. Then clearly there is a small disc $D_\varepsilon(0)$ on which f does not have a zero. Then $h(z) = 1/f(z)$ is holomorphic in the punctured disc $D_\varepsilon(0)^*$. Moreover,

$$\lim_{z \to 0} |h(z)| = \frac{1}{\lim_{z \to 0} |f(z)|} = 0,$$

and hence in particular is bounded near $z = 0$. By Theorem 14.1.1, $h(z)$ actually extends to a holomorphic function to the entire disc $D_\varepsilon(0)$, which we continue to call $h(z)$, and from the limit it is clear that $h(p) = 0$. So $h(z)$ atisfies all the conditions in (2), and this proves that $(1) \implies (2)$.

To show that $(2) \implies (3)$, note that by the theorem on zeroes, since h is not identically zero, there exists an integer m such that for all $z \in D_\varepsilon(0)$,

$$h(z) = z^m g_1(z),$$

where $g_1(p) \neq 0$. Moreover, since $h(p) = 0$, we must necessarily have $m \geq 1$. Now consider the function

$$g(z) = z^m f(z)$$

holomorphic on Ω^*. Then on $D_\varepsilon(0) \setminus \{0\}$, $g(z) = 1/g_1(z)$. Since $g_1(p) \neq 0$ and g_1 is holomorphic on $D_\varepsilon(0)$, we see that $1/g_1(z)$ is bounded on $D_\varepsilon(0)$. Hence by the removable singularity theorem, $g(z)$ extends to a holomorphic function on all of Ω, and satisfies all the conditions in (3).

To show that $(3) \implies (4)$, note that since g is holomorphic near $z = 0$, it will in particular be bounded in a neighborhood. So there exists $M > 0$ such that for all $z \in D_\varepsilon(0)$,

$$|g(z)| \leq M.$$

On the other hand, since $g(p) \neq 0$, by continuity, for the $\varepsilon > 0$ above, there exists a δ such that

$$|g(z)| \geq \delta$$

for all $z \in D_\varepsilon(0)$. Take M large enough so that $1/M < \delta$, then we see that on $D_\varepsilon(0)$,

$$\frac{1}{M} \leq |g(z)| \leq M,$$

and this proves (4).

$(4) \implies (1)$ also holds trivially, thus completing the proof of the Theorem. $\square$

Example 14.2.1. *The function*

$$\cot z = \frac{\cos z}{\sin z}.$$

has poles at all the zeroes of $\sin z$ *(since* $\cos z$ *and* $\sin z$ *do not share any zeroes, there is no "cancellation" of the poles). Let us find the order of the zero at* $z = 0$*. Near* $z = 0$*,* $\sin z \approx z$*. More precisely,*

$$z \cot z = \frac{z \cos z}{\sin z} = \frac{\cos z}{\mathrm{Si}(z)},$$

where $Si(z)$ *is the function from the last section. Then we saw from the power series expansion, that* $\mathrm{Si}(0) = 1$ *and hence* $\cos z/\mathrm{Si}(z) \to 1$ *as* $z \to 0$*. In particular, for a small* $\varepsilon > 0$*,* $1/2 < |\cos z/\mathrm{Si}(z)| < 2$*, and hence*

$$\frac{1}{2z} \leq \frac{\cos z}{\sin z} \leq \frac{2}{z},$$

and so $z = 0$ *is a pole of order* $m = 1$*. It is once again instructive to look at an expansion near* $z = 0$*. For* $z \neq 0$*,*

$$\begin{aligned}
\frac{\cos z}{\sin z} &= \frac{1 - z^2/2 + \cdots}{z - z^3/6 + \cdots} \\
&= \frac{1}{z} \cdot \frac{1 - z^2/2 + \cdots}{1 - z^2/6 + \cdots} \\
&= \frac{1}{z}\Big(1 - \frac{z^2}{2} + \cdots\Big)\Big(1 + \frac{z^2}{6} + \cdots\Big) \\
&= \frac{1}{z} - \frac{z}{3} + \cdots
\end{aligned}$$

From this it is clear that $\cot z$ *has a pole of order* $z = 0$*.*

Remark 14.2.2. The idea of an expansion for a singular function near it's pole can be generalized. Let p be a pole for $f : \Omega_p^* \to \mathbb{C}$. Then from the theorem, we can write

$$f(z) = \frac{g(z)}{(z-p)^m},$$

for some holomorphic $g : \Omega \to \mathbb{C}$ with $g(p) \neq 0$. By analyticity, in a neighborhood of p we can write

$$g(z) = \sum_{n=0}^{\infty} a_n (z-p)^n,$$

with $a_0 \neq 0$. Hence for $z \neq p$, we have the expansion

$$f(z) = \frac{a_0}{(z-p)^m} + \frac{a_1}{(z-p)^{m-1}} + \cdots + a_m + a_{m+1}(z-p) + a_{m+2}(z-p)^2 + \cdots .$$

Such an expansion is called a Laurent series expansion, which we will study in greater detail in the next lecture. The part with the negative powers is called the **principal part** of f near p. In fact, if we denote by

$$Q_p(w) := a_0 w^m + \cdots + a_{m-1} w,$$

then we can write

$$f(z) = Q_p\Big(\frac{1}{z-p}\Big) + h_p(z),$$

where h_p extends to a holomorphic function across p.

14.3 Essential singularities

If $f : \Omega_p^* \to \mathbb{C}$ is holomorphic, then p is called an *essential singularity* if it is neither a removable singularity nor a pole. Unlike in the case of removable singularities and poles, the function behaves rather erratically in any neighborhood around an essential singularity.

Theorem 14.3.1 (Casorati-Weierstrass)**.** *The following are equivalent.*

1. *f has an essential singularity at p.*
2. *For any disc $D_\varepsilon(p)$, $f(D_\varepsilon(p))$ is dense in $\mathbb{C}$, that is for any disc $D_\varepsilon(p)$ and any $a \in \mathbb{C}$, there exists a sequence $\{z_n\} \in D_\varepsilon(p)$ such that*
$$\lim_{n\to\infty} f(z_n) = a.$$

Proof. We first show that $(2) \implies (1)$. If p is a removable singularity, then for some disc $D_\varepsilon(p)$, $f(D_\varepsilon(0))$ is a bounded set in $\mathbb{C}$, and so cannot be dense. On the other hand if p is a pole, then $|f(z)| \to \infty$ as $z \to p$. In particular, there is a disc $D_\varepsilon(p)$ such that for all $z \in D_\varepsilon(p)$,

$$|f(z)| > 1,$$

and hence once again $D_\varepsilon(p)$ cannot be dense in $\mathbb{C}$. This forces p to be an essential singularity.

Conversely, suppose p is an essential singularity. We then have to show that (2) holds. If not, then there is a disc $D_{\varepsilon_0}(p)$ such that $f(D_{\varepsilon_0}(p) \setminus \{p\})$ is not dense in $\mathbb{C}$. Hence there exists an $a \in \mathbb{C}$ and an $r > 0$ such that

$$|f(z) - a| > r$$

for all $z \in D_\varepsilon(p) \setminus \{p\}$. Then define $g : D_\varepsilon(p) \setminus \{p\} \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z) - a}.$$

Since $f(z) \neq a$ on that punctured disc, $g(z)$ is holomorphic. Moreover $|g(z)| \leq 1/r$ in $D_\varepsilon(p) \setminus \{p\}$, and hence by the removable singularity Theorem 14.1.1, there exists an extension $\tilde{g}$ holomorphic on $D_\varepsilon(p)$. There are now two cases.

Case-1. $\tilde{g}(p) \neq 0$. Then by continuity, there is a smaller $r < \varepsilon$ and a $\delta > 0$ such that $|\tilde{g}(z)| > \delta$ on $D_r(p)$. But away from p,

$$f(z) = \frac{1}{g(z)} + a,$$

and so on $D_r(p) \setminus \{p\}$,

$$|f(z)| \leq \frac{1}{|\tilde{g}(z)|} + |a| < \frac{1}{\delta} + |a|,$$

and so $|f(z)|$ is bounded in a neighborhood of p. By the removable singularity theorem, f must have a removable singularity at $z = p$ which is a contradiction.

Case-2. $\tilde{g}(p) = 0$. Then for any $\varepsilon > 0$, there exists a $r > 0$ such that on D_r,

$$|\tilde{g}(z)| \leq \varepsilon.$$

So by triangle inequality, if ε small enough so that $|a| < 1/2\varepsilon$, then on $D_r(p) \setminus \{p\}$ we have

$$|f(z)| = \Big|\frac{1}{g(z)} + a\Big| \geq \Big|\frac{1}{|g(z)|} - |a|\Big| \geq \frac{1}{\varepsilon} - |a| > \frac{1}{2\varepsilon}$$

for all $z \in D_r(p)$. This shows that $\lim_{z\to p} |f(z)| = \infty$, which is a contradiction, completing the proof of the theorem. $\square$

Remark 14.3.1. It is a theorem of Picard's that in any neighbourhood of an essential singularity, the image under f is not only dense in $\mathbb{C}$ but misses at most one point of $\mathbb{C}$!

Example 14.3.2. *The function $f(z) = e^{1/z}$, which is holomorphic on $\mathbb{C}^*$, has an essential singularity at $z = 0$. To see this, we need to rule out the possibilities of f having a removable singularity or a pole at $z = 0$. Since*

$$f(1/n) = e^n \xrightarrow{n\to\infty} \infty,$$

$f(z)$ is not bounded in any neighborhood of $z = 0$, and hence cannot have a removable singularity. On the other hand,

$$f\Big(\frac{1}{2\pi ni}\Big) = e^{2\pi in} = 1.$$

Hence the limit $\lim_{z\to 0} f(z)$ cannot be infinity, and hence f cannot have a pole at $z = 0$. This shows that $f(z)$ has to have an essential singularity at $z = 0$. Again looking at an expansion, we see that for $z \neq 0$,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots.$$

So the expansion has infinitely many terms with negative powers of z. As we will see when we discuss Laurent series, this in fact characterizes essential singularities.

Remark 14.3.3. We finally remark that non-isolated singularities can exist. For instance the function

$$f(z) = \tan\Big(\frac{1}{z}\Big)$$

has singularities at 0 and points $p_n = 2/n\pi$ which converge to 0. The analysis in the present lecture does not apply to such singularities.

Lecture 15

Laurent series

A *Laurent series* centered at $z = a$ is an infinite series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} c_n (z-a)^n \tag{15.1}$$

We can combine this into one infinite sum

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n = \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots,$$

by setting

$$a_n = \begin{cases} b_{-n}, \ n \leq -1 \\ c_n, \ n \geq 0. \end{cases} \tag{15.2}$$

We say that the Laurent series in (15.1) is *convergent* at z if both the infinite series are convergent. The first term above is an infinite series of the form

$$b_1(z-a)^{-1} + \cdots. \tag{15.3}$$

Changing the variable to $w = (z-a)^{-1}$, we can re-write this as a usual power series -

$$b_1 w + b_2 w^2 + \cdots.$$

Then by the fundamental theorem for power series, there exists an R_1 such that the series converges on the disc $|w| < R_1^{-1}$ (or equivalently the annulus $|z| > R_1$), where

$$R_1 = \limsup_{n\to\infty} |b_n|^{1/n}$$

Or equivalently, the series (15.3) converges for $|z-a| > R_1$. On the other hand the second series in (15.1) is a regular power series, and hence setting

$$R_2 = (\limsup_{n\to\infty} |c_n|^{1/n})^{-1},$$

the second series is convergent for $|z-a| < R_2$. Combining this, we have the following theorem.

Theorem 15.0.1. *If R_1, R_2 given by the formulae above satisfy $R_1 < R_2$, then the Laurent series 15.1 converges for all $z \in \mathbb{C}$ such that $R_1 < |z-a| < R_2$. Moreover, the convergence is uniform and absolute in the region $r_1 \leq |z-a| \leq r_2$ for any r_1, r_2 satisfying $R_1 < r_1 < r_2 < R_2$. As a consequence, the limiting function is holomorphic in the annulus $R_1 < |z-a| < R_2$.*

Henceforth if $R_1 < R_2$ we will denote the annulus of inner radius R_1 and outer radius R_2 by

$$A_{R_1,R_2}(a) = \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\}.$$

Our main result in this chapter is a converse.

Theorem 15.0.2. *Let $R_1 < R_2$, and f be holomorphic on a domain containing the closure of the annulus $A_{R_1,R_2}(a)$. Then for all $z \in A_{R_1,R_2}(a)$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta,$$

for any $r \in [R_1, R_2]$. Moreover, the series converges uniformly and absolutely on any compact subset of $A_{R_1,R_2}(a)$.

First we need the following elementary observations.

Lemma 15.0.1. *Let F be holomorphic on any domain containing the closure of the annulus $A_{R_1,R_2}(a)$. Then*

$$\int_{C_r(a)} F(z)\, dz$$

is independent of $r \in [R_1, R_2]$.

Proof. Let $R_1 < r_1 < r_2 < R_2$. For simplicity let us denote $C_{r_i}(a) = C_i$. We claim that $C_1 \sim_{A_{R_1,R_2}(a)} C_2$. The lemma then follows from the generalized Cauchy theorem. To prove the claim, we need to compute indices. Let $w \notin A_{R_1,R_2}(a)$. Then either $|w| > R_2$ or $|w| < R_1$. If it is the former, then $w \notin Int(C_1)$ and $w \notin Int(C_2)$. Hence $n(C_2, w) = n(C_1, w) = 0$. On the other hand, if it is the latter, then $w \in Int(C_1) \subset Int(C_2)$, and so $n(C_1, w) = n(C_2, w) = 1$. In either case, for all $w \notin A_{R_1,R_2}(a)$, $n(C_1, w) = n(C_2, w)$, and hence by definition $C_1 \sim_{A_{R_1,R_2}(a)} C_2$. ☐

Lemma 15.0.2 (CIF for annuli)**.** *Let f be holomorphic on a domain containing the closure of the annulus $A_{R_1,R_2}(a)$. Then for all $z \in \mathbb{C}$ such that $R_1 < |z-a| < R_2$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof. For convenience, we use the notation $A = A_{R_1,R_2}(a)$. Fix $z \in A$, and consider the function

$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta-z}, & \zeta \neq z \\ f'(z), & \zeta = z. \end{cases}$$

Clearly $g(\zeta)$ is holomorphic on the punctured annulus $A \setminus \{z\}$. But it is continuous on the whole of the annulus since f is holomorphic at z. Hence by the theorem on removable singularities, $g(\zeta)$ is holomorphic on all of A. Then by the above lemma

$$\int_{C_{R_2}} g(\zeta)\, d\zeta = \int_{C_{R_1}} g(\zeta)\, d\zeta.$$

Since $z \notin C_{R_2}$ or C_{R_1}, the above is equivalent to

$$\begin{aligned}\int_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \int_{C_{R_2}} \frac{f(z)}{\zeta - z}\,d\zeta - \int_{C_{R_1}} \frac{f(z)}{\zeta - z}\,d\zeta \\ &= f(z)\int_{C_{R_2}} \frac{d\zeta}{\zeta - z} - f(z)\int_{C_{R_1}} \frac{d\zeta}{\zeta - z} \\ &= 2\pi i f(z)(n(C_{R_2}, z) + n(C_{R_1}, z))\end{aligned}$$

Since $z \in Int(C_{R_2})$ but lies in $Ext(C_{R_1})$, $n(C_{R_2}, z) = 1$ and $n(C_{R_1}, z) = 0$, and this completes the proof of the Lemma. □

Proof of theorem 15.0.2

This is similar to the proof of analyticity, and the key tool as before is the geometric series expansion

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n,$$

which is valid in the region $|w| < 1$. By the Lemma above

$$f(z) = \frac{1}{2\pi i}\int_{C_{R_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta := I_2 - I_1.$$

To evaluate I_2, we write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a - (z-a)} = \frac{1}{\zeta - a}\Big(\frac{1}{1 - (z-a)/(\zeta - a)}\Big)$$

Since 'a' is the center of the annulus, if $\zeta \in C_{R_2}$, and $z \in Int(C_{R_2})$, then

$$\frac{|z-a|}{|\zeta - a|} = \frac{|z-a|}{R_2} < 1.$$

Applying the geometric series expansion with $w = (z-a)/(\zeta - a)$ we see that

$$\begin{aligned}I_2 &= \frac{1}{2\pi i}\int_{C_{R_2}} \frac{f(\zeta)}{\zeta - a}\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^n}\,d\zeta \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\int_{C_{R_2}} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta\Big)\cdot (z-a)^n \\ &= \sum_{n=0}^{\infty} a_n (z-a)^n,\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_{R_2}} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta.$$

To analyze I_1, we write

$$\frac{1}{\zeta - z} = -\frac{1}{z-a}\Big(\frac{1}{1 - (\zeta - a)/(z-a)}\Big).$$

But now, if $\zeta \in C_{R_1}$, then for $z \in A_{R_1,R_2}(a)$ we have that

$$\frac{|\zeta - a|}{|z-a|} = \frac{R_1}{|z-a|} < 1,$$

and so again from the geometric series expansion it follows that

$$I_1 = -\frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\zeta)}{z-a}\sum_{k=0}^{\infty}\frac{(\zeta-a)^k}{(z-a)^k}\,d\zeta$$
$$= -\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_{R_1}} f(\zeta)(\zeta-a)^k\,d\zeta\right)(z-a)^{-k-1}.$$

Putting $k+1=-n$, we can write

$$I_1 = -\sum_{n=-1}^{-\infty} a_n(z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta.$$

This completes the proof of the theorem.

15.1 Application to study of isolated singularities

Corollary 15.1.1. *Let $f:\Omega_p^*\to\mathbb{C}$ holomorphic. The for any disc $D_R(p)$ such that $\overline{D_R(p)}\subset\Omega$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-p)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{\partial D_R(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,d\zeta.$$

Proof. Apply the theorem to the annulus $A_{r,R}(p)$ and let $r\to 0$. □

We then have the following characterization of isolated singularities based on the Laurent series expansion.

Theorem 15.1.1. *Let $f:\Omega_p^*$ holomorphic with Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-p)^n$$

around p. Then

1. *p is a removable singularity if and only if $a_n=0$ for all $n<0$.*
2. *p is a pole of order m if and only if $a_n=0$ for all $n<-m$.*
3. *p is an essential singularity if and only if for any $N>0$, there exists an $n<-N$ such that $a_n\neq 0$. That is, there are infinitely many non-zero negative exponent terms in the Laurent series expansion.*

Proof. Note that if $\overline{D_R(p)}\subset\Omega$, then the coefficients are given by the formula

$$a_n = \frac{1}{2\pi i}\int_{\partial D_R(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,d\zeta.$$

So if p is a removable singularity, then for integers $n<0$, $(\zeta-p)^{-n-1}f(\zeta)$ is holomorphic on the entire disc $D_R(p)$, and hence by Cauchy's theorem for discs, $a_n=0$ for all $n<0$. Conversely, if $a_n=0$ for

$n < 0$, the Laurent series reduces to a power series, and we know that power series are holomorphic on the entire disc of convergence.

To prove the characterization of poles, apply the same argument to the function $(\zeta - p)^m f(\zeta)$. The characterization of essential singularities then follows from the definition and the first two parts. □

Remark 15.1.2. Note that in the event the function has only poles, the Laurent series of the function centered at some other points might have infinitely many negative exponent terms. The theorem only states that the Laurent series centered at the isolated singularity can have only finitely many negative exponent terms. For an illustration of this, see Example 15.1.4 below.

Example 15.1.3. *Consider the function*

$$\frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}.$$

It has two singularities at $z = 1$ and $z = 2$ which are clearly poles. We can expand the function as a Laurent series centered at either of the poles. To illustrate this, let us find the Laurent series expansion centered at $z = 1$. One approach is to use the formula for the coefficients in Theorem 15.0.2 and compute out all the integrals. An easier approach is to use the geometric series expansion, namely that

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$$

whenever $|w| < 1$. Note that the function is holomorphic on the annulus $0 < |z - 1| < 1$, and so we can hope to have a Laurent series expansion on that domain. Writing

$$\frac{1}{z-2} = \frac{1}{z-1-1} = -\frac{1}{1-(z-1)}.$$

Since $|z - 1| < 1$, using the geometric series expansion (with $w = z - 1$) we see that

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} (z-1)^n,$$

and so

$$\frac{1}{z^2 - 3z + 2} = -\frac{1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n.$$

Example 15.1.4. *Sticking with the function from the previous example, one can also try to find a Laurent series expansion on other annuli. For instance the function is holomorphic on the annulus $A_{1,2}(0) = 1 < |z| < 2$. We consider each of the terms in the partial fraction decomposition separately. For $z \in A_{1,2}(0)$, $|z| > 1$ and so applying the geometric series expansion above to $w = 1/z$, we see that*

$$\frac{1}{z-1} = \frac{1}{z}\Big(\frac{1}{1-1/z}\Big) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

On the other hand, for $z \in A_{1,2}(0)$, $|z| < 2$ and hence once again applying the geometric series expansion to $w = z/2$,

$$\frac{1}{z-2} = -\frac{1}{2}\Big(\frac{1}{1-z/2}\Big) = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

Putting it all together, we see that on $1 < |z| < 2$,

$$\frac{1}{z^2 - 3z + 2} = -\sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

So even though the function only has poles, it Laurent series centred around $z = 0$ has infinitely many negative exponent terms.

Lecture 16

Meromorphic functions

16.1 Definition and basic properties

A function on a domain Ω is called meromorphic, if there exists a sequence of points $p_1, p_2, \cdots$ with no limit point in Ω such that if we denote $\Omega^* = \Omega \setminus \{p_1, \cdots\}$

- $f : \Omega^* \to \mathbb{C}$ is holomorphic.
- f has poles at $p_1, p_2 \cdots$.

We denote the collection of meromorphic functions on Ω by $\mathcal{M}(\Omega)$. We have the following observation, whose proof we leave as an exercise.

Proposition 16.1.1. *The class of meromorphic function forms a field over $\mathbb{C}$. That is, given any meromorphic functions $f, g, h \in \mathcal{M}(\Omega)$, we have that*

1. $f \pm g \in \mathcal{M}(\Omega)$,
2. $fg \in \mathcal{M}(\Omega)$,
3. $f(g+h) = fg + fh$.
4. $f \pm 0 = f,\ f \cdot 1 = f$,
5. $1/f \in \mathcal{M}$.

Recall that if a holomorphic function has finitely many roots, then it can be "factored" as a product of a polynomial and a no-where vanishing holomorphic function. Something similar holds true for meromorphic functions.

Proposition 16.1.2. *Let $f \in \mathcal{M}(\Omega)$ such that f has only finitely many poles $\{p_1, \cdots, p_n\}$ with orders $\{m_1, \cdots, m_n\}$. Then there exist holomorphic functions $g, h \in \mathcal{O}(\Omega)$ such that for all $z \in \Omega \setminus \{p_1, \cdots, p_n\}$,*

$$f(z) = \frac{g(z)}{h(z)}.$$

Moreover, we can choose g and h such that $f(z)$ and $g(z)$ have the exact same roots with same multiplicities, while $h(z)$ has zeroes precisely at $p_1, \cdots, p_n$ with multiplicities exactly $m_1, \cdots, m_n$.

Proof. We define $g : \Omega \setminus \{p_1, \cdots, p_m\}$ by

$$g(z) = \left(\Pi_{k=1}^{n}(z - p_k)^{m_k}\right) f(z).$$

This is clearly a holomorphic function. Moreover, since $f(z)$ has a pole of order m_k at p_k, $g(z)$ is bounded in a neighbourhood of p_k. Thus, by the theorem on removable singularities, $g(z)$ can be extended as a holomorphic function on Ω. The theorem is then proved with

$$h(z) = (z-p_1)^{m_1}\cdots(z-p_n)^{m_n}.$$

$\square$

Remark 16.1.3. The same is true even if the meromorphic function has infinite number of poles. This is a consequence of Weierstrass' factorization theorem. We will prove this theorem for the special case when $\Omega = \mathbb{C}$. For a general open set the proof requires the use of Runge's approximation theorem.

16.2 Partial fraction decomposition of meromorphic functions on $\mathbb{C}$.

Recall that if f has a pole of order m at p, then the Laurent series expansion can be written as

$$f(z) = Q_p f\Big(\frac{1}{z-p}\Big) + H_p f(z),$$

where $H_p f$ is holomorphic near p, and $Q_p f(w)$ is a polynomial

$$Q_p f(w) = a_{-m}w^m + \cdots a_{-1}w,$$

where for each $n = 1, 2, \cdots, m$ and each $\varepsilon << 1$, we have

$$a_{-n} = \frac{1}{2\pi i}\int_{|z-p|=\varepsilon} f(z)(z-p)^{n-1}.$$

The difference $f(z) - H_p f(z)$ is called the *principal* part of $f(z)$ at p. We then have the following fundamental theorem.

Theorem 16.2.1 (Mittag-Leffler). *Let $\{p_k\}$ be a discrete set of points in Ω, and for each k, let $Q_k(w)$ be a polynomial without a constant term. There there exists a $f \in \mathcal{M}(\Omega)$ with poles at p_n and holomorphic everywhere else, with principle part at p_k given by $Q_k(1/(z-p_k))$. Moreover, all such meromorphic functions are of the form*

$$f(z) = \sum_k \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big) + H(z),$$

where each $q_k(z)$ and $H(z)$ are holomorphic functions on Ω. Furthermore:

1. *If $\{p_k\}$ is a finite sequence, then one could take $q_k \equiv 0$.*
2. *If $\Omega = \mathbb{C}$, and $|p_k| \to \infty$, then one could take each q_k to be a polynomial.*

We will prove parts (1) and (2) in the next lecture. For a general Ω and infinitely many poles, the proof require's Runge's theorem, and an outline will be provided in the appendix to the next lecture.

Remark 16.2.2. Note that if $\sum_k g_k$ is a compactly convergent series on Ω (for instance, take a power series), then $\tilde{q}_k = q_k + g_k$, and $\tilde{H} = H + \sum_k g_k$ will give another representation for the function $f(z)$, and hence q_k and H are by no means unique.

Remark 16.2.3. Note that given a meromorphic function, the theorem does not say whether that particular function has a partial fraction decomposition (unlike say for rational functions, as we will see in the next section, or more generally meromorphic functions with only a finite number of poles). In other words, we are not claiming that the converse holds (even when $\Omega = \mathbb{C}$). It does turn out that the converse holds under some additional conditions on the distribution of poles. But in particular examples, one can get away by a more hands-on approach. We will see a beautiful illustration of this below.

Example 16.2.4. *Consider the meromorphic function $f(z) = \pi^2/\sin^2 \pi z$ which is a meromorphic function with poles at integers. Near zero,*

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{\Big(\pi z + O(z^3)\Big)^2} = \frac{1}{z^2\Big(1 - z^2/6 + \cdots\Big)^2} = \frac{1}{z^2}\Big(1 + \frac{z^2}{6} + \cdots\Big)^2,$$

and so the principal part of $f(z)$ is given by $1/z^2$. Using the identity $\sin^2(\pi(z-n)) = \sin^2 \pi z$, it is easy to see that the principal parts at $z = n$ are given by $(z-n)^{-2}$. Now consider the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

This converges uniformly on all compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, and hence represents a meromorphic function on $\mathbb{C}$ with poles of order two at all integers. Moreover the principle parts at each pole $z = n$ is given by $Q_n((z-n)^{-1})$, where $Q_p(w) = w^2$. It is then easy to see that the difference

$$H(z) := \frac{\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

extends to an entire function.

Claim. $H \equiv 0$.

Proof. Note that the series and the function on the left are both periodic with period 1, and hence so is $H(z)$. That is, $H(z+1) = H(z)$ for all $z \in \mathbb{C}$. Also by Euler's identity, if $z = x + iy$, then

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = \sin(\pi x)\cosh(\pi y) + \sinh(\pi y)\cos(\pi x),$$

and so

$$|\sin \pi z|^2 = \cosh^2(\pi y) - \cos^2(\pi x) \geq \cosh^2(\pi y) - 1 \to \infty$$

uniformly as $|y| \to \infty$. As a consequence $\pi^2/\sin^2 \pi z$ converges uniformly to zero as $|y| \to \infty$. But the infinite series also shares this property. Indeed, since the series converges uniformly on $|y| \geq 1$, we can take pointwise limit, and clearly each $(z-n)^{-2} \to 0$ uniformly as $|y| \to \infty$. The upshot is that $H(z)$ converges uniformly to zero as $|y| \to 0$. In particular, $H(z)$ is bounded on the strip $\{z \in \mathbb{C} \mid 0 \leq Re(z) \leq 1\}$. But then since H is periodic with period one, this means that H is a bounded entire function, and hence a constant by Liouville. But since $\lim_{y\to 0} H(iy) = 0$, we can conclude that $H \equiv 0$. $\square$

Assuming this, we get the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

for all $z \notin \mathbb{Z}$. Plugging in $z = 1/2$, we obtain the identity

$$\frac{\pi^2}{4} = \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Now let, $S = \sum_{m=0}^{\infty} m^{-2}$. Then we have

$$S = \sum_{n=1}^{\infty} \frac{1}{(2m-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$= \frac{\pi^2}{8} + \frac{S}{4}.$$

Solving for S, we get the beautiful identity

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}.$$

Proving this identity was the so-called Basel problem, first "solved" by Euler. But his "proof" would not pass our modern day standards of rigour. Euler used a "facotrization" for sine, but a rigorous development of the theory of infinite product factorizations of entire functions had to wait till Weierstrass came along many decades later. Nevertheless, Euler's insights were of course crucial in all subsequent developments.

16.3 Holomorphic self maps on the extended complex plane

It is often useful to think of $z = \infty$ on the same footing as other points in the complex plane, and to define the *extended complex plane*

$$\overline{\mathbb{C}} = \mathbb{C} \cup \infty.$$

We can then think of meromorphic functions $f : \Omega \setminus \{p_1, \cdots, p_j, \cdots\} \to \mathbb{C}$, as functions $f : \Omega \to \hat{\mathbb{C}}$, by defining

$$f(p_j) = \infty$$

for all the poles p_js.

Definition 16.3.1. Let $\Omega \subset \mathbb{C}$. We say that a map $f : \Omega \to \overline{\mathbb{C}}$ is holomorphic if for any $p \in \Omega$, either $f(p) \neq \infty$ and f is holomorphic near p, or $f(p) = \infty$ and $1/f(z)$ is holomorphic near p.

Proposition 16.3.2. *A function $f : \Omega \to \overline{\mathbb{C}}$ is holomorphic if and only if $f \in \mathcal{M}(\Omega)$ with poles precisely on the polar set $f^{-1}(\infty)$.*

Similarly, in studying meromorphic functions on $\mathbb{C}$, it is also useful to consider the extension of the function themselves to $\hat{\mathbb{C}}$. We say that $z = \infty$ is a pole of order m (resp. removable or essential singularity) if $z = 0$ is a pole of order m (resp. removable or essential singularity) for the function

$$\hat{f}(z) = f(1/z).$$

Similarly we can also define a zero of order m at infinity. We then say that a meromorphic function on $\mathbb{C}$ is meromorphic on the extended plane, if it does not have an essential singularity at $z = \infty$. It turns out that meromorphic functions on $\hat{\mathbb{C}}$ can be classified. Recall that a rational function on $\mathbb{C}$ is a function of the form

$$R(z) = \frac{P(z)}{Q(z)}$$

where both $P(z)$ and $Q(z)$ are polynomials.

Example 16.3.3.

1. *A polynomial $P(z) = a_n z^n + \cdots + a_0$ with $a_n \neq 0$ has a pole of order n at infinity. In fact, conversely, ever entire function $p(z)$ with a pole of order n at infinity is a polynomial of degree n. This follows from the Cacuhy estimates.*
2. *The function e^z has an essential singularity at infinity.*
3. *A rational function has a pole or removable singularity at infinity. In fact a rational function $R(z) = P(z)/Q(z)$ as above has*
   - *a pole of order* $\deg P - \deg Q$ *at infinity if* $\deg(P) > \deg(Q)$,
   - *a removable singularity at infinity if* $\deg(P) \leq \deg(Q)$,

- *a zero of order* $\deg(Q) - \deg(P)$ *if* $\deg(P) < \deg(Q)$.

Theorem 16.3.4. *The only meromorphic functions on* $\hat{\mathbb{C}}$ *are rational functions.*

Proof. Let $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic function.

Claim-1. F has only finitely many poles $\{p_1, \cdots, p_n\}$ in the complex plane C.

To see this, note that $F(1/z)$ has either a pole or zero at $z = 0$. In either case there is a small neighborhood $|z| < \varepsilon$ which has no other pole. Which is the same as saying that F has no finite pole in $|z| > 1/\varepsilon$. But $|z| \leq 1/\varepsilon$ is compact, and since all poles are isolated, this shows that there are only finitely many poles. Now, corresponding to each of the poles $p_k \in \mathbb{C}$ there exists a polynomial Q_k (see Remark 0.2 in Lecture-20) such that

$$F(z) = Q_k\Big(\frac{1}{z - p_k}\Big) + H_k(z),$$

where G_k is holomorphic on a whole neighborhood around p_k (including at the point p_k). Similarly if $|z| > R$, we can write

$$F(z) = Q_\infty(z) + H_\infty\Big(\frac{1}{z}\Big),$$

where as before, $H_\infty(z)$ is holomorphic in a neighborhood of $z = 0$.

Claim-2. The function

$$G(z) = F(z) - Q_\infty(z) - \sum_{k=1}^{n} Q_k\Big(\frac{1}{z - p_k}\Big)$$

is an entire and bounded function.

Assuming the claim, by Liouville's theorem, $G(z)$ is a constant, and hence $F(z)$ must be rational, and the theorem is proved. To prove the claim, first note that clearly, $G(z)$ is holomorphic away from $\{p_1, \cdots, p_n\}$. At some $z = p_k$, $Q_j(1/z - p_j)$ is holomorphic for all $j \neq k$. On the other hand, near p_k,

$$F(z) - Q_k\Big(\frac{1}{z - p_k}\Big) = H_k(z)$$

which is holomorphic. This shows that $G(z)$ is entire. As a consequence, to show boundedness, we only need to show boundedness on $|z| > R$ for some large R. To see, first observe that since Q_k are polynomials,

$$\lim_{z \to \infty} Q_k\Big(\frac{1}{z - p_k}\Big) = 0.$$

Hence it is enough to show that $F(z) - Q_\infty(z)$ is bounded near infinity. But this follows immediately from noting that

$$H_\infty(z) = F\Big(\frac{1}{z}\Big) - Q_\infty\Big(\frac{1}{z}\Big)$$

is holomorphic near $z = 0$ and hence is bounded on $|z| < \varepsilon$ for some $\varepsilon > 0$. In particular $F(z) - Q_\infty(z)$ is bounded on $|z| > 1/\varepsilon$. This proves the claim, and hence completes the proof of the theorem.

□

Remark 16.3.5. A meromorphic function $f \in \mathcal{M}(\overline{\mathbb{C}})$ gives rise to a holomorphic map $F : \mathbb{P}^! \to \mathbb{P}^1$. Conversely, given any map $F : \mathbb{P}^1 \to \mathbb{P}^1$, one gets a meromorphic map from $\overline{\mathbb{C}} \to \mathbb{C}$ with poles at $F^{-1}([0, 1])$. So the theorem can be reformulated in the following way - all holomorphic maps from F : $\mathbb{P}^1 \to \mathbb{P}^1$ are given by rational functions of two variables, where the numerator and denominator are homogenous polynomials.

A simple consequence of the proof is the following theorem on partial fraction decomposition that we take for granted as an important tool in integration theory, but never see the proof of.

Corollary 16.3.6. *For any rational function $R(z) = P(z)/Q(z)$ has a partial fraction decomposition of the form*

$$R(z) = Q_\infty(z) + \sum_{k=1}^{n} Q_k\Big(\frac{1}{z-p_k}\Big),$$

where p_k is a root of $Q(z)$ of order m_k, Q_k is a polynomial of degree m_k, and $\deg Q_\infty = \deg P - \deg Q$ if this number is non-negative. Else we have that $Q_\infty \equiv 0$.

Lecture 17

The theorems of Mittag Leffler and Weierstrass

17.1 Proof of the Mittag-Leffler theorem.

Recall that the Mittag-Leffler theorem was as follows.

Theorem 17.1.1 (Mittag-Leffler). *Let $\{p_k\}$ be a discrete set of points in Ω, and for each k, let $Q_k(z)$ be a polynomial without a constant term. There there exists a $f \in \mathcal{M}(\Omega)$ with poles at p_k and holomorphic everywhere else, with principle part at p_k given by $Q_k(1/(z-p_k))$. Moreover, all such meromorphic functions are of the form*

$$f(z) = \sum_k \Big(Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big) + H(z),$$

where each $q_k(z)$ and $H(z)$ are holomorphic functions on Ω, and q_k depends only on Q_k. Furthermore:

1. *If $\{p_k\}$ is a finite sequence, then one could take $q_k \equiv 0$.*
2. *If $\Omega = \mathbb{C}$, and $|p_k| \to \infty$, then one could each q_k to be a polynomial.*

Proof. The proof of the theorem in case (1) is trivial, and we leave it as an exercise. We prove the theorem only in the case (2) above. For a general Ω with possibly infinite sequence $\{p_k\}$, the proof relies on Runge's theorem and is out of the scope of the present course.

So from now, suppose $\Omega = \mathbb{C}$. Without loss of generality, we can assume that no p_k is equal to 0. Suppose we order them such that $0 < |p_1| \leq \cdots$. By the first part, we can assume that the number of poles is infinite, and hence that $\lim_{k\to\infty} |p_k| = \infty$. Since each $Q_k(1/(z-p_k))$ is holomorphic on $|z| < |p_k|$, it can be expanded as a Taylor series around $z = 0$. Let q_k be the partial sum of degree d_k of this Taylor expansion. Let

$$M_k = \sup_{|z| \leq |p_k|/2} \Big| Q_k\Big(\frac{1}{z-p_k}\Big)\Big|.$$

Claim. For all z such that $|z| \leq |p_k|/4$, we have the estimate

$$\Big| Q_k\Big(\frac{1}{z-p_k}\Big) - q_k(z)\Big| \leq 2M_k \Big(\frac{2|z|}{|p_k|}\Big)^{d_k+1} \leq M_k 2^{-d_k}.$$

We assume the claim for the moment. Now pick $d_k >> 1$ such that $2^{d_k} \geq M_k 2^k$, and consider the series

$$f(z) = \sum_k \left(Q_k\left(\frac{1}{z-p_k}\right) - q_k(z) \right).$$

For any compact set $K \subset\subset \mathbb{C} \setminus \{p_1, \cdots, \}$, there exists a N such that for $k > N$, $K \subset D_{|p_k|/4}(0)$. By the claim and our choice of d_k, for all $k > N$, each term of the infinite series

$$\sum_{k=N+1}^{\infty} \left(Q_k\left(\frac{1}{z-p_k}\right) - q_k(z) \right)$$

is dominated by 2^{-k}, and hence by Weierstarss test the tail, represents a holomorphic function. On the other hand

$$\sum_{k=1}^{N} \left(Q_k\left(\frac{1}{z-p_k}\right) - q_k(z) \right)$$

is a meromorphic function on K with poles at $p_1, \cdots, p_N$ with prescribed principal parts. In particular, $f(z)$ is meromorphic on $\mathbb{C}$ with poles at p_k with principal part $Q_k((z-p_k)^{-1})$. To finish the proof, if $\tilde{f}$ is another such function, then clearly $\tilde{f} - f$ extends to an entire function.

Proof of the claim. Suppose

$$Q_k\left(\frac{1}{z-p_k}\right) - q_k(z) = \sum_{j=d_k+1}^{\infty} a_j z^j$$

in the disc $|z| < |p_k|/2$. By the Cauchy estimate, we have that $|a_j| \leq \frac{2^j M_k}{|p_k|^j}$, and so

$$\begin{aligned}
\left| Q_k\left(\frac{1}{z-p_k}\right) - q_k(z) \right| &\leq M_k \sum_{j=d_k+1}^{\infty} \frac{2^j |z|^j}{|p_k|^j} \\
&\leq M_k \left(\frac{2|z|}{|p_k|}\right)^{d_k+1} \sum_{j=0}^{\infty} \frac{1}{2^j} \\
&\leq 2M_k \left(\frac{2|z|}{|p_k|}\right)^{d_k+1},
\end{aligned}$$

where we used the fact that $|z| \leq |p_k|/4$ in the penultimate line. □

Example 17.1.2. *In the previous lecture, we illustrated the theorem by obtaining an expansion for* $\pi^2/\sin^2 \pi z$. *We now obtain an expansion for* $\pi \cot \pi z$ *which is meromorphic on* $\mathbb{C}$ *with only simple poles at* $z = n \in \mathbb{Z}$. *In fact, the principal part is precisely* $(z-n)^{-1}$. *Unfortunately, the series* $\sum (z-n)^{-1}$ *is divergent, and hence one has to subtract off a polynomial, which in this case turns out to be a constant. Consider the series*

$$\sum_{n \neq 0} \frac{1}{z-n} + \frac{1}{n} = \sum_{n \neq 0} \frac{1}{(z-n)n},$$

which is compactly convergent on $\mathbb{C} \setminus \mathbb{Z}$ *as can be seen by comparing with the series* $\sum n^{-2}$. *Hence the series represents a meromorphic function on* $\mathbb{C}$ *with simple poles at* $z = n$. *Then clearly*

$$H(z) = \pi \cot \pi z - \frac{1}{z} - \sum_{n \neq 0} \frac{1}{z-n} + \frac{1}{n}$$

is an entire function. Moreover, by direct calculation, one can see that for $z \notin \mathbb{Z}$ *(and hence everywhere),*

$$H'(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} = 0$$

by our expansion from previous lecture. Hence $H(z)$ is a constant. Now, rewriting the function as

$$H(z) = \pi \cot \pi z - \lim_{m\to\infty}\Big(\frac{1}{z} + \sum_{n=-m}^{m} \frac{1}{z-n} + \frac{1}{n}\Big) = \pi \cot \pi z - \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

The right hand side is an odd function, and hence $H(z)$ must be zero. Thus we obtain the identity

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\Big(\frac{1}{z-n} + \frac{1}{n}\Big). \tag{17.1}$$

17.2 Infinite products

A infinite product of non-zero complex numbers $\Pi_{n=1}^{\infty} p_n$ is said to *converge* if

$$P := \lim_{n\to\infty} \Pi_{k=1}^{n} p_k$$

exists. If some of the terms are allowed to be zero, then we say that the infinite product converges if the following two conditions hold

1. At most a finite number of terms are zero.
2. If $N > 0$ such that $p_n \neq 0$ for all $k > N$, then $\Pi_{k=N+1}^{\infty} p_n$ converges in the above sense.

. If we denote the n^{th} partial product by $P_n = \Pi_{k=1}^{n} p_k$, then it is clear that it any convergent product $p_n = P_n/P_{n-1}$ converges to 1. Denoting $p_n = 1 + b_n$, we say that the product converges absolutely if $\Pi(1 + |b_n|)$ converges. We then have the following basic fact.

Proposition 17.2.1. *Let $\{b_n\}$ be a sequence of complex functions, none of which is zero. Then the infinite product $\Pi(1 + b_n)$ (absolutely) converges if and only the series $\sum \log(1 + b_n)$ (absolutely) converges, where* $\log$ *is the principal branch of the logarithm.*

One can similarly talk about uniform convergence and compact convergence of infinite products. We then have the following counterpart of the above theorem.

Proposition 17.2.2. *Let $\{f_n\}$ be a sequence of entire. Suppose that for every compact set K, all but finitely many f_ns are zero free in K. Then $\Pi_{n=1}^{\infty}(1 + f_n(z))$ converges compactly (resp. absolutely) on $\mathbb{C}$ if and only if $\sum \log f_n(z)$ converges compactly (resp. absolutely) on $\mathbb{C}$. In such a case the infinite product converges to an entire function.*

Note that absolutely convergent products also satisfy the “rearrangement property".

17.3 Weierstrass factorisation theorem

Theorem 17.3.1. *Let $\{a_n\}$ be an arbitrary sequence of non-zero complex numbers ordered such that $\lim_{n\to\infty} |a_n| = \infty$, if the sequence is infinite. Then there exists an entire function with zeroes at precisely the points a_n. Moreover, every entire function with these and no other zeroes (except possibly at $z = 0$) is given by*

$$f(z) = z^m e^{g(z)} \Pi_{n=1}^{\infty}\Big(1 - \frac{z}{a_n}\Big) e^{q_n(z)}, \tag{17.2}$$

where q_n is a polynomial given by

$$q_n(z) = \frac{z}{a_n} + \frac{1}{2}\Big(\frac{z}{a_n}\Big)^2 + \cdots + \frac{1}{m_n}\Big(\frac{z}{a_n}\Big)^{m_n}.$$

The convergence here is absolute, and uniform on compact sets. Furthermore, if there exists some integer $h > 0$, such that

$$\sum_n \frac{1}{|a_n|^{1+h}} < \infty, \tag{17.3}$$

then we can take $m_n = h$ for all n.

The expression (17.2) above is called the *canonical product* associated with $\{a_n\}$, and the smallest integer h satisfying (17.3) (if it exists) is called the *genus* of the canonical product. Else the genus is said to be infinite. If $g(z)$ above reduces to a polynomial, then we say that $f(z)$ is of finite genus, and the genus of $f(z)$ is defined to be the maximum of the degree of $g(z)$ and h.

Proof. We only prove the second part, and leave the more general statement as an exercise. So from now on assume that our sequence satisfies (17.3). We need to prove the existence of polynomials $q_n(z)$ such that the infinite product

$$\Pi_{n=1}^{\infty}\Big(1 - \frac{z}{a_n}\Big)e^{q_n(z)}$$

converges to an entire function. By Proposition 17.2.2, this holds if and only if the series with the general term

$$r_n(z) = \log\Big(1 - \frac{z}{a_n}\Big) + q_n(z)$$

converges compactly. Let $R > 1$ be arbitrary. We only consider those a_n such that $|a_n| > 2R$. Then on $|z| < R$, since $|z/a_n| < 1$, the first term in $r_n(z)$ has a power series expansion:

$$\log\Big(1 - \frac{z}{a_n}\Big) = -\frac{z}{a_n} - \frac{1}{2}\Big(\frac{z}{a_n}\Big)^2 - \cdots.$$

For the h as in the hypothesis, we let

$$q_n(z) = \frac{z}{a_n} + \frac{1}{2}\Big(\frac{z}{a_n}\Big)^2 + \cdots + \frac{1}{h}\Big(\frac{z}{a_n}\Big)^h,$$

so that

$$r_n(z) = -\frac{1}{1+h}\Big(\frac{z}{a_n}\Big)^{h+1} - \frac{1}{2+h}\Big(\frac{z}{a_n}\Big)^{h+2} - \cdots.$$

By comparison with a geometric series, we obtain the estimate that

$$|r_n(z)| \le \frac{1}{1+h}\Big(\frac{R}{|a_n|}\Big)^{1+h}\Big(1 - \frac{R}{|a_n|}\Big)^{-1} < \frac{2}{1+h}\Big(\frac{R}{|a_n|}\Big)^{1+h}.$$

By hypothesis, $\sum r_n(z)$ converges absolutely and compactly on $|z| \le R$, and hence so must the product. Note that even though the proof required us to work with $|a_n| > 2R$, just multiplying in the terms on the product with $|a_n| \le 2R$ does not affect convergence of the product, since such a_n's are only finite in number. This shows that the product with our choice of q_n converges to an entire function with the required properties. Now suppose $f(z)$ is any other such entire function with a zero of order m at $z = 0$. Then consider the entire function

$$F(z) = \frac{f(z)}{z^m \Pi_{n=1}^{\infty}\Big(1 - \frac{z}{a_n}\Big)e^{q_n(z)}}.$$

Since the zeroes of the numerator and denominator match up, $F(z)$ clearly extends as an entire function which is no-where zero. But then by simple connectivity of $\mathbb{C}$, this implies that one can take a holomorphic branch $g(z) = \log F(z)$. Then clearly $f(z)$ has the expression above. $\square$

Example 17.3.2. *Consider the function* $\sin \pi z$*. This is an entire function with zeroes at* $z = \pm n$*. Since*

$$\sum_n \frac{1}{n^{1+h}}$$

converges for $h = 2$ *(and* 2 *is the smallest such integer), we can apply the above proposition to obtain*

$$\sin \pi z = z e^{g(z)} \Pi_{n \in \mathbb{Z} \setminus \{0\}} \Big(1 - \frac{z}{n}\Big) e^{z/n}.$$

Claim. $e^{g(z)} \equiv \pi$.

To see this, we take a logarithmic derivative. Then

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \Big(\frac{1}{z-n} - \frac{1}{n}\Big).$$

Comparing with the expansion of $\pi \cot \pi z$*, we see that* $g'(z)$ *has to vanish, and hence* $g(z)$ *is a constant. On the other hand,*

$$\pi = \lim_{z \to 0} \frac{\sin \pi z}{z} = e^{g(0)},$$

and so $e^{g(z)} \equiv \pi$*. It follows that* $f(z)$ *is an entire function of genus* 1*. Because of absolute convergence, we can rewrite*

$$\frac{\sin \pi z}{\pi z} = \Pi_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big).$$

Expanding, and comparing the coefficients of z^2*, we once again see that*

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This was the original approach of Euler.

We finish with the following elementary corollary, whose proof we leave as an exercise.

Corollary 17.3.3. *Any meromorphic function in* $\mathbb{C}$ *is the quotient of two entire functions.*

Lecture 18

The Residue theorem

Let f be a holomorphic function in $D_\varepsilon(p) \setminus \{p\}$ with a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-p)^n.$$

The *residue* of $f(z)$ at $z = p$ is then defined by

$$\mathrm{Res}_{z=p} f(z) = a_{-1}.$$

Then by Theorem 2 in Lecture 16, for any $r < \varepsilon$,

$$\mathrm{Res}_{z=p} f(z) = \frac{1}{2\pi i} \int_{|z-p|=r} f(z)\, dz.$$

More generally, we have the following fundamental result.

Theorem 18.0.1 (Residue Theorem). *Let Ω be open, $\{p_k\} \in \Omega$ a sequence of isolated points, and $f \in \mathcal{O}(\Omega^*)$, where $\Omega^* := \Omega \setminus \{p_1, \cdots\}$. Then for any cycle $\gamma \sim_\Omega 0$ in Ω such that no p_k lies on $Supp(\gamma)$, we have*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_k n(\gamma, p_k) \mathrm{Res}_{p_k} f(z).$$

Moreover, for any given γ as above, $n(\gamma, p_k) = 0$ for all but finitely many k, and hence the summation above has only a finite number of non-zero terms.

Proof. First let us assume that there are only a finite number of singular points. Let C_k be a small circle around p_k enclosing a disc D_k, such that $\overline{D_k} \subset \Omega$ and such that C_k does not intersect $Supp(\gamma)$. We now claim that

$$\gamma \sim_{\Omega^*} \sum_k n(\gamma, p_k) C_k. \tag{18.1}$$

Assuming this, we are then done by the generalised Cauchy theorem, since

$$\begin{aligned}\int_\gamma f(z)\, dz &= \sum_k n(\gamma, p_k) \int_{C_k} f(z)\, dz \\ &= 2\pi i \sum_k n(\gamma, p_k) \mathrm{Res}_{z=p_k} f(z).\end{aligned}$$

To prove (18.1) let $a \notin \Omega^*$. We need to show that

$$n(\gamma, a) = \sum_k n(\gamma, p_k) n(C_k, a). \tag{18.2}$$

If $a \notin \Omega$, then by construction $a \notin \overline{D_k}$, and since a disc is simply connected, $n(C_k, a) = 0$. On the other hand, since $\gamma \sim_\Omega 0$, we also have that $n(\gamma, a) = 0$, and hence (18.2) is trivially satisfied. If $a \in \Omega$, then $a = p_j$ for some j. Once again, as above, $n(C_k, a) = 0$ for all $k \neq j$. On the other hand, $n(C_j, a) = 1$. This verifies (18.2) and completes the proof of (18.1).

Finally, suppose the number of singularities is infinite in number. It is enough to prove that $n(\gamma, p_k) = 0$ for all but finitely many k. Note that

$$U_0 := \{a \in \mathbb{C} \mid n(\gamma, a) = 0\}$$

is an open set (since the index is locally constant). Moreover it contains the annulus $A_{R.\infty}(0)$ for R large enough. As a consequence, the set $U^c = \mathbb{C} \setminus U_0$ is a compact set and must contain only finitely many p_k (since the singularities are isolated). □

An important corollary is the following.

Corollary 18.0.2. *Let f be holomorphic in Ω except possibly at isolate points $\{p_1, p_2, \cdots, \}$ in Ω, and let γ be a positively oriented, simple, closed curve in Ω not passing through any of the singularities. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p_k \in Int(\gamma)} \mathrm{Res}_{z=p_k} f(z).$$

Proof. This follows from the residue theorem and the fact that

$$n(\gamma, p_k) = \begin{cases} 1, & z \in Int(\gamma) \\ 0, & z \in Ext(\gamma). \end{cases}$$

□

Our next result helps in computing the residue at poles.

Proposition 18.0.3. *Let f have a pole of order m at p. Then*

$$\mathrm{Res}_{z=p} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\Big|_{z=p} (z-p)^m f(z).$$

Proof. If f has a pole of order m, then $(z-p)^m f(z)$ has a removable singularity at $z = p$. Moreover, if the Laurent series expansion for f at p is given by

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-p)^n,$$

then

$$(z-p)^m f(z) = \sum_{k=0}^{\infty} a_{k-m} (z-p)^k,$$

and hence

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\Big|_{z=p} (z-p)^m f(z) = a_{m-1-m} = a_{-1} = \mathrm{Res}_{z=p} f(z).$$

□

Example 18.0.4. *Let us evaluate*

$$\int_{S_R} \frac{e^{\pi/2z}}{1+z^2}\, dz,$$

where S_R is a square of side length $2R$ centred at the origin and oriented in an anti-clockwise direction. Let $f(z)$ be the integrand. Then it has three isolated singularities, namely an essential one at 0 and poles of order one at $\pm i$. Let us compute the residue at each of the singularities.

- **Residue at $z = 0$.** *The Laurent series expansion is given by*

$$\frac{e^{(\pi/2z)}}{z^2+1} = \Big(\sum_{n=0}^{\infty} \frac{\pi^n}{2^n n!} z^{-n}\Big)\Big(\sum_{m=0}^{\infty} (-1)^{2m} z^{2m}\Big),$$

hence the residue, which is the coefficient of z^{-1} is given by

$$\mathrm{Res}_{z=0} f(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{\pi}{2}\right)^{2m+1} = \sin\left(\frac{\pi}{2}\right) = 1.$$

- **Residue at $z = i$.** *By Proposition 18.0.3, the residue is given by*

$$\mathrm{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)\frac{e^{\pi/2z}}{z^2+i} = \frac{e^{\pi/2i}}{2i} = -\frac{1}{2}.$$

- **Residue at $z = -i$.** *Once again by Proposition 18.0.3, the residue is given by*

$$\mathrm{Res}_{z=-i} f(z) = \lim_{z\to -i}(z+i)\frac{e^{\pi/2z}}{z^2+i} = \frac{e^{-\pi/2i}}{-2i} = -\frac{1}{2}.$$

Then by the residue theorem, we have

$$\int_{S_R} \frac{e^{\pi/2z}}{1+z^2}\, dz = \begin{cases} 1, & R < 1, \\ 0, & R > 1. \end{cases}$$

18.1 The argument principle

Theorem 18.1.1 (The argument principle)**.** *Let Ω be a domain and $f \in \mathcal{M}(\Omega)$ zeroes at $\{a_j\}$ or orders $\{m_j\}$ and poles at $\{b_k\}$ of orders $\{n_k\}$. Then for every cycle $\gamma \sim_\Omega 0$ which does not pass through any zeroes or poles, we have that*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_j n(\gamma, a_j) m_j - \sum_k n(\gamma, b_k) n_k.$$

Furthermore the two summations are finite summations.

As a simple corollary we have the following.

Corollary 18.1.2. *Let Ω be a simply connected domain and $f \in \mathcal{O}(\Omega)$ with zeroes at $\{a_j\}$ or orders $\{m_j\}$. Then for any simple, closed, positively oriented curve γ no passing through any of the roots, we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{a_j \in Int(\gamma)} m_j.$$

Proof of the argument principle. Note that f'/f has poles precisely at the zeroes and poles of $f(z)$, and is holomorphic everywhere else. So the integral can be computed by using the residue theorem. To do so, we need to compute the residues of f'/f. There are two cases.

1. **Residue of f'/f at $z = a_j$.** Near a_j, say on $D_{\varepsilon_j}(a_j)$, we can write $f(z) = (z - a_j)^{m_j} g_j(z)$, where $g_j(z)$ is holomorphic and zero free on $D_{\varepsilon_j}(a_j)$. Then
$$\frac{f'(z)}{f(z)} = \frac{m_j}{z - a_j} + \frac{g_j'(z)}{g_j(z)}.$$
Since g_j'/g_j is holomorphic, we have that
$$\mathrm{Res}_{z=a_j} \frac{f'(z)}{f(z)} = m_j.$$

2. **Residue of f'/f at $z = b_k$.** Near b_k, say on $D_{\varepsilon_k}(b_k)$, we can write $f(z) = (z - b_k)^{-n_k} g(z)$, where $g_k(z)$ is holomorphic and zero free on $D_{\varepsilon_k}(b_k)$. Then
$$\frac{f'(z)}{f(z)} = -\frac{n_k}{z - a_j} + \frac{g_k'(z)}{g_k(z)}.$$
Once again, since g_k'/g_k is holomorphic, we have that
$$\mathrm{Res}_{z=b_k} \frac{f'(z)}{f(z)} = -n_k.$$

The theorem then follows by an application of the residue theorem. □

Remark 18.1.3. More generally, if f is holomorphic, and we take $f(z) - w$, then for any simple closed curve γ such that $w \notin f(\mathrm{Supp}(\gamma))$,
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w}\, dz = \sum_{a_j \in Int(\gamma), f(a_j) = w} m_j,$$
where m_j is the order of the zero of $f(z) - w$ at a_j.

18.1.1 Argument principle as an index calculation

If γ is a short curve such that $|\gamma(0)| = |\gamma(1)|$, and not passing through the origin, then the index $n(\gamma, 0)$ computes the change in the argument of $\gamma(t)$ (upto a factor of 2π). To see this, note that is γ is short enough, then one can define a holomorphic branch of the logarithm $\log z$ in a neighborhood of $\mathrm{Supp}(\gamma)$. Then
$$n(\gamma, 0) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} = \frac{\log \gamma(1) - \log \gamma(0)}{2\pi i} = \frac{\arg \gamma(1) - \arg(0)}{2\pi}.$$

Now, let $\gamma : [0, 1] \to \Omega$ be a curve and $f \in \mathcal{O}(\Omega)$. Then $\Gamma(t) = f(\gamma(t))$ defines a curve in $\mathbb{C}$ with $\Gamma'(t) = f'(\gamma(t))\gamma'(t)$. Then
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)\, dt = \frac{1}{2\pi i} \int_0^1 \frac{\Gamma'(t)}{\Gamma(t)}\, dt = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w},$$
and hence we conclude that
$$n(\Gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$
So the integral of f'/f along γ essentially measures the change in argument of $f(z)$. More generally, together with the remark above, we see that for any $w \notin f(\mathrm{Supp}(\gamma))$,
$$n(\Gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w}\, dz.$$

Lecture 19

Applications of the argument principle

19.1 Local mapping properties of holomorphic functions

Theorem 19.1.1. *Let Ω be connected, and $f \in \mathcal{O}(\Omega)$ be a non-constant holomorphic function. Suppose $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order m at z_0. Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that whenever w with $0 < |w - w_0| < \delta$, the equation $f(z) = w$ has exactly m distinct solutions in $B_\varepsilon(z_0)$ each of multiplicity one.*

Proof. First we choose $\varepsilon_0 > 0$ such that

1. $f(z) - w_0$ has no other root in $\overline{D_{\varepsilon_0}(z_0)}$, and
2. For all $z \in D_{\varepsilon_0}(z_0)$, $f'(z) \neq 0$.

The first condition can be achieved since $f(z) - w_0$ is not constant, and zeroes are isolated. For the second condition, if $m = 1$, then $f'(z_0) \neq 0$, and hence an $\varepsilon_0 > 0$ as above can be picked by continuity of $f'(z)$. If $m > 1$, then $f'(z_0) = 0$. But $f'(z)$ is also holomorphic, and hence it's zeroes must also be isolated. Let γ be the circle $|z - z_0| = \varepsilon$ oriented in the positive sense, and let $\Gamma = f \circ \gamma$. Now $w_0 \notin \mathrm{Supp}(\Gamma)$ by propert (1) above, and hence there exists a $\delta > 0$ such that $\overline{D_\delta(w_0)} \subset \mathbb{C} \setminus \mathrm{Supp}(\Gamma)$. For any $w \in D_\delta(w_0)$, since the index is locally constant, $n(\Gamma, w) = n(\Gamma, w_0)$. By our discussion in the previous lecture, $n(\Gamma, w_0)$ counts the number of zeroes of $f(z) - w_0$ (with multiplicity) in the interior of γ, which in this case is m. Hence $n(\Gamma, w) = m$, and so $f(z) - w$ also has exactly m solutions in $D_\varepsilon(z_0)$ counted with multiplicity. Now, look at $g(z) = f(z) - w$. Since $g'(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$, none of the roots of $g(z)$ can have multiplicity more than one. Hence $f(z) = w$ has exactly m distinct solutions in $D_\varepsilon(z_0)$, each with multiplicity one. □

Remark 19.1.2. The theorem essentially says that locally, holomorphic functions are "branched" or "ramified" covers. That is if $f(z_0) = w_0$ with multiplicity m, and with ε, δ as above, the map $f : D_\varepsilon(z_0) \setminus \{z_0\} \to D_\delta(w_0) \setminus \{w_0\}$ is $m : 1$ covering map, and the m branches come together at z_0. If $m > 1$, we say that z_0 is a branch point, and that m is the branching order. The prototypical example that one should keep in mind is $f(z) = z^m$. Then in any small neighbourhood of $z = 0$ (excluding at zero), then function is $m : 1$. Namely, for any $w \neq 0$, then if $w = re^{i\theta}$, $f(r^{1/m} \zeta_m^k e^{i\theta/m}) = w$ for $k = 0, 1, \cdots, m - 1$, where $\zeta_m = e^{2\pi i/m}$ is the primitive m^{th} root of unity.

Corollary 19.1.3 (Open mapping theorem)**.** *Let U be and open set, and $f : U \to \mathbb{C}$ be any non-constant holomorphic function. Then $f(U)$ is an open subset of $\mathbb{C}$.*

Proof. Let $w_0 \in f(U)$. Then there exists a $z_0 \in U$ such that $f(z_0) = w_0$. By the above theorem, there exists a $\varepsilon > 0$ and $\delta > 0$ such that $D_\varepsilon(z_0) \subset U$ and $f(z) = w$ has at least one solution in $D_\varepsilon(z)$ for each $w \in D_\delta(w_0)$. In particular, $D_\delta(w_0) \subset U$, and since w_0 was arbitrary, $f(U)$ is open. □

Remark 19.1.4. This is of course not true in the real setting, even for polynomials, much less real analytic functions. For instance, consider $f(x) = x^2$ on $(-1, 1)$. Then $f((-1, 1)) = [0, 1)$ which is not open.

Given two open sets U and V, we say that $f : U \to V$ is a *biholomorphism* if f is bijective, holomorphic, and it's inverse $f^{-1} : V \to U$ is also holomorphic.

Corollary 19.1.5 (Inverse function theorem)**.** *Let $f \in \mathcal{O}(\Omega)$, and $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, and put $w_0 = f(z_0)$. Then there exist $\varepsilon, \delta > 0$ such that for every $w \in D_\delta(w_0)$ there exists a unique $z_w \in D_\varepsilon(z_0)$ such that $f(z_w) = w$. Moreover we have the following explicit formula for z_w:*

$$z_w = \frac{1}{2\pi i} \int_{|z-z_0|=r} z \frac{f'(z)}{f(z) - w} \, dz, \tag{19.1}$$

where $|z_w - z_0| < r < \varepsilon$. In particular, if we set $U = f^{-1}(D_\delta(w_0)) \cap D_\varepsilon(z_0)$, then $f : U \to D_\delta(w_0)$ is a biholomorphism with $f^{-1}(w) = z_w$ and $(f^{-1})'(w) = 1/f'(z_w)$.

Proof. Since $f'(z_0) \neq 0$, the multiplicity of $f(z) = w_0$ is exactly one at $z = z_0$. By Theorem 19.1.1, there exists $\varepsilon, \delta > 0$ such that for all $w \in D_\delta(w_0)$, there is a unique z_w such that $f(z_w) = w$ in the disc $D_\varepsilon(z_0)$. Also note that $f'(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$. locally To prove the formula for z_w, we use the residue theorem. Consider the function

$$H_w(z) = \frac{z f'(z)}{f(z) - w}.$$

Then since $f(z) = w$ has a unique solution in $|z - z_0| < \varepsilon$, $H_w(z)$ has a pole exactly order one at $z = z_w$, and is holomorphic everywhere else. Also note that $f'(z_w) \neq 0$. This follows since $f(z) - w$ has a zero of multiplicity one at z_w. We then compute the residue

$$\begin{aligned} \operatorname{Res}_{z=z_w} H_w(z) &= \lim_{z \to z_w} (z - z_w) \frac{z f'(z)}{f(z) - w} \\ &= z_w f'(z_w) \lim_{z \to z_w} \frac{z - z_w}{f(z) - f(z_w)} = z_w. \end{aligned}$$

Then (19.1) is proved by an application of the residue theorem. In particular, as in the statement of the theorem, if we set $U = f^{-1}(D_\delta(w_0)) \cap D_\varepsilon(z_0)$, then $f : U \to D_\delta(w_0)$ is an injective function with a well defined inverse function $f^{-1} : D_\delta(w_0) \to U$. By the open mapping theorem this inverse function is continuous. In fact since in the formula for f^{-1}, the integrand depends holomorphically on w, an argument similar to the proof of the CIF for derivative, shows that f^{-1} is holomorphic. By the chain rule then $(f^{-1})'(w) = 1/f'(z_w)$.

□

Remark 19.1.6. Another proof can be obtained by using the inverse function theorem from multivariable calculus. Recall that if $J_f(z_0)$ is the Jacobian (determinant) of f when thought of as a map from subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, then $J_f(z_0) = |f'(z_0)|^2 \neq 0$. Hence from the inverse function theorem in calculus, there exists a local inverse f^{-1} on an open neighbourhood V of w_0 with continuous partial derivatives. Possibly by shrinking V one can assume that that $f'(z) \neq 0$ on U. All one needs to do is to show that $f^{-1} : V \to U$ is holomorphic. It is enough to prove that f^{-1} satisfies CR equations. By chain rule,

$$0 = \frac{\partial}{\partial \bar{w}} f \circ f^{-1} = \frac{\partial f}{\partial z} \frac{\partial f^{-1}}{\partial \bar{w}} + \frac{\partial f}{\partial \bar{z}} \frac{\partial \overline{f^{-1}}}{\partial \bar{w}} = f'(z) \frac{\partial f^{-1}}{\partial \bar{w}},$$

since f is holomorphic. But then since $f'(z) \neq 0$, we see that $\partial f^{-1} / \partial \bar{w} = 0$ at each point.

An elementary but important consequence of the proof is the following.

Corollary 19.1.7. *A holomorphic function is locally injective on an open set U if and only if for all $z \in U$, $f'(z) \neq 0$.*

Proof. Suppose $f'(z)$ is never zero, then the inverse function theorem implies that the function is locally injective. Conversely, suppose the function is injective on some $D_r(z_0)$, but $f'(z_0) = 0$. Then by Theorem 19.1.1 there exists a $\delta > 0$, and $w \in D_\delta(f(z_0))$ such that $f(z) = w$ has at least two distinct solutions in $D_r(z_0)$ contradicting injectivity. ☐

Once again, the counterpart in real variable theory is false, as can be seen by considering the function $f(x) = x^3$. This is globablly (and hence locally) injective, but $f'(0) = 0$.

19.2 The maximum modulus principle

The next theorem says that for non-constant holomorphic functions $f(z)$, $|f|$ cannot have local maximums.

Theorem 19.2.1 (Max modulus principle). *Let Ω be connected and $f \in \mathcal{O}(\Omega)$. If there exists a $z_0 \in \Omega$ and a neighbourhood U such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$, then $f(z)$ is a constant.*

Proof. By assumption $|f(z_0)| = \sup_{z\in U} |f(z)|$. If $f(z)$ is non-constant on U, then by the open mapping theorem $f(U)$ is an open set. In particular, there exists a $\delta > 0$ such that for any $w \in D_\delta(f(z_0))$, there exists $z \in U$ such that $f(z) = w$. Now simply pick w_1 such that $|w_1| > |f(z_0)|$. Then there exists a $z_1 \in U$ such that $|f(z_0)| < |f(z_1)|$ which is a contradiction. Hence $f(z)$ must be a constant on U. But then by analytic continuation, $f(z)$ must be a constant on all of Ω. ☐

As a consequence we have the following estimate.

Corollary 19.2.2. *Let Ω be a bounded set and $f \in \mathcal{O}(\Omega)$ such that f extends continuously to the boundary $\partial\Omega$. Then*

$$\sup_{z\in\Omega} |f(z)| \leq \sup_{z\in\partial\Omega} |f(z)|.$$

Proof. It is enough to assume that Ω is connected (or else one could work with each connected component). Since $\overline{\Omega}$ is compact, there exists a $z_0 \in \overline{\Omega}$ such that $|f(z_0)| = \sup_{z\in\overline{\Omega}} |f(z)|$. If $z_0 \in \partial\Omega$, there is nothing to prove. If not, then by the above theorem, z_0 is an interior local maximum for $|f|$ and hence $f(z)$ must be a constant. But in that case the above inequality is trivial. ☐

Note that a *minimum principle* does not hold, as can be seen easily by considering the function $f(z) = z$ on any neighbourhood of the origin. It turns out that this is only way in which a minimum principle can fail. Recall that $\mathcal{O}^*(\Omega)$ stands for holomorphic functions that are nowhere vanishing on Ω.

Corollary 19.2.3 (Minimum principle). *Let Ω be connected and $f \in \mathcal{O}^*(\Omega)$. If there exists a $z_0 \in \Omega$ and a neighbourhood U such that $|f(z)| \geq |f(z_0)|$ for all $z \in U$, then $f(z)$ is a constant.*

Proof. Simply apply the maximum modulus principle to the holomorphic function $g(z) = 1/f(z)$. ☐

Remark 19.2.4. A function u is said to be *subharmonic* if $\Delta u \geq 0$ and *superharmonic* if $\Delta u \leq 0$. It is a general fact that subharmonic functions satisfy a maximum principle while super harmonic functions satisfy a minimum principle. In particular, harmonic functions satisfy both a minimum and a maximum principle. If $f(z)$ is holomorphic, we can compute that

$$\Delta|f|^2 = \frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}}|f(z)|^2 = \frac{1}{4}\frac{\partial}{\partial z}f(z)\overline{f'(z)} = |f'(z)|^2 \geq 0.$$

Hence $|f(z)|^2$ is subharmonic, and must satisfy a maximum principle. Hence $|f(z)|$ satisfies a maximum principle. On the other hand if $|f(z)|$ is nowhere vanishing, then $\log|f(z)|^2$ is smooth function, and in fact is harmonic as can be seen from the following computation

$$\Delta \log|f|^2 = \frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}}\log|f(z)|^2 = \frac{1}{4}\frac{\partial^2}{\partial z\partial\bar{z}}(\log f(z) + \log\overline{f(z)}) = 0.$$

Note that since $f(z)$ is no-where vanishing at least locally near z one can define a holomorphic branch of $\log$. The upshot is that $\log|f(z)|^2$ must satisfy a minimum principle, and hence must $|f(z)|$ (since $\log$ is increasing).

19.3 Rouche's theorem

Theorem 19.3.1. *Let γ be a simple closed curve in Ω and $f, g \in \mathcal{O}(\Omega)$ such that for all $z \in Supp(\gamma)$,*

$$|f(z) - g(z)| < |g(z)|.$$

Then $f(z)$ and $g(z)$ have the same number of zeroes in $Int(\gamma)$.

Proof. Firstly, note that $f(z)$ and $g(z)$ have no zero on γ (the strictness of the inequality above is crucial precisely for this purpose). Moreover, for all $z \in Supp(\gamma)$, we have

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1.$$

Put $F(z) = f(z)/g(z)$. Then $F(z) \in \mathcal{M}(\Omega)$. Moreover, at the points where $f(z)$ and $g(z)$ are non-zero (in particular on $Supp(\gamma)$), one can easily see that

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}.$$

A quick way to see this is that in the nieghbourhood of such points, $\log F(z)$ is well defined and holomorphic, and moreover, $\log F(z) = \log f(z) - \log g(z)$. Now consider $\Gamma := F \circ \gamma$, then Γ is a close curve in $D_1(1)$. Since $D_1(1)$ is simply connected, and $0 \notin D_1(1)$, $n(\Gamma, 0) = 0$. By the argument principle,

$$0 = n(\Gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

Once again by argument principle, we see that $f(z)$ and $g(z)$ must have the same number of zeroes in $Int(\gamma)$. □

Typically, as can be seen in the example below, the theorem is applied to count the number of zeroes of $f(z)$. The heart of the matter is to come up with a suitable $g(z)$, whose zeroes can be counted easily, and such that the above (strict) inequality can be satisfied.

Example 19.3.2. *Consider the polynomial $p(z) = z^4 - 6z + 3$. We claim that all it's roots are contained in the disc $D_2(0)$, and three of them are contained in the annulus $A_{1,2}(0)$. We divide the proof into the following two cases.*

- **The disc $|z| < 2$.** *On the circle $|z| = 2$ we have the following estimate*

$$\begin{aligned}|p(z) - z^4| &= |6z - 3| \\ &\le 6|z| + 3 = 12 + 3 = 15 < |z|^4.\end{aligned}$$

  By Rouche's theorem, $p(z)$ has the same number of roots as z^4 in $|z| < 2$, and hence has four roots in that disc. But $p(z)$ is also a polynomial of degree four, and hence these four roots must include all the possible roots of $p(z)$.

- **The disc** $|z| < 1+\varepsilon$ **for** $\varepsilon << 1$. *Lets take* $\varepsilon < 1/10$. *On the circle* $|z| = 1+\varepsilon$ *we have the following estimate*

$$\begin{aligned}|p(z) - (-6z)| &= |z^4 + 3| \\ &\leq |z|^4 + 3 = (1+\varepsilon)^4 + 3 < 4.5 < 6(1+\varepsilon) = |-6z|.\end{aligned}$$

Once again by Rouche's theorem, $p(z)$ *has exactly one root in* $|z| < 1+\varepsilon$, *and hence has exactly three roots in* $1+\varepsilon \leq |z| < 2$. *Since this is true for all* $\varepsilon << 1$, *in particular, it has exactly three roots in* $A_{1,2}(0)$.

Remark 19.3.3. Rouche's theorem can be used to give another proof of the fundamental theorem of algebra. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a general degree n polynomial (so $a_n \neq 0$). It is easy to see that for $R >> 1$, if $|z| = R$, then

$$|p(z) - a_n z^n| < |a_n||z^n|.$$

This is essentially because $p(z) - a_n z^n$ is a polynomial of a strictly lower degree. Now by Rouche's theorem $p(z)$ and $a_n z^n$ have the same number of roots on $|z| < R$. In particular, $p(z)$ must have exactly n roots on $|z| < R$. In fact it can be shown easily (by induction for instance) that it cannot have any further zeroes.

19.4 Appendix : details left out in the proof of Corollary 19.1.5

To spell out the details on the holomorphicity of f^{-1} and that the derivative is $1/f'(z_w)$, we first note that

$$\begin{aligned}\frac{f^{-1}(w+h) - f^{-1}(w)}{h} &= \frac{1}{2\pi i h}\int_{|z-z_0|=r} zf'(z)\Big(\frac{1}{f(z)-w-h} - \frac{1}{f(z)-w}\Big)\,dz \\ &= \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{zf'(z)}{(f(z)-w-h)(f(z)-w)}\,dz\end{aligned}$$

Now the integrand is continuous and bounded for $|h| << 1$, and hence we can take compute the limit by swapping the integral and the limit. That is,

$$\lim_{h\to 0}\frac{f^{-1}(w+h) - f^{-1}(w)}{h} = \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{zf'(z)}{(f(z)-w)^2}\,dz.$$

Another application of the residue theorem shows that the second integral is precisely $1/f'(z_w)$. To see this, we observe that

$$\frac{zf'(z)}{(f(z)-w)^2} = \frac{(z-z_w)f'(z)}{(f(z)-w)^2} + z_w\frac{f'(z)}{(f(z)-w)^2}.$$

From the geometric series expansion, one can see that the second term is of the form

$$z_w\frac{f'(z)}{(f(z)-w)^2} = \frac{z_w}{f'(z_w)^2}(z-z_w)^{-2} + g(z),$$

where $g(z)$ is holomorphic near z_w. This relies on the fact that $f'(z_w) \neq 0$, and hence the numerator has a non-zero constant term in it's Taylor expansion. The upshot is that the second term does not contribute to the residue. The advantage now is that the first term has a simple pole at $z = z_w$, and hence we can compute the residue as

$$\mathrm{Res}_{z=z_w}\frac{zf'(z)}{(f(z)-w)^2} = \lim_{z\to z_w}\frac{(z-z_w)^2 f'(z)}{(f(z)-w)^2} = \frac{1}{f'(z_w)}.$$

Lecture 20

Contour Integration - I

In the next two lectures, we'll systematically study applications of the residue theorem to the computation of real variables integrals. We will follow the exposition in Ahlfor's book [1] very closely. For instance the classification of the integrals into Types is taken directly from [1].

20.1 Type-I: Integrals of rational functions

In this section we study integrals of the form

$$\int_{-\infty}^{\infty} R(x)\, dx,$$

where $R(x)$ is a rational function.

Assumption. $R(x) = P(x)/Q(x)$ with $\deg(Q) \geq \deg(P) + 2$ and $Q(x)$ has no real root. Recall that by definition,

$$\int_{-\infty}^{\infty} R(x)\, dx = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} R(x)\, dx.$$

By the hypothesis on degree and no real root of $Q(x)$, the integral is absolutely convergent and it follows that in fact,

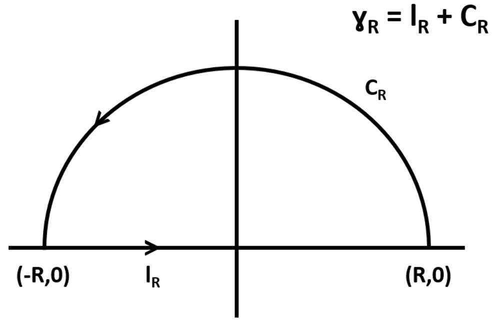
$$\int_{-\infty}^{\infty} R(x)\, dx = \lim_{R \to \infty} \int_{-R}^{R} R(x)\, dx.$$

The method. We consider the contour γ_R consisting of a semi-circle of radius R centred at the origin and traversed in the anti-clockwise direction. We can decompose $\gamma_R = l_R + C_R$, where l_R is the interval $(-R, R)$ and C_R is the semi-circular part, and hence

$$\int_{-R}^{R} R(x)\, dx = \int_{\gamma_R} R(z)\, dz - \int_{C_R} R(z)\, dz.$$

If $R >> 1$, then all roots of $Q(x)$ in the upper half plane lie in the interior of γ_R. By the residue theorem,

$$\int_{\gamma_R} R(z)\, dz = 2\pi i \sum_{\substack{Q(\alpha)=0 \\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z).$$



On the other hand, since $\deg Q \geq \deg P + 2$, when $R >> 1$,

$$|R(z)| \leq \frac{M}{R^2}$$

for some $M > 0$. Hence,

$$\Big| \int_{C_R} R(z)\, dz \Big| \leq \frac{2\pi M}{R} \xrightarrow{R \to \infty} 0.$$

Putting it all together we have

$$\int_{-\infty}^{\infty} R(x)\, dx = 2\pi i \sum_{\substack{Q(\alpha)=0 \\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z).$$

Example 20.1.1. *Consider the integral*

$$\int_0^{\infty} \frac{x^2}{1+x^6}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\, dx = \pi i \sum_{\substack{1+\alpha^6=0 \\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z),$$

where $R(z) = z^2/(1+z^6)$. The roots of the denominator are given by $\alpha_k = e^{\pi i/6 + 2\pi i k/6}$ for $k = 0, 1, \cdots, 5$. Of these only $\alpha_0 = e^{\pi i/6}, \alpha_1 = e^{\pi i/2} = i$ and $\alpha_3 = e^{5\pi i/6}$ are in the upper half region. Each of these is a simple pole, and hence we can compute that

$$\mathrm{Res}_{z=\alpha_k} R(z) = \lim_{z \to \alpha_k} (z - \alpha_k) \frac{z^2}{1+z^6} = \alpha_k^2 \lim_{z \to \alpha_k} \frac{z - \alpha_k}{1+z^6} = \frac{1}{6\alpha_k^3} = \begin{cases} -i/6, & k = 0 \\ i/6, & k = 1 \\ -i/6, & k = 2. \end{cases}$$

Finally we get that

$$\int_0^{\infty} \frac{x^2}{1+x^6}\, dx = \pi i \Big(-\frac{i}{6} \Big) = \frac{\pi}{6}.$$

20.1.1 A variation

One also often encounters integrals (as in the above example) of the form

$$\int_0^{\infty} R(x)\, dx.$$

If $R(x)$ is an even function as in the example above, then one can simply convert it into an integral on all of $\mathbb{R}$ at the cost of an additional factor. But this trick will not work in general, so one might have to get more creative.

Example 20.1.2. *Consider the integral*

$$I = \int_0^\infty \frac{x}{1+x^4}\,dx = \lim_{R\to\infty}\int_0^R \frac{x}{1+x^4}\,dx.$$

As usual, we let I_R denote the integral on the right (inside the limit), and we let $f(z) = z/(1+z^4)$. We take the contour $\gamma_R = \gamma_{1,R} + C_R + \gamma_{2,R}$, where $\gamma_{1,R}$ consists of the straight line from (0.0) to $(R,0)$, C_R is the quadrant of the circle from $(R,0)$ to $(0,R)$, and $\gamma_{2,R}$ is the straight line from $(0,R)$ to $(0,0)$. By the residue theorem, since the only pole of $f(z)$ in the interior of γ_R is a simple pole at $z = e^{i\pi/4}$, we have

$$\begin{aligned}\int_{\gamma_R} f(z)\,dz &= 2\pi i \mathrm{Res}_{z=e^{i\pi/4}} f(z)\\ &= 2\pi i \lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})z}{1+z^4}\\ &= \frac{2\pi i e^{i\pi/4}}{4e^{3i\pi/4}} = \frac{\pi}{2}.\end{aligned}$$

Next, we observe that

$$\int_{\gamma_{1,R}} f(z)\,dz = I_R,$$

and by the discussion above,

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

To compute the integral over $\gamma_{2,R}$ we parametrize $\gamma_{2,R}(t) = -it$, where $t \in (-R,0)$. Then $\gamma'_{2,R}(t) = -i$, and so

$$\int_{\gamma_{2,R}} f(z)\,dz = \int_{-R}^0 \frac{-it}{1+t^4}(-idt) = I_R.$$

Hence if $R > 1$, by the residue computation above,

$$2I_R + C_R = \frac{\pi}{2}.$$

Taking the limit as $R \to \infty$ we see that

$$I = \frac{\pi}{4}.$$

Remark 20.1.3. This example was merely for illustration, since the integral can of course be computed in a more elementary way by a change of variables $x^2 = u$. The reader can challenge himself/herself with the following:

$$\int_0^\infty \frac{x^3}{1+x^5}.$$

20.2 Type-II: Rational functions of sine and cosine

The type 2 integrals are of the following kind:

$$\int_0^{2\pi} R(\cos\theta, sin\theta)\,d\theta,$$

where $R(x)$ is again a rational function.

Assumption. No pole for $\theta \in [0, 2\pi)$.

The method. Consider $z = z(\theta) = e^{i\theta}$. Then

$$\begin{cases} \cos\theta = \frac{z+\bar{z}}{2} = \frac{z+1/z}{2} \\ \sin\theta = \frac{z-\bar{z}}{2i} = \frac{z-1/z}{2i}. \end{cases}$$

Moreover, $dz = z'(\theta)\theta = ie^{i\theta}\theta$, and hence

$$d\theta = -i\frac{dz}{z}.$$

One can then transform the integral to

$$\int_0^{2\pi} R(\cos\theta, sin\theta)\, d\theta = -i\int_{|z|=1} R\Big(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\Big)\frac{dz}{z}.$$

Example 20.2.1. *Consider the integral*

$$I := \int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta},\ a > 1.$$

Using the above transformations we can re-write the second integral as

$$\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = -i\int_{|z|=1}\frac{1}{a+\frac{z+1/z}{2}}\frac{dz}{z} = -2i\int_{|z|=1}\frac{dz}{z^2+1+2az},$$

and so our required integral is

$$I = -i\int_{|z|=1}\frac{dz}{z^2+2az+1}.$$

Now the denominator has two real roots, $\alpha = -a+\sqrt{a^2-1}$ *and* $\beta = -a-\sqrt{a^2-1}$. *Since* $a > 1$, *it is easy to see that* $|\alpha| < 1$ *while* $|\beta| > 1$. *So*

$$I = (-i)(2\pi i)\mathrm{Res}_{z=\alpha}\frac{1}{z^2+1+2az} = 2\pi\lim_{z\to\alpha}\frac{z-\alpha}{z^2+2az+1} = \frac{2\pi}{\alpha-\beta} = \frac{\pi}{\sqrt{a^2-1}}.$$

Type-III: Products of rational functions and triganometric functions

These are integrals of the kind

$$\int_{-\infty}^{\infty} R(x)\cos x\, dx,\ \int_{-\infty}^{\infty} R(x)\sin x,\ dx.$$

These can be combined into the analysis of one single integral

$$\int_{-\infty}^{\infty} R(x)e^{ix}\, dx.$$

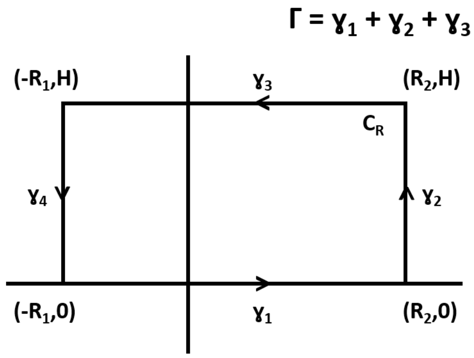
Assumptions. These come in two sub-types where the analysis substantially differs. As before, we let $R(x) = P(x)/Q(x)$.

- Type-III(a). We assume that $\deg Q \geq \deg P + 2$, and no root of Q is real.
- Type-III(b). Here we assume that $\deg Q = \deg P + 1$, and no root of Q is real.

The method. For Type-III(a), one proceeds exactly as in Type-I, and we do not spend additional time on this. For Type-III(b), it is not clear at all if the integral converges. It certainly will not absolutely converge, and hence we cannot use the semi-circle contour. But in many cases, the oscillating factor e^{ix} might make the integral converge conditionally. By definition, if the integral converges, then

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx = \lim_{R_1,R_2\to\infty}\int_{-R_1}^{R_2} R(x)e^{ix}\,dx.$$

Now let Γ be the rectangle with vertices $(-R_1,0)$, $(R_2,0)$, (R_2,H) and $(-R_1,H)$. with sides given by straight-line curves $\gamma_1,\gamma_2,\gamma_3$ and γ_4 (see figure below). If R_1, R_2 and H are big enough, the rectangle will



contain all the roots of Q with positive imaginary part. Hence

$$\sum_{i=1}^{4}\int_{\gamma_i} R(z)\,dz = 2\pi i \sum_{\substack{Q(\alpha)=0\\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z)e^{iz}.$$

The idea is to first fix R_1, R_2 and let $h\to\infty$. And then let $R_1, R_2\to\infty$. We illustrate with an example.

Example 20.2.2. *Consider the integral*

$$I = \int_{-\infty}^{\infty}\frac{x\sin x}{1+x^2} = \frac{1}{i}\int_{-\infty}^{\infty}\frac{xe^{ix}}{1+x^2}\,dx,$$

since $x\cos x/(1+x^2)$ is an odd function and hence the integral will be zero. We let $f(z)=ze^{iz}/(1+z^2)$. If $H>1$, then the rectangle will contain the only pole of $f(z)$ with positive imaginary part, namely $z=i$. Hence

$$\int_{\Gamma} f(z)\,dz = 2\pi i\mathrm{Res}_{z=i} f(z) = 2\pi i\lim_{z\to i}(z-i)\frac{ze^{iz}}{1+z^2} = 2\pi i\Big(\frac{ie^{i^2}}{i+i}\Big) = \frac{\pi i}{e}.$$

Now let us analyze each smooth curve in Γ.

- **The curve** γ_1. *This is the integral we are interested in*

$$\int_{\gamma_1} f(z)\,dz = \int_{-R_1}^{R_2}\frac{xe^{ix}}{1+x^2}\,dx.$$

- **The curve** γ_2. *On γ_2, $|z|>R_2$, and hence if $R_2>>1$, then*

$$\Big|\frac{z}{1+z^2}\Big| \le \frac{2}{R_2}.$$

Then parametrizing the curve by $\gamma_2(t) = R_2 + it$, and noting that $|e^{iz}| = e^{-t}$, we see that

$$\Big|\int_{\gamma_2} f(z)\,dz\Big| \le \frac{2}{R_2}\int_0^H e^{-t}dt \le \frac{2}{R_2}.$$

- **The curve γ_3.** *Again if $H >> 1$, we will have the estimate*

$$\Big|\frac{z}{1+z^2}\Big| \le \frac{2}{H}.$$

Also $|e^{iz}| = e^{-H}$ on γ_3. Hence $|f(z)| \le 2e^{-H}/H$, and we have

$$\Big|\int_{\gamma_2} f(z)\,dz\Big| \le \frac{2e^{-H}(R_2+R_1)}{H}.$$

- **The curve γ_4.** *As for the second curve, we once again have the bound*

$$\Big|\int_{\gamma_4} f(z)\,dz\Big| \le \frac{2}{R_1}.$$

Together, if we first let $H \to \infty$, and then let $R_1, R_2 \to \infty$ the integrals on the last three curves converge to 0, and hence

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2}\,dx = \int_\Gamma f(z)\,dz = \frac{\pi i}{e}.$$

Our original integral is then

$$\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2} = \frac{\pi}{e}.$$

Lecture 21

Contour Integration - II

This is a continuation of our previous lecture on computing real variable integrals using the residue theorem. In the present lecture, we'll focus on integrations involving branch cuts. But first we refine a method introduced in the previous lecture.

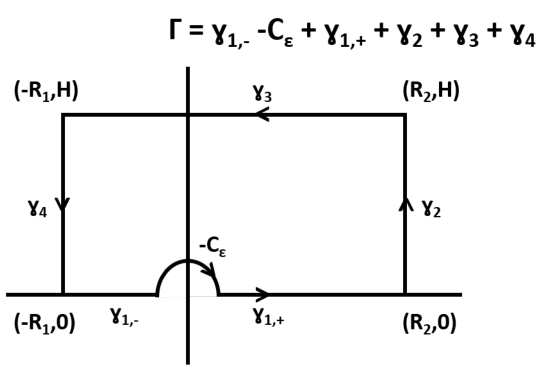
21.1 Type-III(b): Principal values

Now suppose that $Q(x)$ has a simple zero on the real line. In this case even though the integral of $R(x)e^{ix}$ will not converge, the integrals of $R(x)\sin x$ or $R(x)\cos x$ might converge if the simple zero happens to coincide with a zero of $\sin x$ and $\cos x$ respectively. To illustrate the work-around in this case, suppose we wish to evaluate the integral

$$\int_{-\infty}^{\infty} R(x)\sin x.$$

Assumption. $\deg Q(x) = \deg P(x) + 1$ and $Q(x)$ has a simple zero at $x = 0$.

The method. Consider the contour Γ in the figure below. Using the analysis from the final section of the



previous lecture, we can see that

$$\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\,dx + \int_{\varepsilon}^{\infty} R(x)e^{ix}\,dx - \int_{C_\varepsilon} R(z)e^{iz}\,dz = 2\pi i \sum_{\substack{Q(\alpha)=0 \\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z)e^{iz}.$$

To compute the third integral, note that near $z = 0$,

$$R(z)e^{iz} = \frac{A}{z} + R_0(z),$$

where $R_0(z)$ is holomorphic near $z = 0$ (and hence bounded) and $A = \mathrm{Res}_{z=0} R(z)e^{iz}$. Then by an explicit calculation it is easy to see that

$$\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon} R(z)e^{iz}\, dz = \pi i A,$$

and hence

$$\lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\, dx + \int_{\varepsilon}^{\infty} R(x)e^{ix}\, dx \right) = 2\pi i \Big(\frac{1}{2} \mathrm{Res}_{z=0} R(z)e^{iz} + \sum_{\substack{Q(\alpha)=0 \\ Im(\alpha)>0}} \mathrm{Res}_{z=\alpha} R(z)e^{iz} \Big). \tag{21.1}$$

The <u>*principal value*</u> of the integral of $R(x)e^{ix}$ on $\mathbb{R}$ is defined to be

$$\mathrm{p.v.}\Big(\int_{-\infty}^{\infty} R(x)e^{ix}\, dx \Big) := \lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} R(x)e^{ix}\, dx + \int_{\varepsilon}^{\infty} R(x)e^{ix}\, dx \right),$$

<u>if the limit exists</u>. Note that if $Q(x)$ has a simple pole at $x = 0$, then the above limit will exist, as the analysis above shows. Moreover, under these assumptions, clearly $R(z)\sin z$ is integrable near zero and infinity, and so

$$\int_{-\infty}^{\infty} R(x)\sin x\, dx = Im\Big(\mathrm{p.v.}\Big(\int_{-\infty}^{\infty} R(x)e^{ix}\, dx \Big) \Big),$$

and the latter principal value can be computed by the analysis above. Let us illustrate this via a famous integral.

Example 21.1.1. *Consider the integral of*

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

Then by the above analysis,

$$I = Im\Big(\pi i \mathrm{Res}_{z=0} \frac{e^{iz}}{z} \Big),$$

and so

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Remark 21.1.2. This is the standard example of a function whose improper integral is finite, and yet the Lebesgue integral does not converge. I cannot resist the temptation to include another method of computing the integral, using the so called "Feynman technique". This is essentially just differentiation under the integral sign, but was popularised by Feynman as his way of handling integrals that others needed residue calculus for! Let I be the integral above. We introduce a parameter '$a > 0$' and consider the the family of integrals

$$I(a) = \int_0^{\infty} \frac{\sin(x)e^{-ax}}{x}\, dx.$$

One can check that $I(a)$ is a differentiable function of a, and that differentiation under the integral works. Then,

$$I'(a) = -\int_0^{\infty} e^{-ax} \sin x\, dx.$$

We can compute the integral on the right, by applying integration by parts twice. Indeed

$$\begin{aligned}
J(a) := \int_0^\infty e^{-ax} \sin x \, dx &= -\frac{1}{a} \int_0^\infty \sin x \, de^{-ax} \\
&= \frac{1}{a} \int_0^\infty e^{-ax} d \sin x \\
&= \frac{1}{a} \int_0^\infty e^{-ax} \cos x \, dx \\
&= -\frac{1}{a^2} \int_0^\infty \cos x \, de^{-ax} \\
&= \frac{1}{a^2} + \int_0^\infty e^{-ax} \, d \cos x \\
&= \frac{1}{a^2} - \frac{J(a)}{a^2}.
\end{aligned}$$

Solving for $J(a)$ we get that

$$I'(a) = -\frac{1}{1+a^2},$$

and hence

$$I(a) = -\arctan(a) + C.$$

Since $\lim_{a\to\infty} I(a) = 0$, clearly $C = \pi/2$. But then taking $a \to 0^+$ we see that

$$I = \lim_{a\to 0^+} I(a) = C = \frac{\pi}{2}.$$

21.2 Type-IV: Products of rational functions and powers of x.

In this section we study integrals of the form

$$\int_{-\infty}^\infty x^\alpha R(x) \, dx,$$

where $R(x) = P(x)/Q(x)$ is a rational function and $\alpha \in (0, 1)$.

Assumption. $\deg Q \geq \deg P + 2$, and $Q(x)$ has a simple zero at $x = 0$ and no other real zero.

The method. Note that the assumption implies that the integral is absolutely convergent, and so

$$\int_{-\infty}^\infty x^\alpha R(x) \, dx = \lim_{R\to\infty} \int_{-R}^R x^\alpha R(x) \, dx.$$

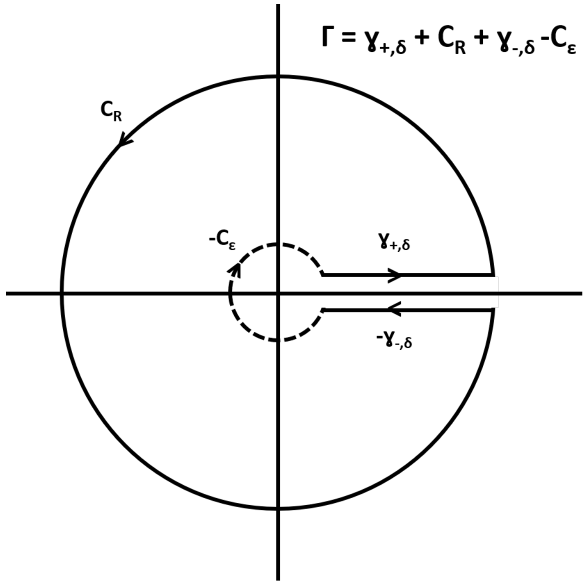
Consider the contour in the figure below. We integrate the function $z^\alpha R(z)$ on the contour. Since we are dealing with fractional powers, we have to make a choice of a branch cut and a a corresponding branch of the power. Given the geometry of the contour, it is clear that we have to use the branch cut $(0, \infty)$. Recall that $z^\alpha = e^{\alpha \log z}$, where $\log z = \log |z| + i \arg(z)$, and $\arg(z) \in (0, 2\pi)$.

We first have the following observations:

Lemma 21.2.1. *With the orientations as in the figure,*

$$\lim_{\delta\to 0} \int_{\gamma_{+,\delta}} z^\alpha R(z) \, dz = \int_\varepsilon^R x^\alpha R(x) \, dx,$$

$$\lim_{\delta\to 0} \int_{\gamma_{-,\delta}} z^\alpha R(z) \, dz = e^{2\pi i\alpha} \int_\varepsilon^R x^\alpha R(x) \, dx.$$



The proof relies on the fact that continuous functions on compact sets are uniformly continuous. For a fixed ε and R, and for small $\delta > 0$, the function $z^\alpha R(z)$ is continuous, and hence uniformly continuous, on the rectangle with vertices $(-\delta, \varepsilon)$, $(-\delta, R)$, (δ, R) and (δ, ε). The $e^{2\pi i \alpha}$ factor in the second integral is due to the fact that there is a jump of $e^{2\pi i \alpha}$ in the value of z^α across the branch cut $z > 0$. We leave the details to the reader.

By the residue theorem,

$$\int_{\gamma_{+,\delta}} z^\alpha R(z)\,dz + \int_{C_R} z^\alpha R(z)\,dz - \int_{\gamma_{-,\delta}} z^\alpha R(z)\,dz - \int_{C_\varepsilon} z^\alpha R(z)\,dz = 2\pi i \sum_{\substack{Q(\beta)=0\\ \beta\neq 0}} \mathrm{Res}_{z=\beta} z^\alpha R(z).$$

Letting $\delta \to 0$, by the Lemma above,

$$(1 - e^{2\pi i \alpha}) \int_\varepsilon^R x^\alpha R(x)\,dx = 2\pi i \sum_{\substack{Q(\beta)=0\\ \beta\neq 0}} \mathrm{Res}_{z=\beta} z^\alpha R(z) - \int_{C_R} z^\alpha R(z)\,dz + \int_{C_\varepsilon} z^\alpha R(z)\,dz.$$

Finally, letting $\varepsilon \to 0^+$ and $R \to \infty$ one proves that the integrals on the right converge to zero under the given assumptions. We illustrate with an example.

Example 21.2.2. *Consider the integral*

$$I := \int_0^\infty \frac{x^{-a}}{1+x}\,dx,\ a \in (0,1).$$

To get it into the above form, we re-write this as

$$I = \int_0^\infty \frac{x^{1-a}}{x(1+x)}\,dx,$$

and let $I_{\varepsilon,R}$ be the corresponding integral from ε to R. Let $f(z) = z^{1-a}/(z(1+z))$. Recall that we are using the branch $z^{1-a} = e^{(1-a)\log z}$, where $\log(re^{i\theta}) = \log r + i\theta$ and $\theta \in (0, 2\pi)$. Now $f(z)$ has simple poles at $z = 0$ and $z = -1$. By the above discussion,

$$(1 - e^{2\pi i(1-a)}) I_{\varepsilon,R} = 2\pi i \mathrm{Res}_{z=-1} \frac{z^{-a}}{1+z} - \int_{C_R} \frac{z^{1-a}}{z(1+z)}\,dz + \int_{C_\varepsilon} \frac{z^{1-a}}{z(1+z)}\,dz \tag{21.2}$$

We first estimate the two remaining integrals on the right.

- **The integral on C_R.** *When $|z| = R >> 1$, we have $|1+z| \geq |z| - 1 > R/2$, and so*

$$\left|\frac{z^{-a}}{1+z}\right| = \frac{R^{-a}}{|1+z|} \leq \frac{2R^{-a}}{R} = \frac{2}{R^{1+a}}.$$

  The integral then satisfies

$$\left|\int_{C_R} \frac{z^{1-a}}{z(1+z)}\,dz\right| \leq \frac{4\pi R}{R^{1+a}} = \frac{4\pi}{R^a} \xrightarrow{R\to\infty} 0.$$

- **The integral on C_ε.** *When $|z| = \varepsilon << 1$, we have $|1+z| \geq 1 - |z| > 1/2$, and so*

$$\left|\frac{z^{-a}}{1+z}\right| = \frac{\varepsilon^{-a}}{|1+z|} \leq 2\varepsilon^{-a}.$$

  The integral then satisfies

$$\left|\int_{C_\varepsilon} \frac{z^{1-a}}{z(1+z)}\,dz\right| \leq 4\pi\varepsilon^{1-a} \xrightarrow{\varepsilon\to 0^+} 0.$$

Finally we compute the residue at $z = -1$,

$$\mathrm{Res}_{z=-1} f(z) = \lim_{z\to -1} z^{-a} = e^{-a\log(-1)} = e^{-i\pi a}.$$

Putting all of this together, taking $\varepsilon \to 0^+$ and $R \to \infty$ in (21.2) *we see that*

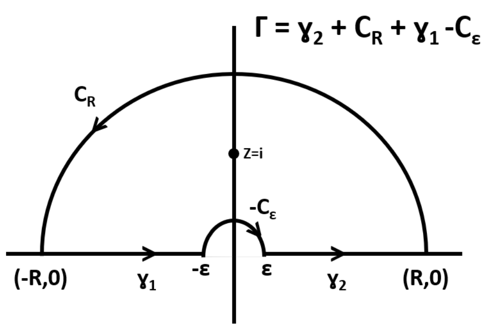
$$I = \frac{2\pi i e^{-i\pi a}}{(1 - e^{2\pi i(1-a)})} = \frac{2\pi i e^{-i\pi a}}{(1 - e^{-2\pi i a})} = \frac{2\pi i}{(e^{\pi i a} - e^{-\pi i a})} = \frac{\pi}{\sin \pi a}.$$

21.3 A bonus integral

As a final integral, let us compute

$$I := \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx.$$

We denote the corresponding integral over (ε, R) by $I_{\varepsilon,R}$. As an exercise, the reader should attempt to use the contour above too evaluate this integral. We will instead use a semicircular contour Γ in the figure below. We use the branch cut $\{iy \mid y \in (-\infty, 0)\}$, and the branch of the logarithm defined by



$\log(re^{i\theta}) = \log r + i\theta$, with $\theta \in (-\pi/2, 3\pi/2)$. Let

$$f(z) = \frac{(\log z)^2}{1+z^2}.$$

Then clearly,

$$\int_{\gamma_1} \frac{(\log z)^2}{1+z^2} = \int_{\varepsilon}^{R} \frac{(\log x)^2}{1+x^2}\,dx = I_{\varepsilon,R},$$
$$\int_{\gamma_2} \frac{(\log z)^2}{1+z^2} = \int_{-R}^{-\varepsilon} \frac{(\log|t|+i\pi)^2}{1+t^2}\,dt.$$

The limits follow from the fact that the integral is absolutely convergent. For the second integral above we parametrize $\gamma_2(x) = t = |t|e^{i\pi}$ with $t \in (-R,-\varepsilon)$. Putting $x = -t$ int he second integral, we see that

$$\begin{aligned}\int_{\gamma_2} \frac{(\log z)^2}{1+z^2} &= \int_{-R}^{-\varepsilon} \frac{(\log|t|+i\pi)^2}{1+t^2}\,dx \\ &= \int_{\varepsilon}^{R} \frac{(\log x+i\pi)^2}{1+x^2}\,dx \\ &= I_{\varepsilon,R} + 2\pi i \int_{\varepsilon}^{R} \frac{\log x}{1+x^2}\,dx - \pi^2 \int_{\varepsilon}^{R} \frac{dx}{1+x^2}.\end{aligned}$$

Notice that the third integral is $\arctan(R/\varepsilon)$, and so letting $\varepsilon \to 0^+$ and $R \to \infty$,

$$\lim_{\substack{\varepsilon\to 0^+\\ R\to\infty}} \Big(\int_{\gamma_1} \frac{(\log z)^2}{1+z^2} + \int_{\gamma_1} \frac{(\log z)^2}{1+z^2} \Big) = 2I - \frac{\pi^3}{2} + 2\pi i \int_{\varepsilon}^{R} \frac{\log x}{1+x^2}\,dx.$$

By the exact analysis as the previous section, one can prove that

$$\lim_{R\to\infty} \int_{C_R} \frac{(\log z)^2}{1+z^2}\,dz = \lim_{\varepsilon\to 0^+} \int_{C_\varepsilon} \frac{(\log z)^2}{1+z^2}\,dz = 0,$$

and so by the residue theorem,

$$\begin{aligned}2I - \frac{\pi^3}{2} + 2\pi i \int_{\varepsilon}^{R} \frac{\log x}{1+x^2}\,dx &= 2\pi i \mathrm{Res}_{z=i} \frac{(\log z)^2}{1+z^2} \\ &= 2\pi i \lim_{z\to i} (z-i) \frac{(\log z)^2}{1+z^2} \\ &= 2\pi i \frac{(\log i)^2}{2i} \\ &= -\frac{\pi^3}{4}.\end{aligned} \tag{21.3}$$

Note that in the penultimate line, we used the fact that for our chosen branch of the logarithm, we have $\log i = i\pi/2$. Equating the real parts, and solving for I, we get that

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

Remark 21.3.1. Notice that since $2\pi i$ times the residue above was completely real, the imaginary part in equation (21.3) above has to be zero. This yields a curious integral identity

$$\int_0^\infty \frac{\log x}{1+x^2} = 0.$$

A simple change of variables yields an explanation. Namely, denoting the integral by J, if we let $t = 1/x$, we have

$$J = \int_\infty^0 \frac{-t^2\log t}{1+t^2} \Big(\frac{-dt}{t^2} \Big) = -J,$$

and hence $J = 0$.

Part III

Geometry of holomorphic maps

Lecture 22

Conformality

In this lecture, we will begin our study of some geometric properties of holomorphic maps.

22.1 Conformal maps

Loosely speaking, a conformal map is a map that preserves angles. We now try to make this concept more rigorous, and give a precise definition of conformal maps. Throughout this introductory part, we'll work with either $\mathbb{R}^2$ or $\mathbb{C}$, as is convenient, always remembering the natural identification. Let $f : \Omega \to \mathbb{R}^2$ be a C^1 map ie. the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ exist and are continuous. Given two curves $\gamma_1(t) = (x_1(t), y_1(t)) : (-\varepsilon, \varepsilon) \to \mathbb{C}$ and $\gamma_2(t) = (x_2(t), y_2(t)) : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = p$, we define the angle between them as

$$\measuredangle \gamma_1(0), \gamma_2(0)$$

Recall that if $\vec{v_1}$ and $\vec{v_2}$ are non-zero vectors in $\mathbb{R}^2$, then the angle between them is defined to be

$$\measuredangle \vec{v_1}, \vec{v_2} = \arccos\Big(\frac{\langle \vec{v_1}, \vec{v_2}\rangle}{|\vec{v_1}||\vec{v_2}|}\Big),$$

where $\langle \cdot, \cdot \rangle$ is the usual dot product on $\mathbb{R}^2$ and $|\cdot|$ is the usual norm (given by the square-root of the dot-product of the vector with itself). Motivate by this, we say that a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ *preserves angles* or is *conformal* if $\det(T) > 0$ [1], and for any pair of non-zero vectors $\vec{v}, \vec{w} \in \mathbb{R}^2 \setminus \{\vec{0}\}$,

$$\frac{\langle \vec{v}, \vec{w}\rangle}{|\vec{v}||\vec{w}|} = \frac{\langle T\vec{v}, T\vec{w}\rangle}{|T\vec{v}||T\vec{w}|}.$$

More generally, we sat that a C^1 mapping $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is *conformal at* $(x_0, y_0) \in \Omega$ if the total derivative $\mathbf{D}_{(x_0,y_0)} f : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal. We say that f is *conformal* if it is conformal at all points in Ω. Recall that in the standard basis the matrix representing $\mathbf{D}_{(x_0,y_0)} f$ is given by the Jacobian matrix

$$\mathbf{D}_{(x_0,y_0)} f = \begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix},$$

where (u, v) are the components of f ie. in complex notation $f = u + iv$.

[1]This condition is equivalent to the map being orientation preserving. Some authors choose to not impose this extra condition on conformal maps

A slightly more geometric insight is obtained by looking at curves. Consider a pair of curves $\gamma_1(t), \gamma_2(t) : (-\varepsilon, \varepsilon) \to \mathbb{C}$ intersecting at $\gamma_1(0) = \gamma_2(0) = z_0$. We say that they intersect at an angle θ at z_0 if the angle between the tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ is θ. The a mapping is conformal at z_0 if and only if for any two curves γ_1 and γ_2 as above, the angle between them is equal to the angle between their images $f(\gamma_1(t))$ and $f(\gamma_2(t))$ at $f(z_0)$. To see this, it is enough to note that if $\gamma(t) : (-\varepsilon, \varepsilon) \to \Omega$ is a curve with $\gamma(0) = z_0 = (x_0, y_0)$ and $\gamma'(0) = \vec{v}$, then

$$\mathbf{D}_{(x_0.y_0)} f(\vec{v}) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)).$$

We are now ready to prove our main observation.

Proposition 22.1.1. *Let $f : \Omega \to \mathbb{C}$ be a C^1 map. Then f is conformal at z_0 if and only if f is complex differentiable at z_0 and $f'(z_0) \neq 0$. In particular, f is conformal if and only if it is holomorphic with nowhere vanishing complex derivative.* [2].

We first need two elementary lemmas from linear algebra.

Lemma 22.1.2. *A linear map $C : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal if and only if $C = \lambda Q$, where Q is an orientation preserving orthogonal transformation and $\lambda > 0$.*

Recall that Q is orthogonal if and only if $|Q\vec{v}| = |\vec{v}|$ for all $\vec{v}$ or equivalently $Q^T Q = I$, where Q^T is the transpose of Q, and I is the identity matrix.

Proof. Let $\vec{e_1}$ and $\vec{e_2}$ denote the standard basis vectors in the x and y directions, and let $C^T C = (a_{ij})$ be the matrix representation of $C^T C$ in this basis. By conformality, Then

$$0 = \langle \vec{e_1}, \vec{e_2} \rangle \implies 0 = \langle C\vec{e_1}, C\vec{e_2} \rangle = \vec{e_1}^T (C^T C) \vec{e_2} = a_{12}.$$

By the symmetry of $C^T C$, we also have that $a_{21} = 0$, and hence $C^T C$ is diagonal. On the other hand,

$$\begin{aligned} 0 = \langle \vec{e_1} - \vec{e_2}, \vec{e_1} + \vec{e_2} \rangle \implies 0 &= \langle C(\vec{e_1} - \vec{e_2}), C(\vec{e_1} + \vec{e_2}) \rangle \\ &= \langle C\vec{e_1}, C\vec{e_1} \rangle - \langle C\vec{e_2}, C\vec{e_2} \rangle \\ &= a_{11} - a_{22}. \end{aligned}$$

That is, the diagonal terms in $C^T C$ are equal, and hence $C^T C = \mu I$ for some $\mu \in \mathbb{R}$. Since $\det C > 0$ and the diagonal terms of $C^T C$ have to be non-negative, we have that $\mu > 0$ Now, let $Q = \mu^{-1/2} C$. Then

$$C^T C = \mu I \implies Q^T Q = I,$$

and this proves the claim with $\lambda = \sqrt{\mu}$. The converse, that a matrix $C = \lambda Q$ is conformal if Q is orthogonal is trivial. □

Lemma 22.1.3. *Any orientation preserving orthogonal 2×2 matrix Q is a rotation matrix, that is, it is given by*

$$Q = R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for some $\theta \in [0, 2\pi)$.

Proof. Since $Q^T Q = I$ and $\det Q > 0$, we have that $\det Q = 1$. Suppose

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

[2] If we drop orientation preserving from the definition of conformality, then a conformal map is either a holomorphic or anti-holomorphic map

Then the combination of $Q^T Q = I$ and $\det Q = 1$ gives us the three equations

$$\begin{cases} a^2 + c^2 = 1 \\ b^2 + d^2 = 1 \\ ad - bc = 1. \end{cases}$$

In particular, $(a-d)^2 + (b+c)^2 = 0$, and hence $a = d$ and $c = -b$. Since $a^2 + c^2 = 1$, we can set $a = \cos\theta$ and $c = \sin\theta$. The result then follows. $\square$

Proof.

- Suppose f is conformal at z_0. Then $C := \mathbf{D}_{z_0} f$ is a conformal linear map, and by the above two lemmas, $C = \lambda R_\theta$ for some θ. In particular, the partial derivatives of f satisfy the Cauchy-Riemann equations, and since f is a C^1 map (in particular the total derivative exists), this implies that f is complex differentiable at z_0. Moreover, $0 < \det C = |f'(z_0)|^2$, and hence $f'(z_0) \neq 0$.
- Conversely, suppose f is complex differentiable at z_0 and $f'(z_0) \neq 0$. Then $C := \mathbf{D}_{z_0} f$ satisfies $\det C = |f'(z_0)|^2 > 0$. Moreover, it can be checked by direct calculation (using the fact that the Cauchy-Riemann equations are satisfied) that $Q = |f'(z_0)|^{-1} C$ is orthogonal. Then by Lemma 22.1.2, C is a conformal linear map, and hence by definition, f is a conformal map.

$\square$

Corollary 22.1.4. *Any holomorphic, injective function is conformal.*

Proof. Let $f \in \mathcal{O}(\Omega)$ and injective, but <u>not</u> conformal. By Proposition 22.1.1 there exists a point $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. If $w_0 = f(z_0)$, then the equation $f(z) = w_0$ has a root of multiplicity $m > 1$ at $z = z_0$. By our fundamental theorem on the local mapping properties of holomorphic functions (Theorem 1 in Lecture-19), there exists $\varepsilon, \delta > 0$ such that for each w such that $0 < |w - w_0| < \delta$, there exists at least m distinct points in $B_\varepsilon(z_0)$ such that $f(z) = w$. In particular f is not injective, which is a contradiction. $\square$

Remark 22.1.5. Of course, there are plenty of conformal maps which are not injective. For example $f(z) = z^n$ on $\mathbb{C}^*$ is conformal, but not injective if $n > 1$. Other examples include e^z, $\sin z$, $\cos z$ etc.

22.2 A survey of elementary mappings

Given two regions Ω_1 and Ω_2, we wish to construct a conformal map one to the other. A reasonable strategy is to first try to map Ω_1 into the unit disc, and then to map the unit disc into Ω_2. A priori, t his strategy might seem limiting. After all why would there exist such a conformal map from Ω_1 into the unit disc. In a couple of lectures time, we'll prove a deep fact - The Riemann mapping theorem - that for a large class of regions, namely simply connected strict subsets of $\mathbb{C}$, such a mapping always exists. In view of this it is important to build up a toolkit of familiar mappings, so that more complicated mapping can be constructed by taking products, compositions etc. So in this section we'll familiarise ourselves with the mapping properties of complex powers, exponentials and logarithms. A good strategy in finding the image of a certain region under a conformal mapping is to find the image of the boundary. Our convention is to use $z = x + iy$ as the complex coordinate in the domain, and $w = u + iv$ as the complex coordinate on the image.

22.2.1 Rotations and dilations

Clearly rotations $R_\theta(z) = e^{i\theta} z$, dilations and $T_\lambda(z) = \lambda z$ are examples of conformal maps. In fact these form a subgroup of the full group of Mobius transformations.

22.2.2 Complex powers

Complex powers are useful in mapping sectors and half planes to each each other. We illustrate this using two examples.

- **Case-1:** $f(z) = z^n,\ n \in \mathbb{N}$. An example of such a mapping is in Figure 22.1

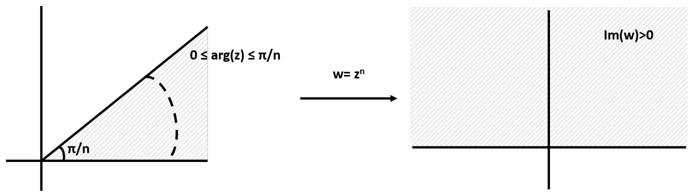


Figure 22.1: integer powers turn a sector into a half plane

- **Case-1:** $f(z) = z^n,\ n \in \mathbb{N}$. An example of such a mapping is in Figure 22.2

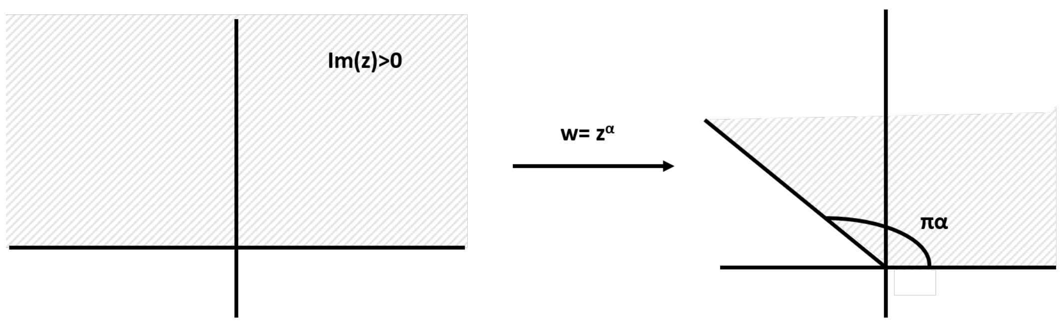


Figure 22.2: fractional powers can turn a half plane into a sector

.

22.2.3 The logarithm

If $L(z) = \log z$ is a branch of the logarithm, then $L'(z) = 1/z \neq 0$ on it's domain of definition. Hence it defines a conformal map. As an illustration, in the Figures 22.3 and 22.4, we consider the branch of the logarithm

$$\log z = \log |z| + i \arg(z),$$

where $\arg(z) \in (-\pi/2, 3\pi/2)$. That is, our branch cut is the line $\{z = iy \mid y < 0\}$ or the $-ve$ y axis.

In the Figure 22.3, the boundary of the domain $\mathbb{H}$ contained in the domain of definition of $\log z$ consists of two components, namely the negative and positive x (or real) axis. The negative axis consists of points with $\arg(z) = \pi$, and hence is mapped via the log to the line $v = \pi$. Similarly the positive x-axis has $\arg(z) = 0$, and hence is mapped to $v = 0$. Hence $\mathbb{H}$ is mapped by the above branch of logarithm to the infinite strip $\{0 < v < \pi\}$. The reader should similarly work out the mapping in Figure 22.4.

22.2.4 The exponential map

The map $f(z) = e^z$ is clearly a conformal map by Proposition 22.1.1 since it's derivative never vanishes. Figures 22.5 and 22.6 illustrate some of the mapping properties of the exponential. The reader should try to work out why the images of the two mappings below are given by the figures on the left. As in the discussion above, the trick is to work out the images of the various boundary components.

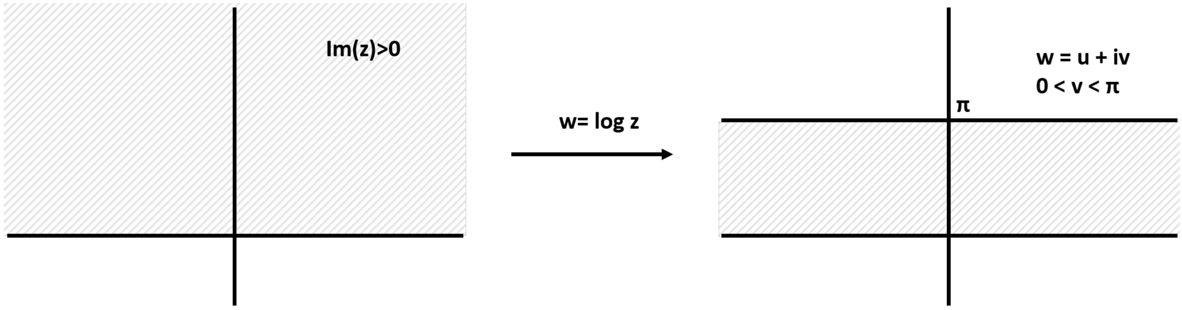


Figure 22.3: The logarithm mapping the $\mathbb{H}$ into an infinite horizontal strip

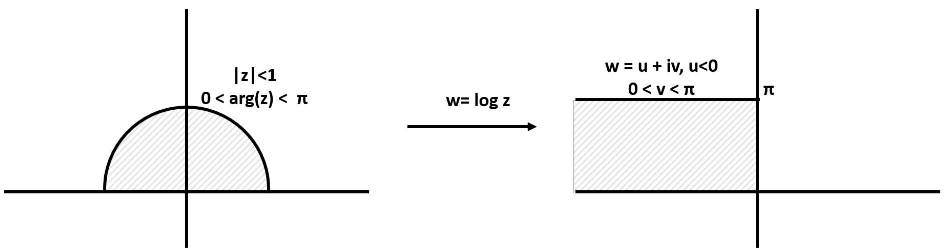


Figure 22.4: The logarithm mapping a semi-circular region into a half infinite strip

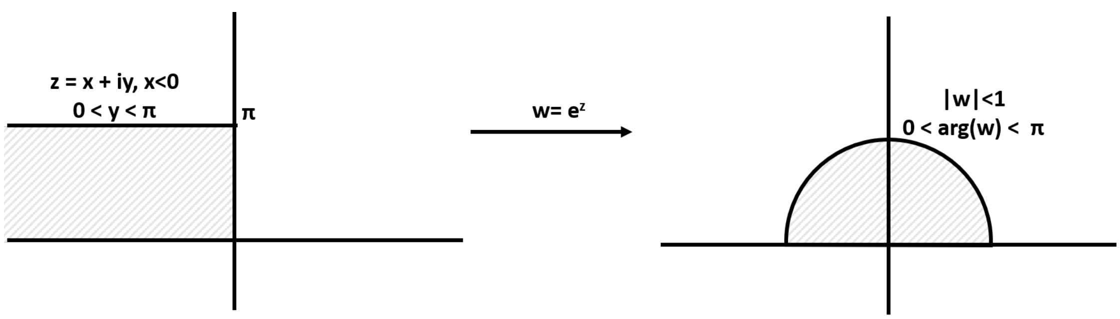


Figure 22.5: The exponential mapping a half infinite strip to a semi-circular region.

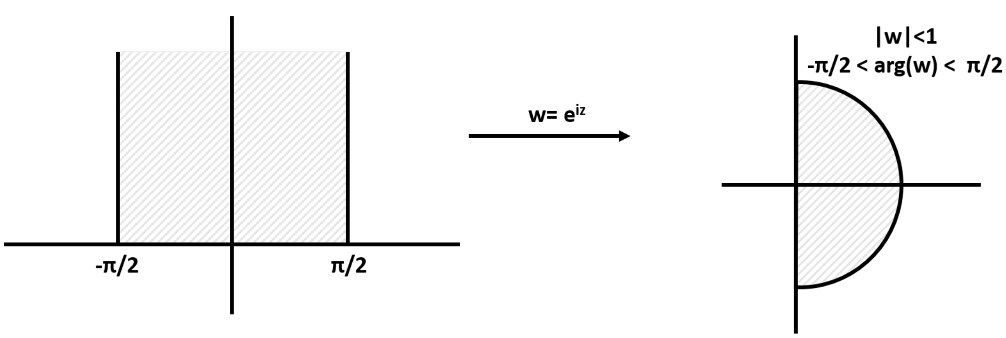


Figure 22.6: e^{iz} mapping a half infinite strip to a semi-circle.

Lecture 23

Mobius transformations

23.1 Mobius transformations

A *fractional linear transformation* or a *Mobius transformation* is a map of the form

$$w = T(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$. Clearly if each of a, b, c and d are scaled by the same complex number, then $T(z)$ remains invariant. Hence it is often convenient to normalize so that $ad - bc = 1$. The set of Mobius transformations is denoted by $\mathrm{Mob}(\mathbb{C})$. Note that the map is defined and holomorphic for all z except $z = -d/c$. Moreover,

$$\lim_{z \to \infty} T(z) = \frac{a}{c}.$$

In view of this, it is sometimes more convenient to define $T(-d/c) = \infty$ and $T(\infty) = a/c$ and think of $T(z)$ as a map from the extended complex plane $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$ to itself. We say that a map $T : \overline{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic, and write $T \in \mathcal{O}(\overline{\mathbb{C}})$ if the following three conditions hold:

1. If $R\Big|_{\mathbb{C}}$ is a meromorphic function and
2. $F(z) = T(1/z)$ is holomorphic in a neighbourhood of $z = 0$.

Lemma 23.1.1. *$\mathcal{O}(\hat{\mathbb{C}})$ can be identified with the set of rational functions on $\mathbb{C}$.*

This is essentially Theorem 0.2 in Lecture 16.

Theorem 23.1.2.
1. *A Mobius transformation T is a holomorphic, bijective map from $\hat{C}$ onto $\hat{C}$, and it's inverse is also a Mobius transformation.*
2. *In particular if $T(z_0) = \infty$ for some $z_0 \in \mathbb{C}$, then $T\Big|_{\mathbb{C}\setminus\{z_0\}} \to \mathbb{C} \setminus \mathbb{C}$ is a conformal map.*
3. *Moreover, if T and S are Mobius transformations, then so is $S \circ T$. In other words, $\mathrm{Mob}(\mathbb{C})$ forms a group under the law of composition with the identity element given by the transformation $I(z) = z$.*

Proof. Since T is a rational function, it is clearly in $\mathcal{O}(\overline{\mathbb{C}})$. The second point follows from the first easily. Let $T(z)$ be as above.

- T **is injective.** Suppose $T(z_1) = T(z_2)$ and neither of z_1 or z_2 is infinity or $-d/c$. Then rearranging, it is easy to see that $(ad - bc)z_1 = (ad - bc)z_2$, and since $ad - bc \neq 0$, we have $z_1 = z_2$. Now, suppose $z_1 = \infty$ then $T(z_1) = \frac{a}{c} = T(z_2)$. This forces z_2 to be infinity and hence $z_1 = z_2$. On the other hand, if $z_1 = -d/c$, then $T(z_1) = \infty = T(z_2)$. Again, this means that $z_2 = -d/c = z_1$.

- T **is surjective.** To prove this, we can simply solve the equation $w = T(z)$. That is, $w = T(z)$, if and only
$$z = \frac{dw - b}{a - wc}.$$
Combining this with the injectivity, we then have a well defined map $T^{-1} : \hat{C} \to \hat{C}$ defined by
$$T^{-1}(w) = \frac{dw - b}{a - wc}$$
which is again a Mobius transformation.

- $S \circ T$ **is a Mobius transformation.** Suppose T is as above, and S is another Mobius transformation
$$S(w) = \frac{pw + q}{rw + s}.$$
Then a simply computation gives
$$S \circ T(z) = \frac{(pa + cq)z + pb + qd}{(ra + cs)z + rb + ds}.$$
Moreover,
$$(pa + cq)(rb + ds) - (ra + cs)(pb + qd) = (ad - bc)(pr - qs) \neq 0,$$
and so $S \circ T$ is again a Mobius transformation.

□

23.1.1 The group $PSL(2, \mathbb{C})$.

For a Mobius transformation, if we write the coefficients as a matrix, we get what we call the coefficient matrix
$$M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
Since $ad - bc \neq 0$, the matrix $M(T)$ is an invertible matrix, that is $M(T) \in GL(2, \mathbb{C})$, the group of all invertible 2×2 complex valued matrices. If we normalize so that $ad - bc = 1$, then $M(T) \in SL(2, \mathbb{C})$, the so-called special linear group of 2×2 complex valued matrices. Even with this normalization, a particular Mobius transformation actually corresponds to 2 matrices, namely $M(T)$ and $-M(T)$, and hence a Mobius transformation actually corresponds to an *equivalence class* of matrices. To make this more precise we define the *projective special linear group* as
$$PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/A \sim \pm A.$$
We denote any element of $PSL(2, \mathbb{C})$ as $[A]$ where A is a matrix in $SL(2, \mathbb{C})$. One can prove that $PSL(2, \mathbb{C})$ forms a group with the multiplication
$$[A] \cdot [B] := [AB],$$
where AB is the usual matrix multiplication. One has to of course check that if you pick different representatives in $[A]$ (ie. -A instead of A) and/or $[B]$, then the multiplication gives the same element in $PSL(2, \mathbb{C})$. With this definition in place, we then obtain a map $\Phi : Mob(\mathbb{C}) \to PSL(2, \mathbb{C})$, given by
$$\Phi(T) = [M(T)].$$

By Theorem 23.1.2, part(3), it is clear that

$$[M(S \circ T)] = [M(S)M(T)].$$

where the multiplication on the right is simply the usual matrix multiplication. In fancier language, this says that the map Φ is a group homomorphism. In fact we have the following.

Theorem 23.1.3. *The map Φ is an isomorphism between the groups $Mob(\mathbb{C})$ and $PSL(2,\mathbb{C})$.*

Remark 23.1.4. One can also define $PGL(2,\mathbb{C})$ as $GL(2,\mathbb{C})/\sim$ where $A \sim \lambda A$ where $\lambda \in \mathbb{C}^*$. It is then easy to see that $PGL(2,\mathbb{C})$ is isomorphic as a group to $PSL(2,\mathbb{C})$.

Remark 23.1.5. A convenient way to represent Mobius transformations is using homogenous coordinates, and this makes the role of $PSL(2,\mathbb{C})$ much more transparent. The extended complex plane $\overline{\mathbb{C}}$ can be identified with the set $\mathbb{P}^1$ of complex lines passing through the origin in $\mathbb{C}^2$. The identification is given by the complex slope. A line L in $\mathbb{C}^2$ passing through the origin is determined by a point $(\xi_1, \xi_2) \neq (0,0)$, and any other point on the line is given by $(t\xi_1, t\xi_2)$ for $t \in \mathbb{C}$. Hence we represent points in $\mathbb{P}^1$ as equivalence classes of these points $[\xi_1 : \xi_2]$. The complex slope is then given by $z = \xi_1/\xi_2$, and is a well defined number in $\overline{\mathbb{C}}$. For instance the points $[0:1]$ and $[1:0]$ correspond to the points 0 and ∞ respectively in $\hat{\mathbb{C}}$. We say that $[\xi_1 : \xi_2]$ are the homogenous coordinates of z. Note that homogenous coordinates are unique, only up to scaling ie. both (ξ_1, ξ_2) and $t\xi_1, t\xi_2)$ for $t \neq 0$, represent the same point z in $\overline{\mathbb{C}}$.

With this identification, if $w = \zeta_1/\zeta_2$ and $z = \xi_1/\xi_2$, we can rewrite $w = Tz$ as

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

So the action of the Mobius transformation on $\hat{\mathbb{C}}$ is exactly the linear action of a 2×2 matrix on $\mathbb{C}^2 \setminus \{(0,0)\} = \{\text{set of homogenous coordinates}\}$.

23.1.2 The cross ratio

Given any four numbers z_1, z_2, z_3, z_4 in the extended complex plane $\hat{\mathbb{C}}$, the *cross ratio* is defined to be

$$(z_1, z_2, z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}.$$

Note that if one of the points is infinity, then the cross ratio is defined by taking a limit. For instance, if $z_1 = \infty$, then

$$(\infty, z_2, z_3, z_4) = \frac{z_4 - z_2}{z_3 - z_2}.$$

The importance of the cross ratio comes form the following theorem.

Theorem 23.1.6. *1. Given any three points z_2, z_3, z_4, there exists a unique Mobius transformation mapping these points to $1, 0$ and ∞ respectively. In fact we can take*

$$S(z) = (z, z_2, z_3, z_4).$$

2. If T is any Mobius transformation, then

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4).$$

Proof. 1. Clearly, $S(z) = (z, z_2, z_3, z_4)$ is a Mobius transformation that takes (z_2, z_3, z_4) to $(1, 0, \infty)$. Let T be another such Mobius transformation. Then $S \circ T^{-1}$ takes $(1, 0, \infty)$ to itself. Then it is not hard to prove that $S \circ T^{-1}(z) = z$, and hence $S(z) = T(z)$.

2. Let $S(z) = (z, z_2, z_3, z_4)$. Then ST^{-1} carries (Tz_2, Tz_3, Tz_4) to $(1, 0, \infty)$. Then by part (1), $ST^{-1}(z) = (z, Tz_2, Tz_3, Tz_4)$. Applying this to Tz_1, we see that

$$(z_1, z_2, z_3, z_4) = Sz_1 = ST^{-1}Tz_1 = (Tz_1, Tz_2, Tz_3, Tz_4).$$

□

As a consequence we have the following.

Corollary 23.1.7. *Given any pair of three points (z_1, z_2, z_3) and (w_1, w_2, w_3), there exists a unique Mobius transformation that takes the first triple to the second.*

Proof. Let S and T be the Mobius transformations that take (z_1, z_2, z_3) and (w_1, w_2, w_3) to $(1, 0, \infty)$ respectively. Then $T \circ S^{-1}$ is the Mobius transformation that we need. Uniqueness follows from the fact that there is unique transofrmation (namely the identity) that takes $(1, 0, \infty)$ to itself. □

There is another, more geometric, application of the cross ratio. A *generalised circle* in $\mathbb{C}$ is either a circle (given by an equation $|z - z_0| = r$ or a straight line (given by the equation $\bar{a}z + a\bar{z} + b = 0$, where $b \in \mathbb{R}$). In principle, a straight line is being thought of as a circle with infinite radius. An additional justification for this terminology is that both circles and straight lines in $\mathbb{C}$ correspond to circles on the Riemann sphere via the stereographic projection (or rather via it's inverse). A key observation is that three points determine a unique generalised circle (in the case of a line, one of the points will be at infinity).

Theorem 23.1.8. *The cross ratio of of (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a generalised circle. Consequently, a Mobius transformation maps generalised circles to generalised circles.*

Proof. The second part follows from the first part, and the fact that the cross ratio is invariant under Mobius transformations. So we focus on proving the first part. The key is the following claim.

Claim. If T is a Mobius transformation, then T^{-1} maps the (extended) real axis $\hat{\mathbb{R}}$ to a generalised circle.

Assuming this we complex the proof. There are two directions.

- $\implies$ **.** Suppose the cross ratio (z_1, z_2, z_3, z_4) is real. Consider $Tz = (z, z_2, z_3, z_4)$. Then $Tz_1 \in \mathbb{R}$. Moreover, $(Tz_2, Tz_3, Tz_4) = (1, 0, \infty)$ and hence they already lie on the real line. Since T^{-1} maps $\mathbb{R}$ to a generalised circle, z_1, z_2, z_3 and z_4 lie on a generalised circle.
- $\impliedby$ **.** Suppose z_1, z_2, z_3, z_4 lie on a generalised circle. Once again consider $Tz = (z, z_2, z_3, z_4)$. Once again, $(Tz_2, Tz_3, Tz_4) = (1, 0, \infty)$ and hence they already lie on the real line. So z_2, z_3, z_4 lie on the generalised circle $T^{-1}(\hat{\mathbb{R}})$. Hence z_1 also lies on $T^{-1}(\hat{\mathbb{R}})$, and so $Tz_1 \in \mathbb{R}$. In particular the cross ratio is real.

Proof of the claim: Let $z = T^{-1}(w)$. If w is real, then $Tz = \overline{Tz}$. If $T = \frac{az+b}{cz+d}$, then this condition translates to

$$\frac{az + b}{cz + d} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}.$$

Cross multiplying and simplifying, we get

$$(a\bar{c} - c\bar{a})|z|^2 + (a\bar{d} - c\bar{b})z + (b\bar{c} - d\bar{a})\bar{z} + b\bar{d} - d\bar{b} = 0.$$

There are two cases:

- **Case-1:** $a\bar{c} - c\bar{a} = 0$. Then since $ad - bc \neq 0$ we have that $a\bar{d} - c\bar{b} \neq 0$. Hence the equation above represents a straight line.

- **Case-2:** $a\bar{c} - c\bar{a} =\neq 0$. Completing the square, we can rewrite the equation as

$$\left|z + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a}\right| = \left|\frac{ad - bc}{\bar{a}c - \bar{c}a}\right|,$$

  which clearly defines a circle.

□

23.1.3 The Cayley transform

Perhaps the most important Mobius transformation is the the *Cayley transform*:

$$\beta(z) = \frac{z - i}{z + i}.$$

To compute it's inverse, we set $w = \beta(z)$. and solve for z. It is easy to see that

$$\beta^{-1}(w) = i\frac{1 + w}{1 - w}.$$

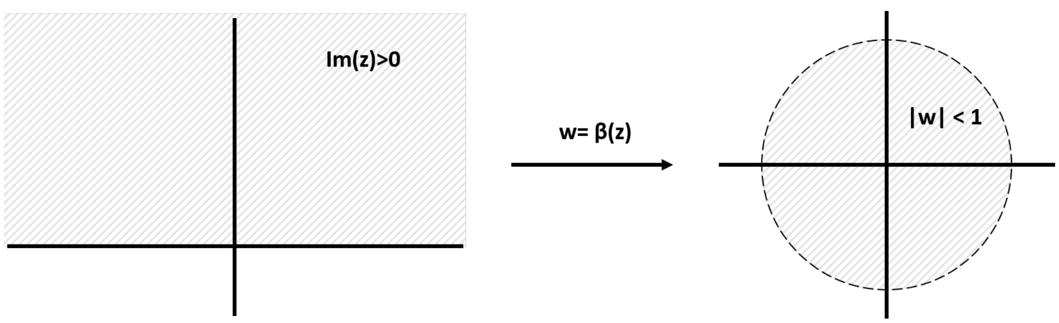


Figure 23.1: The Cayley transform

Lemma 23.1.9. *The Cayley transform is a biholomorphism from the upper half plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$. Moreover, it maps the boundary $\partial\mathbb{H}$ bijectively to $\partial\mathbb{D}$.*

Proof.

- **$\beta(z)$ maps $\mathbb{H}$ into $\mathbb{D}$ and $\partial\mathbb{H}$ to $\partial\mathbb{D}$.** To see this, we compute

$$|\beta(z)|^2 = \frac{1 + |z|^2 + i(z - \bar{z})}{1 + |z|^2 - i(z - \bar{z})} = \frac{1 + |z|^2 - 2Im(z)}{1 + |z|^2 + 2Im(z)} < 1$$

  if $Im(z) > 0$ i.e. if $z \in \mathbb{H}$. On the other hand it is also clear from the computation that $|\beta(z)| = 1$ if and only if $Im(z) = 0$.

- **$\beta(z)$ is surjective from $\mathbb{H}$ onto $\mathbb{D}$.** It is easy to see by direct computation that $\beta^{-1}(w)$ given by the above formula is an inverse. It is a nice exercise to check that indeed $\beta^{-1}(w)$ is in the upper half plane if $w \in \mathbb{D}$.

□

Lecture 24

Some automorphism groups

In this lecture, we compute the automorphism groups of the disc and the complex plane.

24.1 Biholomorphisms and automorphisms

A holomorphic function $f : \Omega \to \Omega'$ is said to be a *bi-holomorphism* if it is a bijective function, and f^{-1} is also holomorphic.

Lemma 24.1.1. *A holomorphic function $f : \Omega \to \Omega'$ is a bi-holomorphism if and only if it is bijective.*

Proof. If f is a bi-holomorphism, then it is automatically bijective from the definition. Conversely suppose $f : \Omega \to \Omega'$ is a bijective, holomorphic function. Injectivity implies that $f'(z) \neq 0$ for all $z \in \Omega$. Then the holomorphic inverse function theorem implies that the inverse is holomorphic. $\square$

For any domain $\Omega \subset \mathbb{C}$, a bi-holomorphic function $f : \Omega \to \Omega$ is called an *automorphism* of Ω.

Lemma 24.1.2. *The set of automorphisms of a domain Ω*

$$\mathrm{Aut}(\Omega) := \{f : \Omega \to \Omega \mid f \textit{ is holomorphic and bijective}\},$$

forms a group under the law of composition.

We call $\mathrm{Aut}(\Omega)$ the automorphism group of Ω. Note that if Ω and Ω' are bi-holomorphic, then $\mathrm{Aut}(\Omega)$ and $\mathrm{Aut}(\Omega')$ are isomorphic as groups. In fact, if $\varphi : \Omega \to \Omega'$ is a biholomorphism, then $\Phi : \mathrm{Aut}(\Omega) \to \mathrm{Aut}(\Omega')$ defined by

$$\Phi(f) = \varphi \circ f \circ \varphi^{-1},$$

is the required group isomorphism.

The aim of this note is to compute the automorphism groups of the complex plane, the punctured plane and the disc (and hence the upper half plane).

24.2 Automorphism group of the disc

In this section, we compute $\mathrm{Aut}(\mathbb{D})$.Recall that in Problem-5 from Assignment-1, you were asked to prove that for any $|\alpha| < 1$,

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha} z}$$

is a biholomorphism from the disc to itself. Our main theorem is that upto rotation these are the only automorphisms. En route, we'll also provide a short proof that ψ_α is indeed an element in the automorphism group, using some of the tools we have developed over the past few months, but which were of course unavailable when you were asked to solve the problem!

Theorem 24.2.1. *The automorphism group of the disc is given by*

$$\mathrm{Aut}(\mathbb{D}) = \{\psi_{\alpha,\theta}(z) = e^{i\theta}\frac{\alpha - z}{1 - z\bar{\alpha}} \mid \alpha \in \mathbb{D},\ \theta \in [0, 2\pi)\}.$$

Moreover, the automorphism $\psi_{\alpha,\theta}$ is precisely the automorphism that takes $z = 0$ to $z = \alpha$. In particular, the automorphisms of the disc fixing the origin are all given by $z \to e^{i\theta}z$ for some fixed θ.

The key tool in the proof is the Schwarz lemma, whose utility extends well beyond the computation of the automorphism groups of the disc.

Lemma 24.2.2 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map such that $f(0) = 0$. Then*

- $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.
- $|f'(0)| \leq 1$.

Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$

Proof. Since $f(0) = 0$, there exists a holomorphic function

$$g : \mathbb{D} \to \mathbb{D}$$

such that

$$f(z) = zg(z).$$

For any fixed $z \in \mathbb{D}$, let $1 > r > |z|$. Then by the maximum modulus principle,

$$|g(z)| \leq \max_{|w|=r} \frac{|f(w)|}{r} < \frac{1}{r}.$$

Letting $r \to 1^-$ we see that $|g(z)| < 1$ for all $z \in \mathbb{D}$, and hence

$$|f(z)| < |z|$$

for all $z \in \mathbb{D}$. This directly implies that $|f'(0)| \leq 1$. Now, suppose that $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}\backslash\{0\}$. Then $|g(z_0)| = 1$, and hence by the maximum modulus principle, $g(z)$ must be a constant, and hence $f(z) = az$ for some a with $|a| = 1$.

Finally, suppose $|f'(0)| = 1$. Then there exists a sequence $z_n \to 0$, $z_n \neq 0$ such that

$$1 - \frac{1}{n} \leq \frac{|f(z_n)|}{|z_n|} \leq 1.$$

By the definition of g, we then have that $1 - 1/n \leq |g(z_n)| \leq 1$, and hence by continuity, $|g(0)| = 1$. Then 0 is an interior maximum point for $|g(z)|$, and hence again by maximum modulus principle, $g(z)$ is a constant. That is, $g(z) = a$, where $a = g(0)$ satisfies $|a| = 1$, and once again $f(z) = az$.

□

Proof of the theorem. There are two steps in the proof. We let $\psi_\alpha := \psi_{\alpha,0}$ as above (ie. ψ_α is the map with rotation by zero angle). Note that ψ_α exchanges 0 and α. That is, $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$.

- We'll first prove that each $\psi_{\alpha,\theta}$ is actually an automorphism. To see this, first we observe that

$$|\psi_{\alpha,\theta}(z)|^2 = \psi_{\alpha,\theta}(z)\overline{\psi_{\alpha,\theta}}(z) = \frac{|\alpha|^2 + |z|^2 - z\bar{\alpha} - \bar{z}\alpha}{1 + |\alpha|^2 - z\bar{\alpha} - \bar{z}\alpha},$$

  and so $|\psi_{\alpha,\theta}(z)| = 1$ if $|z| = 1$. But then by the maximum principle, since $\psi_{\alpha,z}$ is clearly non-constant, $|\psi_{\alpha,\theta}(z)| < 1$ for all $z \in \mathbb{D}$. Hence $\psi_{\alpha,\theta}$ maps $\mathbb{D}$ into itself. Next, consider $\varphi_\alpha := \psi_\alpha \circ \psi_\alpha$. Clearly, $\varphi_{\alpha,\theta}(0) = 0$ and $\varphi_\alpha(\alpha) = \alpha$, and φ_α maps $\mathbb{D}$ into itself. By equality in Schwarz lemma, $\varphi_\alpha(z) = z$ for all $z \in \mathbb{D}$. In particular, ψ_α is surjective and injective, and hence a biholomorphism, with inverse function given by itself ie. $\psi_\alpha^{-1} = \psi_\alpha$. But then $\psi_{\alpha,\theta} = e^{i\theta}\psi_\alpha$ is also clearly a biholomorphism.

- It remains to show that these are the only automorphisms. Let $F \in \mathrm{Aut}(\mathbb{D})$ such that $F(0) = 0$. By the Schwarz lemma, $|F(z)| \leq |z|$. But the same also holds true for $F^{-1}(z)$, and so

$$|z| = |F^{-1}(F(z))| \leq |F(z)| \leq |z|,$$

  and hence all inequalities must be equalities. That is, $|F(z)| = |z|$ for all $z \in \mathbb{D}$. By the equality part of Schwarz lemma, we have that $F(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$. Now suppose $F \in \mathrm{Aut}(\mathbb{D})$ such that $F(\alpha) = 0$. Consider the automorphism $F_\alpha(z) := F(\psi_\alpha(z))$. Then $F_\alpha(0) = 0$, and hence by the above argument, $F_\alpha(z) = e^{i\theta}z$ for some θ. But then, since $\psi_\alpha^{-1} = \psi_\alpha$, we have that

$$F(z) = F_\alpha(\psi_\alpha^{-1}(z)) = e^{i\theta}\psi_\alpha(z) = \psi_{\alpha,\theta}(z).$$

This completes the proof of the theorem. □

24.3 Automorphism groups of $\mathbb{C}$ and $\mathbb{C}^*$

Theorem 24.3.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be an injective holomorphic map. Then*

$$f(z) = az + b$$

for some $a, b \in \mathbb{C}$ with $a \neq 0$.Since any such map is automatically surjective, we have that

$$\mathrm{Aut}(\mathbb{C}) = \{az + b \mid a, b \in \mathbb{C}, a \neq 0\}.$$

Theorem 24.3.2. *Any injective holomorphic map $f : \mathbb{C}^* \to \mathbb{C}^*$ is given by either*

$$f(z) = az \text{ or } f(z) = \frac{a}{z},$$

for some $a \in \mathbb{C}$ and $a \neq 0$. Since any such map is automatically surjective, we have that

$$\mathrm{Aut}(\mathbb{C}^*) = \{az, \frac{a}{z} \mid a \in \mathbb{C}, a \neq 0\}.$$

The key lemma needed to compute both the automorphism groups is the following.

Lemma 24.3.3. *Let $g : \mathbb{C}^* \to \mathbb{C}$ be holomorphic and injective. Then g cannot have an essential singularity at $z = 0$.*

Proof. Suppose, for the sake of contradiction, g has an essential singularity at $z = 0$. Consider the disc $D_1(2)$ of radius 1 around the point $z = 2$, and the unit disc $\mathbb{D}$. Note that both these discs are disjoint from each other. By the open mapping theorem, $V = g(D_1(2))$ is an open neighborhood of $g(2)$. By the Casorati-Weierstrass theorem, $g(\mathbb{D})$ is dense in $\mathbb{C}$. That means there is some $w \in g(D_1(2))$ and some $z_1 \in \mathbb{D}$ such that

$$g(z_1) = w.$$

But since $w \in g(D_1(2))$, there is already a $z_2 \in D_1(2)$ such that

$$g(z_2) = w.$$

Since the open discs $D_1(2)$ and $\mathbb{D}$ are disjoint sets, $z_1 \neq z_2$, but $g(z_1) = w = g(z_2)$. This contradicts the injectivity of g. Hence $z = 0$ cannot be an essential singularity. □

We also need an elementary generalization of Liouville's theorem, which was a homework problem sometime back. We provide a proof for the sake of completeness.

Lemma 24.3.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that*

$$|f(z)| \leq M(1 + |z|^n)$$

for some $M > 0$ and all $z \in \mathbb{C}$. Then f is a polynomial of degree less than or equal to n.

Proof. By the Cauchy estimates (Corollary 3 in Lecture-8), if $R > 1$, we have that,

$$|f^{(k)}(0)| \leq \frac{k!M(1 + R^n)}{R^k}. \leq 2Mk!R^{n-k}.$$

But this holds no matter what R is chosen. So letting $R \to \infty$, if $k > n$, the right hand side goes to zero. Hence

$$f^{(k)}(0) = 0$$

for all $k = n + 1, n + 2, \cdots$. But since f is entire, it has a power series expansion whose coefficients are given by

$$a_k = \frac{f^{(k)}(0)}{k!} = 0,$$

for $k > n$. Hence the power series terminates, and f is a polynomial of degree less than or equal to n. □

We are now ready to compute the automorphism groups of $\mathbb{C}$ and $\mathbb{C}^*$.

Proof of Theorem 24.3.1. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective, holomorphic map. The key idea is to study the function at infinity. That is, define a holomorphic function $g : \mathbb{C}^* \to \mathbb{C}$ by

$$g(z) := f\Big(\frac{1}{z}\Big).$$

Then it is easy to see that g is also injective. Applying lemma 24.3.3, g either has a zero or a pole at $z = 0$. In any case, this means that there exists a constant $M > 0$ and integer $n > 0$ such that $|z|^n|g(z)| < \frac{1}{M}$ on $|z| < 1$. Transferring the estimates to f, we see that

$$|f(z)| < M|z|^n,$$

for $|z| > 1$. On the other hand, since $|z| \leq 1$ is compact, we actually get that (possibly by choosing a bigger M), that

$$|f(z)| \leq M(1 + |z|^n),$$

for all $z \in \mathbb{C}$. Then by lemma 24.3.4, $f(z)$ is a polynomial. By the fundamental theorem of algebra, $f(z)$ has at most n roots. We claim that $n = 1$. To see this, note that by injectivity, all the roots have to be identical, or equivalently, $f(z) = a(z - \alpha)^n$ for some $a, \alpha \in \mathbb{C}$ with $c \neq 0$. If $n > 1$, then $f'(\alpha) = 0$, and hence $f(z)$ cannot even be locally injective (see Theorem 1 or Corollary 0.3 in Lecture 19), and hence we must have $n = 1$. But then clearly $f(z) = az + b$, where we put $b = -a\alpha$. This proves the first part of the theorem. For the second part, notice that any linear polynomial is surjective, and hence the f above will automatically be surjective, and hence give an automorphism. $\square$

Proof of Theorem 24.3.2. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be an injective map. Then by lemma 24.3.3, $z = 0$ is either a removable singularity or a pole.

Case 1. Suppose $z = 0$ is a removable singularity. Then f extends to $\tilde{f} : \mathbb{C} \to \mathbb{C}$. We then claim that $\tilde{f}$ is also injective. Suppose not, then since f is injective, the only possibility is that there are $z_0, w_0 \in \mathbb{C}$ with $z_0 \neq 0$ such that

$$\tilde{f}(z_0) = \tilde{f}(0) = w_0.$$

By the argument principle there is a neighborhood U of w_0 and disjoint discs $D_r(z_0)$, $D_\rho(0)$ around z_0 and 0 respectively such that for any $w \in U$, $w \neq w_0$ there are solutions $z_1 \in D_r(z_0)$ and $z_2 \in D_\rho(0)$ to

$$\tilde{f}(z) = w.$$

. But $z_2 \neq 0$ since $w \neq w_0$. So the two distinct solutions are actually solutions to

$$f(z) = w,$$

contradicting the injectivity of f. This proves that the extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is an injective holomorphic map. But then by Theorem 24.3.1, $\tilde{f}(z) = az + b$ for some $a, b \in \mathbb{C}$ and $a \neq 0$. All that is needed now is to show that $b = 0$. If not, then $0 = \tilde{f}(-b/a) = f(-b/a)$ which is a contradiction since f takes non-zero values being a map from $\mathbb{C}^*$ into $\mathbb{C}^*$. To sum up, in this case $f(z) = az$.

Case-2: Suppose $z = 0$ is a pole. Then if we define

$$h(z) = \frac{1}{f(z)},$$

then h extends to an holomorphic function $\tilde{h} : \mathbb{C} \to \mathbb{C}$. Then by the proof in the first case, we can see that

$$\tilde{h}(z) = cz,$$

for some $c \neq 0$. But then this shows that $f(z) = z/c$, and proves the theorem with $a = 1/c$. $\square$

24.4 Automorphism group of the extended complex plane

. Recall that a map $F : \overline{\mathbb{C}} \to \hat{\mathbb{C}}$ is called holomorphic if

1. If $F\Big|_{\mathbb{C}}$ is a meromorphic function and
2. $G(z) = F(1/z)$ is holomorphic in a neighbourhood of $z = 0$.

Theorem 24.4.1. *The automorphism group of the extended complex plane is given by*

$$\mathrm{Aut}(\overline{\mathbb{C}}) = \Big\{T(z) = \frac{az+b}{cz+d} \mid ad - bc = 1\Big\},$$

and hence $\mathrm{Aut}(\overline{\mathbb{C}}) \cong PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/ \pm I$.

Proof. From our discussion in the previous lecture, it is clear that any $T(z)$ defined as above is an automorphism of $\hat{C}$, and hence the set on the right is contained in $\mathrm{Aut}(\overline{\mathbb{C}})$. To show the reverse containment, let $T \in \mathrm{Aut}(\hat{\mathbb{C}})$. Then there is a unique point $z_0 \in \overline{\mathbb{C}}$ such that $T(z_0) = \infty$, and a unique point w_0 such that $T(\infty) = w_0$.

- **Case-1:** $z_0 = \infty$**.** In this case, $w_0 = \infty$, and so $F(z) = T\Big|_{\mathbb{C}}$ is a bi-holomorphism of $\mathbb{C}$, and by Theorem 24.3.1, $F(z) = az + b$ for some a, b and $a \neq 0$. But then extending this to infinity, $T(z) = az + b$ and hence we are done.

- **Case-2:** $z_0 \neq \infty$**.** In this case, since T is one-one, $w_0 \neq \infty$ (else it will map both z_0 and ∞ to ∞, contradicting the one-one property). Now consider the function $F : \mathbb{C}^* \to \mathbb{C}$ defined by

$$F(z) = T(z + z_0) - w_0.$$

It is easy to check that F maps $\mathbb{C}^*$ to $\mathbb{C}^*$, and is infact an automorphism of $\mathbb{C}^*$ is (this follows essentially from the fact that F has a zero at infinity). Moreover, since $z = 0$ is a pole for F, by Theorem 24.3.2, there exists an $a \in \mathbb{C}^*$ such that $F(z) = a/z$. But then solving for z,

$$T(z) = \frac{a}{z - z_0} + w_0,$$

and is a Mobius transformation.

$\square$

Lecture 25

The Riemann mapping theorem

Recall that two domains are called conformally equivalent or biholomorphic if there exists a holomorphic bijection from one to the other. This automatically implies that there is an inverse holomorphic function. The aim of this lecture is to prove the following deep theorem due to Riemann. Denote by $\mathbb{D}$ the unit disc centered at the origin.

Theorem 25.0.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected set that is not all of $\mathbb{C}$. Then for any $z_0 \in \Omega$, there exists a unique biholomorphism $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0, \quad \text{and } F'(z_0) > 0.$$

Here $F'(z_0) > 0$ stands for $F'(z_0)$ being real and positive, and can be thought of as a normalization, to ensure that the above map is unique. The precise normalization by itself is not very important. The reader should try to test her/his understanding of the proof by coming up with other normalizations that work, and also some that do not work (for instance, you might not be able to impose that $F'(z_0) = 1$). Note that by Liouville's theorem, such a statement is patently false if $\Omega = \mathbb{C}$, and so the hypothesis that Ω is a *proper* subset is a necessary condition. As a consequence of the Theorem, we have the following corollary.

Corollary 25.0.1. *Any two proper, simply connected subsets for $\mathbb{C}$ are conformally equivalent.*

Proof of uniqueness in Theorem 25.0.1. Let $F_1 : \Omega \to \mathbb{D}$ and $F_2 : \Omega \to \mathbb{D}$ be two such mappings. Then $f = F_2 \circ F_1^{-1}$ satisfies the following properties

- $f : \mathbb{D} \to \mathbb{D}$ is injective and onto.
- $f(0) = 0$.
- $f'(0) > 0$.
- f^{-1} also satisfies both these properties.

By Schwarz lemma, $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f^{-1}(w)| \leq |w|$ for all $w \in \mathbb{D}$. Let $w = f(z)$, then second inequality gives $|z| \leq |f(z)|$, and hence $|z| = |f(z)|$. But then by the equality part of Schwarz lemma, we see that $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$. But then $f'(0) = a$, which forces $a = 1$ (since $f'(0) > 0$). Hence $f(z) = z$ for all $z \in \mathbb{D}$ or equivalently $F_2(w) = F_1(w)$ for all $w \in \Omega$. ☐

25.1 Montel's and Hurwitz's theorems

The proof relies on two theorems on sequences of holomorphic functions. Recall that we say that a sequence of functions f_n converges compactly on Ω to f if it converges uniformly on any compact subset

$K \subset \Omega$. More precisely, for every compact set $K \subset \Omega$ and $\varepsilon > 0$, there exists an $N = N(\varepsilon, K)$ such that

$$\sup_{z \in K} |f_n(z) - f(z)| < \varepsilon$$

whenever $n > N$. In Lecture 9 we proved the following theorem:

Theorem 25.1.1. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge compactly to $f : \Omega \to \mathbb{C}$, then $f(z)$ is holomorphic. Moreover*

$$f_n^{(k)} \to f^{(k)}$$

compactly on Ω for all $k \in \mathbb{N}$.

We say that family of continuous functions $\mathcal{F}$ on an open set Ω is *normal* if every sequence of functions in $\mathcal{F}$ has a subsequence that converges compactly on Ω. Note that the definition does not require the limiting function to be contained in $\mathcal{F}$. On the other hand, by Theorem 25.1.1 above, the limiting function will certainly be holomorphic. The family is said to be *locally uniformly bounded* if for any $K \subset \Omega$ compact, there exists a constant M_K such that

$$\sup_{z \in K} |f(z)| < M_K$$

for all $f \in \mathcal{F}$.

Theorem 25.1.2 (Montel's theorem)**.** *A family $\mathcal{F}$ of holomorphic functions on Ω is normal if and only if it is locally uniformly bounded.*

To prove this, we first recall the Arzela-Ascoli theorem. Recall that family $\mathcal{F}$ of continuous functions on Ω is said to *locally equicontinuous* if for all $a \in \Omega$ and all $\varepsilon > 0$ there exists a $\delta = \delta(a, \varepsilon)$ such that

$$z, w \in D_\delta(a) \implies |f(z) - f(w)| < \varepsilon,$$

for all $f \in \mathcal{F}$. Then we have the following basic theorem, which we state without proof.

Theorem 25.1.3 (Arzela-Ascoli)**.** *If a family of functions is locally equicontinuous and locally uniformly bounded, then for every sequence of functions $\{f_n\} \in \mathcal{F}$, there exists a continuous function f and a subsequence $\{f_{n_k}\}$ which converges to f compactly on Ω.*

Remark 25.1.1. Generally, Arzela-Ascoli is stated for compact sets assuming equicontinuity. One can prove the above theorem by taking an exhaustion of Ω by compact sets ie. $K_1 \subset K_2 \cdots \subset K_n \cdots$ such that $\Omega = \cup K_n$. Local equicontinuity will then imply that the family is genuinely equicontinuous on each compact set K_n. Then apply the standard Arzela-Ascoli to $\mathcal{F}$ restricted to each K_n, and use Cantor diagonalization argument.

Proof of Montel's theorem. First, suppose that $\mathcal{F}$ is a locally uniformly bounded family of holomorphic functions. By the Arzela-Ascoli theorem, if we show that $\mathcal{F}$ is automatically locally equicontinuous, then for every sequence, there will exist a subsequence which converges compactly on Ω to a continuous function f. By Theorem 25.1.1, the limit function will then be holomorphic, and hence $\mathcal{F}$ would be a normal family.

Hence it is enough to show that the family $\mathcal{F}$ is locally equicontinuous. To do this, we use the Cauchy integral formula. Fix an $a \in \Omega$ and $\varepsilon > 0$. We need to choose a delta that works. Let $r > 0$ be such that $\overline{D_{2r}(a)} \subset \Omega$, and let M_r such that

$$|f(\zeta)| \leq M_r,$$

for all $\zeta \in \overline{D_{2r}(a)}$ and all $f \in \mathcal{F}$. By the Cauchy estimates (see Corollary 3 from Lecture 8), we have that for any $\zeta \in D_r(a)$,

$$|f'(\zeta)| \leq \frac{2M_r}{r}.$$

Then by the fundamental theorem for complex integrals, for any $z, w \in D_r(a)$,

$$|f(z) - f(w)| = \Big| \int_{l_{w,z}} f'(\zeta)\, d\zeta \Big| \leq \frac{2M_r}{r}|z - w|.$$

Given an $\varepsilon > 0$, let us pick $\delta < 2M_r\varepsilon/r$. Then whenever $|z - w| < \delta$, we have $|f(z) - f(w)| < \varepsilon$. This proves local equicontinuity.

Conversely, suppose $\mathcal{F}$ is a normal family, but not locally uniformly bounded. Then there exists a compact set $K \subset \Omega$, and a sequence of functions $f_n \in \mathcal{F}$ such that

$$\sup_{z \in K} |f_n(z)| \geq n.$$

Since the family is normal, there exists a subsequence f_{n_k} which converges uniformly on K. But then $\sup_{z\in K} |f_{n_k}|$ would be a bounded sequence which is a contradiction. □

We also need the following theorem due to Hurwitz on the limit of injective holomorphic functions.

Theorem 25.1.4 (Hurwitz)**.** *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic, injective functions on an open connected subset, which converge uniformly on compact subsets to $F : \Omega \to \mathbb{C}$. Then either F is injective, or is a constant.*

Proof. We argue by contradiction. So suppose F is non constant and not injective. Then for some $w \in \mathbb{C}$, there exists $a, b \in \Omega$ such that $F(a) = F(b) = w$. Let $f_n(a) = w_n$, then $w_n \to w$. Choose an $r > 0$ small enough so that there does not exist any $z \in \overline{D_r(b)}$ such that $F(z) = w$. This is possible by the principle of analytic continuation since we are assuming that F is non-constant. In particular $a \notin \overline{D_r(b)}$. Since f_n is injective for any n, there exists no solution to

$$f_n(z) = w_n$$

in the closure of the disc $D_r(b)$, and so by the argument principle applied to $f_n(z) - w_n$, we see that

$$\frac{1}{2\pi i} \int_{|\zeta - b| = r} \frac{f_n'(\zeta)}{f_n(\zeta) - w_n}\, d\zeta = 0.$$

But since $f_n \to F$ uniformly on compact sets, in particular, on the compact set $\overline{D_r(a)}$ we have $f_n'(\zeta) \to F'(\zeta)$ and $f_n(\zeta) - w_n \to F(\zeta) - w$ uniformly. Hence the integral also converges uniformly, and from this we conclude that

$$\frac{1}{2\pi i} \int_{|\zeta - b| = r} \frac{F'(\zeta)}{F(\zeta) - w}\, d\zeta = 0.$$

This integral calculates the number of zeroes of $F(\zeta) - w = 0$ in $D_r(b)$ which we know is at least one (counting multiplicity) since $F(b) = w$. This is a contradiction, and hence if F is non-constant, it has to be injective. □

25.2 Proof of Riemann mapping theorem

25.2.1 Some further preparation

For a fixed $z_0 \in \Omega$, we define a family $\mathcal{F}$ of holomorphic function functions by

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ holomorphic and injective, } f(z_0) = 0\}.$$

The required biholomorphic map will be obtained by maximizing the modulus of the derivative at z_0, amongst all functions in this family. We first show that this family is non-empty.

Lemma 25.2.1. *There is an injective holomorphic function $f : \Omega \to \mathbb{D}$ such $f(z_0) = 0$. That is, $\mathcal{F} \neq \phi$.*

Proof. Since $\Omega \neq \mathbb{C}$, there is an $a \in \mathbb{C} \setminus \Omega$. Then $z - a$ is never zero on Ω. Since Ω is simply connected, we can choose a holomorphic branch of $\log(z - a)$, or in other words, there is a holomorphic function $l : \Omega \to \mathbb{C}$ such that

$$e^{l(z)} = z - a$$

for all $z \in \Omega$. Clearly $l(z)$ is injective. Moreover, if $z_1, z_2 \in \Omega$ and $z_1 \neq z_2$ then $l(z_2) - l(z_1) \notin 2\pi i\mathbb{Z}$, i.e their difference cannot be an integral multiple of $2\pi i$. In particular, $l(z) \neq l(z_0) + 2\pi i$. We in fact claim that $|f(z) - (f(z_0) + 2\pi i)|$ is bounded strictly away from zero. That is,

Claim. There exists an $\varepsilon > 0$ such that $|l(z) - (l(z_0) + 2\pi i)| > \varepsilon$ for all $z \in \Omega$.

To see this, assume the claim is false. Then there is a sequence $\{z_n\} \in \Omega$ such that $l(z_n) \to l(z_0)+2\pi i$. But then exponentiating, since the exponential function is continuous, we see that $z_n \to z_0$. But then, since $l(z)$ is continuous, this implies that $l(z_n) \to l(z_0)$ contradicting the assumption that $l(z_n) \to l(z_0) + 2\pi i$. This proves the claim.

Now consider the function

$$\tilde{f}(z) = \frac{1}{l(z) - l(z_0) - 2\pi i}.$$

By the claim, this is a bounded, injective and holomorphic function on Ω, and hence $\tilde{f} : \Omega \to D_R(0)$, where R can be taken to be $R = 1/\varepsilon$ where ε is from the claim above. Suppose $\tilde{f}(z_0) = a$, then

$$f(z) = \frac{\tilde{f}(z) - a}{R + |a|}$$

is the required function. ☐

Next, let

$$\lambda = \sup_{f \in \mathcal{F}} |f'(z_0)|.$$

We claim that $\lambda > 0$. To see this, consider the $f \in \mathcal{F}$ constructed above. Since $f(z)$ is injective, by Corollary 0.3 from Lecture 19, $|f'(z_0)| > 0$ and hence $\lambda > 0$.

Lemma 25.2.2. *There is a function $F \in \mathcal{F}$ such that $|F'(z_0)| = \lambda$. In particular, λ is also finite.*

Proof. Let $f_n \in \mathcal{F}$ be a sequence of functions that maximize $|f'(z_0)|$; that is

$$\lim_{n\to\infty} |f_n'(z_0)| = \lambda.$$

Since $|f_n(z)| < 1$, by Montel's theorem there is a subsequence that converges uniformly on compact sets to a holomorphic function F satisfying $|F(z)| \leq 1$, and $F(z_0) = 0$. Moreover since the derivatives also converge, we must have $|F'(z_0)| = \lambda \neq 0$. In particular, F cannot be a constant. Then by the maximum modulus principle, $|F(z)| < 1$ for all $z \in \Omega$, since otherwise, there will be a point $z \in \Omega$ with $|F(z)| = 1$, and hence will be an interior maximum point for $|F|$. Finally to show that $F \in \mathcal{F}$, we need to show that F is injective. But this follows from Hurwitz's theorem since F is non-constant and all $f_n \in \mathcal{F}$ are injective. ☐

25.2.2 Completion of the proof of Riemann mapping

Since $F'(z_0) \neq 0$, by composing with a suitable rotation, we can assume that $F'(z_0)$ is real and positive. We claim that this F is the required bi-holomorphism. We already know that $F : \Omega \to \mathbb{D}$, $F(z_0) = 0$ and F is injective. To complete the proof, we need to show that F is surjective. If not, then there exists a $\alpha \in \mathbb{D}$ such that $F(z) = \alpha$ has no solution in Ω. We then exhibit a $G \in \mathcal{F}$ with $|G'(z_0)| > |F'(z_0)|$ contradicting the choice of F. To do this, consider $\psi_\alpha : \mathbb{D} \to \mathbb{D}$ defined by

$$\psi_\alpha(w) = \frac{\alpha - z}{1 - \bar{\alpha} z},$$

and let

$$g(z) = \sqrt{\psi_\alpha \circ F(z)}.$$

Since $\psi_\alpha(z) = 0$ if and only if $z = \alpha$, we see that $\psi_\alpha \circ F$ is always zero free, and so a holomorphic branch of $\log \psi_\alpha \circ F$ can be defined since Ω is simply connected. We can then choose a holomorphic branch for $g(z)$ by letting

$$g(z) = e^{\frac{1}{2} \log \psi_\alpha \circ F(z)}.$$

Note that $g(z_0) = \sqrt{\alpha}$. To construct a member of the family, we need to bring this back to the origin, and hence we define

$$G(z) = \psi_{\sqrt{\alpha}} \circ g(z).$$

Then $G(z_0) = 0$. Moreover, $G(z)$ is also injective since $\psi_{\sqrt{\alpha}}$ and $g(z)$ are injective, and so $G \in \mathcal{F}$.

Claim. $|G'(z_0)| > |F'(z_0)|$.

To see this, observe that

$$F(z) = \psi_\alpha^{-1} \circ s \circ \psi_{\sqrt{\alpha}}^{-1} \circ G(z) = \Phi \circ G(z),$$

where $s(w) = w^2$ is the squaring function and $\Phi = \psi_\alpha^{-1} \circ s \circ \psi_{\sqrt{\alpha}}^{-1} : \mathbb{D} \to \mathbb{D}$. We compute that $\Phi(0) = \psi_\alpha^{-1}(s(\psi_{\sqrt{\alpha}}^{-1}(0)) = \psi_\alpha^{-1}(s(\sqrt{\alpha})) = \psi_\alpha^{-1}(\alpha) = 0$. By Schwarz lemma, $|\Phi(z)| \leq |z|$, and so

$$|\Phi'(0)| \leq 1.$$

We claim that $|\Phi'(0)| < 1$. Suppose, $|\Phi'(0)| = 1$, then by the second part of Schwarz lemma, $\Phi(z) = az$ for some unit complex number a. In particular, $\Phi(z)$ is injective. But Φ cannot be injective since $s(z)$ is a $2-1$ function ie. sends two points to a single point, and ψ_α and $\psi_{\sqrt{\alpha}}$ are injective. This shows that $|\Phi'(0)| < 1$. But then $F'(z_0) = \Phi'(G(z_0)) \cdot G'(z_0) = \Phi'(0) \cdot G'(z_0)$, and hence $|F'(z_0)| < |G'(z_0)|$ which proves the claim, and completes the proof of the theorem.

□

25.3 Green's functions and a generalization of the Riemann mapping theorem

For the purposes of this section, we assume that Ω is bounded and has a sufficiently nice boundary $\partial\Omega$ (for concreteness, assume that $\partial\Omega$ is a union of piecewise regular curves). Often one is interested in solving the Dirichlet problem on Ω: Namely, given any smooth (real valued) function f on Ω and a continuous function u_0 on $\partial\Omega$, to find a smooth function u such that

$$\begin{cases} \Delta u = f \\ u\Big|_{\partial\Omega} = u_0. \end{cases}$$

This problem arises in many (seemingly) different areas in mathematics and physics. For instance, one interpretation of solutions of the above problem, is that u represents the voltage distribution on a conductor Ω with charge distribution given by $f(z)$ and the boundary being held at voltage u_0.

A *Green's function* for Ω based at $z_0 \in \Omega$ is a function $G_{z_0} : \Omega \to \mathbb{R}$ such that

1. $G_{z_0}(z)$ is a harmonic function on $\Omega \setminus \{z_0\}$.
2. $G_{z_0}\Big|_{\partial\Omega} \equiv 0$.
3. $G_{z_0}(z) - \frac{1}{2\pi} \log|z - z_0|$ is bounded in a neighbourhood of z_0.

Note that $\log|z - z_0|$ is a harmonic function in a punctured neighbourhood of z_0. By an analog of the Riemann removable singularity theorem for harmonic functions, condition (3) is equivalent to $G_{z_0}(z) - \frac{1}{2\pi} \log|z - z_0|$ extending as a harmonic function in a disc $D_r(z_0)$. It is often useful to think of the Green's function as a function of two variables $G : \Omega \times \Omega \to \mathbb{R}$, where we set

$$G(z, w) := G_w(z).$$

It turns out that the function G is actually symmetric in the two variables. The reason why Green's function is important is that it is a *fundamental solution* to the Dirichlet problem on Ω. That is, with f and u_0 as above, a solution to the Dirichlet poblem is given by

$$u(z) = \int_\Omega G(z, w)f(w)\, dwd\bar{w} - \int_{\partial\Omega} \frac{dG}{d\nu}(z, w)g(w)\, d\sigma,$$

where $dG/d\nu$ is the outward normal derivative of G on the boundary (where differentiation is with respect to the variable w), $dwd\bar{w}$ is the usual Lebesgue (or Euclidean) measure on Ω, and $d\sigma$ is the surface measure on $\partial\Omega$.

Conversely, if one can Dirichlet problems with continuous data, then one can construct a Green's function. The idea is to simply find a harmonic function $H_{z_0}(z)$ with boundary data $u_0 = -\log|z - z_0|$. The Green's function $G_{z_0}(z)$, will then be

$$G_{z_0}(z) = \frac{1}{2\pi} \log|z - z_0| + \frac{1}{2\pi} H_{z_0}(z).$$

Now suppose that Ω is simply connected. Fix a $z_0 \in \Omega$, and as above, let $H_{z_0}(z) := 2\pi G_{z_0}(z) - \log|z - z_0|$, which is harmonic by property (3) above. Since Ω is simply connected, $H_{z_0}(z)$ has a *harmonic conjugate*, that is a function $H^*_{z_0}(z) : \Omega \to \mathbb{R}$ which is harmonic, and such that $f(z) = H_{z_0(}z) + iH^*_{z_0}(z)$ is a holomorphic function on Ω (The proof is essentially the same as that for a disc, and this was an exam problem on the midterm). We let $F(z) = (z - z_0)e^{f(z)}$. For any $z \in \partial\Omega$, $G_{z_0}(z) = 0$, $H_{z_0}(z) = -\log|z - z_0|$, and so $|F(z)| = 1$. By the maximum principle, $|F(z)| < 1$ for all $z \in \Omega$. Hence F is a map from Ω to the unit disc $\mathbb{D}$. Furthermore, $F(z)$ has only one zero in Ω, and that too a simple one, namely at $z = z_0$. Next, let $w_0 \in \mathbb{D}$ such that $|w_0| < 1 - \varepsilon < 1$, and let γ be the curve given by $|F(z)| = 1 - \varepsilon$ and $\Gamma = F \circ \gamma$. Then Γ is of course the circle $|w| = 1 - \varepsilon$ (but possibly traversed multiple times). Since $F(z) = 0$ has only one solution in Ω, by the argument principle (rather the index version of it), we see that

$$\int_\gamma \frac{F'(z)}{F(z) - w_0}\, dz = n(\Gamma, w_0) = n(\Gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{F'(z)}{F(z)} = 1.$$

This shows that $F(z) = w_0$ has a unique solution for any $w_0 \in \mathbb{D}$, and hence shows both surjectivity and injectivity of $F(z)$. Finally, we can compose F with a rotation to ensure that $F'(z_0)$ is real and positive.

The reader is encouraged to read a detailed and complete account of a proof of the Riemann mapping theorem along the lines of Riemann's original "proof" from `https://link.springer.com/article/10.1186/s40627-016-0009-7`.

We end with a vastly more general Riemann mapping theorem. Recall that a domain Ω is said to be n-connected, if $\overline{\mathbb{C}} \setminus \Omega$ has n connected components. For instance, Ω is 1-connected if and only if it is simply connected.

Theorem 25.3.1. *Let Ω be an n-connected domain in $\mathbb{C}$ such that no component of $\overline{\mathbb{C}} \setminus \Omega$ consists of a single point. Then there exists a biholomorphism $F : \Omega \to \mathcal{D}$, where*

1. *$\mathcal{D} = \mathbb{D}$ if $n = 1$,*
2. *$\mathcal{D}$ is an annulus $A_{r,R}(0) = \{z \in \mathbb{C} \mid r < |z| < R\}$ if $n = 2$,*
3. *$\mathcal{D}$ is $A_{r,R}(0) \setminus \cup_{i=1}^{n-2} Supp(\gamma_i)$ if $n > 2$, where γ_i are concentric arcs lying on circles $|\zeta| = r_i$, with $r < r_i < R$.*

A historical note. One of the first pushes towards making Dirichlet's problem and harmonic functions a part of mainstream mathematics arose out of Riemann's (faulty) proof of his theorem on conformal mappings into the disc. In fact the first systematic and rigorous study of the Dirichlet problem was to fix the error in Riemann's original proof. By the turn of the twentieth century, the vastly more general uniformization theorem had also been proved using similar methods, and elliptic partial differential equations and calculus of variations (of which the above problem is the simplest example) had become a part of mainstream mathematics. So much so that, they were the subject of two of Hilbert's problems in his 1900 address to the Congress of mathematicians. Finally, this whole circle of ideas of using solutions of partial differential equations to say something about the topology continues to be a fruitful area of mathematical research. Some of the spectacular successes include Hodge theory (characterizing cohomology groups via harmonic forms) and Donaldson theory (characterising smooth structures on four manifolds via solving Yang-Mills equations, which are a non-linear generalization of Dirichlet's problem).

Part IV

Special functions

Lecture 26

Gamma and Zeta functions

In this lecture, we study two important functions, namely the Gamma function and the Zeta function. Each function is initially defined in a certain region in the complex plain; the Gamma function by an integral and the Zeta function by an infinite series. Both the functions are then extended to obtain meromorphic functions on the entire complex plain. The key techinical tool is the principal of analytic continuation.

26.1 Revisiting the principle of analytic continuation

Recall that the principal of analytic continuation says that if two functions agree on some open set, then they must agree on the entire connected component containing the open set. Given a holomorphic function $f : \Omega \to \mathbb{C}$ it is natural to ask for the biggest possible open set Ω' containing Ω on which f has a holomorphic extension. It is in fact much more natural to ask for *meromorphic* extensions. So we pose the following question.

Question 26.1.1. *Given $f : \Omega \to \mathbb{C}$ holomorphic, what is the biggest Ω' containing Ω such that there exists a meromorphic function $F : \Omega' \to \overline{\mathbb{C}}$ such that*

$$F_{|\Omega} = f.$$

Is the extension unique.

The uniqueness part is answered in the affirmative by the following extension of the principle of analytic continuation to meromorphic functions.

Lemma 26.1.2. *Let Ω be a connected open set, and $f, g : \Omega \to \overline{\mathbb{C}}$ be meromorphic functions with poles at isolated sets S_f and S_g respectively. Let $S = S_f \cup S_g$. Suppose there is a sequence of pairwise distinct points $z_n \in \Omega \setminus S$ such that $z_n \to z_0 \in \Omega$ and $f(z_n) = g(z_n)$ for all n, then $S_f = S_g = S$, the poles of f and g are of the same order and the Laurent series expansions match up, and $f = g$ as meromorphic functions.*

Proof. Since $f, g \in \mathcal{O}(\Omega \setminus S)$, by the usual principle of analytic continuation,

$$f\Big|_{\Omega\setminus S} = g\Big|_{\Omega\setminus S}.$$

To complete the proof, we need to show that $S_f = S_g = S$. Let $p \in S_f$. Then there is an $\varepsilon > 0$ such that $D_\varepsilon(p)$ does not contain any point of S apart from p. That is, $D_\varepsilon(p) \setminus \{p\} \subset \Omega \setminus S$, and hence $f(z) = g(z)$ for all $z \in D_\varepsilon(p) \setminus \{p\}$. Since p is a pole of f, $f(z) \to \infty$ as $z \to p$, and hence $g(z) \to \infty$ as $z \to p$. This shows that $S_f \subset S_g$. By symmetry we get the reverse inclusion and this proves that $S_f = S_g$. Since f

and g are equal in the complement, it is also clear that the poles will be of the same order, and the Laurent series expansions match up. Hence $f = g$ as meromorphic functions. □

Example 26.1.3. *Consider the function defined by the power series*

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

This defines a holomorphic function on the unit disc $\mathbb{D}$. Moreover, by the summation formula for geometric series, the function is precisely $f(z) = (1-z)^{-1}$. On the other hand, the function $F(z) = (1-z)^{-1}$ is meromorphic on the entire complex plane with a single pole of order one at $z = 1$. So $F(z)$ defines a meromorphic extension of $f(z)$ to $\mathbb{C}$ and an analytic continuation to $\mathbb{C} \setminus \{1\}$. Note that $F(-1) = 1/2$, since $F(z)$ is an analytic continuation of $f(z)$, naively (and rather thoughtlessly) one might be tempted to plug in $z = -1$ in the power series and write

$$1 - 1 + 1 - 1 + \cdots \text{``}=\text{''} \frac{1}{2}. \tag{26.1}$$

Of course as stated the above equality is meaningless since the series on the left is a divergent series. The correct way to make sense out of this is by analytic continuation. There are other ways to make sense out of the series, for instance by using Cesaro summability. G.H Hardy wrote an entire book on divergent series, and this was quite a hot topic for research in Britain in the early 20th century.

26.2 Analytic continuation of the Mellin transform

In this section we answer the above question completely in the following model case, for the (truncated) *Mellin transform.* Recall that a function $\varphi : [-1, 1] \to \mathbb{R}$ is called smooth, if derivatives of all orders exist on $(-1, 1)$, and are continuous on $[-1, 1]$.

Theorem 26.2.1. *Let $\varphi(t)$ be a smooth function on the unit interval $[-1, 1]$, and consider the function*

$$f(z) = \int_0^1 x^z \varphi(x)\, dx,$$

where we use the principal branch, namely $x^z = e^{z \ln x}$. Then

1. *$f(z)$ defines a holomorphic function on $Re(z) > -1$.*
2. *$f(z)$ admits a unique extension as a meromorphic function on $\mathbb{C}$ with at most simple poles at the negative integers with*
$$\mathrm{Res}_{z=-n} f(z) = \frac{\varphi^{(n-1)}(0)}{(n-1)!},$$
where $\varphi^{(k)}(0)$ as usual denotes the k^{th} derivative.

Note that the usual Mellin transform involves the integral over all of $(0, \infty)$, and hence we call the above a truncated Mellin transform (possibly non standard terminology).

Proof. To prove this, we first show that the integral is absolutely convergent for $\mathrm{Re}(z) > -1$. We only need to worry about convergence near $x = 0$. It is easy to see that $|x^z| = |e^{z \ln x}| = x^{\mathrm{Re}(z)}$, and so if $|\varphi(x)| < M$ on $[0, 1]$, then $|x^z \varphi(x)| < M x^{\mathrm{Re}(z)}$, which, by the p-test, is integrable near $x = 0$ if $\mathrm{Re}(z) > -1$. So by the comparison theorem, $f(z)$ is well defined for $\mathrm{Re}(z) > -1$. To show that it is holomorphic, we look at the difference quotient. Note that x^z is holomorphic for all $x \in (0, 1)$ with derivative

$$\frac{dx^z}{dz} = x^z \ln x.$$

We claim

$$f'(z) = \int_0^1 x^z \varphi(x) \ln x \, dx.$$

To prove this, it is enough to show the following.

Claim. For all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\Big| \frac{f(z+h) - f(z)}{h} - \int_0^1 x^z \varphi(x) \ln x \, dx \Big| < \varepsilon,$$

whenever $|h| < \delta$.

Note that

$$\begin{aligned} \Big| \frac{f(z+h) - f(z)}{h} - \int_0^1 x^z \varphi(x) \ln x \, dx \Big| &= \Big| \int_0^1 (\frac{x^{z+h} - x^z}{h} - x^z \ln x) \varphi(x) \, dx \Big| \\ &\leq \int_0^1 |x^z \varphi(x)| \Big| \frac{e^{h \ln x} - 1}{h} - \ln x \Big| \, dx. \end{aligned} \tag{26.2}$$

By the power series expansion of e^z, we have that

$$\frac{e^{h \ln x} - 1}{h} - \ln x = h(\ln x)^2 \sum_{n=0}^{\infty} \frac{h^n (\ln x)^n}{(n+2)!}.$$

Now the infinite series on the right is convergent. In fact we have that

$$\Big| \sum_{n=0}^{\infty} \frac{h^n (\ln x)^n}{(n+2)!} \Big| \leq \sum_{n=0}^{\infty} \Big| \frac{h^n (\ln x)^n}{n!} \Big| < e^{|h| |\ln x|} = x^{-|h|},$$

where we used the fact that $|\ln x| = \ln(1/x)$ since $x \in (0,1)$. From the power series expansion above, we then have the estimate

$$\Big| \frac{e^{h \ln x} - 1}{h} - \ln x \Big| \leq h(\ln x)^2 x^{-|h|} \leq C_\eta h x^{-|h| - \eta},$$

for any $\eta > 0$ and $|h|$ small. Here C_η is a constant that possibly depends on η but is independent of h. This estimate holds because $\lim_{x \to 0} x^\eta (\log x)^2$ for any $\eta > 0$.

Now suppose $M = \sup_{x \in [0,1]} |\varphi(x)|$, then going back to the integral estimate in (26.2) we see that if $|h| < Re(z)$, then

$$\int_0^1 |x^z \varphi(x)| \Big| \frac{e^{h \ln x} - 1}{h} - \ln x \Big| \, dx \leq h C_\eta M \int_0^1 x^{Re(z) - |h| - \eta} \, dx.$$

We choose $\eta > 0$ small enough $Re(z) - 2\eta > -1$. Suppose $|h| < \eta$, then

$$\int_0^1 x^{Re(z) - |h| - \eta} \, dx \leq \int_0^1 x^{Re(z) - 2\eta} \, dx := A_\eta.$$

Note that A_η of course depends on z, but z is fixed throughout this argument, and hence we hide the dependence of A_η on z. So putting all of this together with (26.2)

$$\Big| \frac{f(z+h) - f(z)}{h} - \int_0^1 x^z \varphi(x) \ln x \, dx \Big| \leq M A_\eta C_\eta |h| < \varepsilon,$$

if $|h| < \varepsilon/MC_\eta A_\eta$. So the claim is proved by choosing

$$\delta = \min\Big(\frac{\varepsilon}{MC_\eta A_\eta}, \eta\Big).$$

Next, we show that a meromorphic extension exists on all of $\mathbb{C}$. For any given integer $N > 0$, we can write the Taylor expansion of φ around $x = 0$ as

$$\varphi(x) = \sum_{j=0}^{N-1} \frac{\varphi^j(0)}{j!} x^j + E_N(x),$$

where $E_N(x)$ is a smooth function on $[-1, 1]$ such that $|E_N(x)| \le C|x|^N$ for some constant $C > 0$. So for $\mathrm{Re}(z) > -1$,

$$\begin{aligned} f(z) &= \sum_{k=0}^{N-1} \int_0^1 \frac{\varphi^k(0)}{k!} x^{k+z} + \int_0^1 E_N(x) x^z \, dx \\ &= \sum_{k=0}^{N-1} \frac{\varphi^k(0)}{k!} \cdot \frac{1}{z+k+1} + \int_0^1 E_N(x) x^z \, dx \end{aligned}$$

Since $|E_N(x)| \le C|x|^N$, $\int_0^1 E_N(x)x^z \, dx$ is convergent for $\mathrm{Re}(z+N) > -1$, and hence defines a holomorphic function on $\mathrm{Re}(z) > -(N+1)$ by the first part. On the other hand, putting $k+1 = j$, the first term on the right defines a meromorphic function on all of $\mathbb{C}$ with simple poles at $z = -j$, $j = 1, 2, \cdots, N$ with residue $\varphi^{j-1}(0)/(j-1)!$. So the right hand side defines a meromorphic function $f_N(z)$ on $\mathrm{Re}(z) > -(N+1)$, which restricts to $f(z)$ on $\mathrm{Re}(z) > -1$. By uniqueness of meromorphic extensions, for $M > N$, the restriction of f_N and f_M to $\mathrm{Re}(z) > -(N+1)$ are equal, and hence letting $N \to \infty$, f_N converges to a meromorphic function on all of $\mathbb{C}$ with simple poles at $z = -n$ with residue $\varphi^{(n-1)}(0)/(n-1)!$. □

Remark 26.2.1. Note that if $\varphi^{(n-1)}(0) = 0$ for some n, then the residue of $f(z)$ at $z = -n$ would be zero. And since $z = -n$ can at most be a simple pole, this would imply that $z = -n$ is in fact not a pole at all, but is a removable singularity.

26.3 The Gamma function

The Gamma function $\Gamma(s)$ is defined on $\mathrm{Re}(s) > 0$ as the Mellin transform of e^{-x}. That is,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx.$$

It is easy to see (exercise!) that the integral is convergent on $\mathrm{Re}(s) > 0$ and hence is well defined and finite on this region. Note also that $\Gamma(1) = 1$.

Theorem 26.3.1. *With $\Gamma(s)$ defined as above for $\mathrm{Re}(s) > 0$, we have the following.*

1. *There exists a meromorphic extension of $\Gamma(s)$ on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \cdots$ with residue*

$$\mathrm{Res}_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}.$$

2. *(Functional equation) For $s \neq -n$, $n = 0, 1, 2, \cdots$,*

$$\Gamma(s+1) = s\Gamma(s),$$

and hence for integers n, $\Gamma(n+1) = n!$.

Proof. For $\mathrm{Re}(s) > 0$, we can write

$$\Gamma(s) = \int_0^1 e^{-x} x^{s-1}\, dx + \int_1^\infty e^{-x} x^{s-1}\, dx.$$

The second integral is convergent for all s, and hence defines an entire function by similar arguments as in the proof of the first part of Theorem 26.2.1. The first integral, by Theorem 26.2.1 (applied to $s - 1 = z$), can be extended to a meromorphic function with simple poles at $s = -n$, for $n = 0, 1, 2, \cdots$. The residue also comes from the first term. Applying theorem 26.2.1 with $\varphi = e^{-x}$, and $s - 1 = z$, we see that the residue at $s = -n$ (or $z = -(n+1)$) is given by

$$\mathrm{Res}_{s=-n}\Gamma(s) = \frac{1}{n!} \cdot \frac{d^n}{dx^n}\Big|_{x=0} e^{-x} = \frac{(-1)^n}{n!}.$$

We first prove part (2) when $\mathrm{Re}(s) > 0$. In this range we can use the integral formula,

$$\begin{aligned}
\Gamma(s+1) &= \int_0^\infty e^{-x} x^s\, dx \\
&= -\int_0^\infty x^{s-1}\, de^{-x} \\
&= e^{-x} x^s \Big|_{x=0}^{x=\infty} + \int_0^\infty e^{-x}\, dx^s \\
&= s\int_0^\infty e^{-x} x^{s-1}\, dx = s\Gamma(s).
\end{aligned}$$

To prove the equality over the entire complex plane, consider $F(s) = \Gamma(s+1) - s\Gamma(s)$.

Claim. $F(s)$ extends to an entire function.

Assuming this, since $F(s) \equiv 0$ on $Re(s) > 0$, by the principle of analytic continuation, $F(s)$ is identically zero, and we are done.

Proof of the Claim. Clearly $F(s)$ is holomorphic everywhere except possibly at the negative integers and $s = 0$. Moreover, F can only have simple poles at these points. At $s = 0$, $\Gamma(s+1)$ is holomorphic, and so is $s\Gamma(s)$ since Γ has a simple pole at $s = 0$. So F can have pole at only negative integers. To rule this out, let us calculate the residue. For $n \in \mathbb{N}$,

$$\mathrm{Res}_{s=-n}\Gamma(s+1) = \mathrm{Res}_{z=-(n-1)}\Gamma(z) = \frac{(-1)^{n-1}}{(n-1)!}.$$

On the other hand,

$$\mathrm{Res}_{s=-n}s\Gamma(s) = \lim_{s\to -n} s(s+n)\Gamma(s) = -n\mathrm{Res}_{s=-n}\Gamma(s) = -\frac{(-1)^n}{(n-1)!},$$

and hence $\mathrm{Res}_{s=-n}F(s) = 0$. Since $s = -n$ is a simple pole, this implies that $s = -n$ is a removable singularity. $\square$

Theorem 26.3.2 (Euler reflection formula)**.** *The Gamma function satisfies the identity*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Proof. By analytic continuation, it enough to prove the identity for $s \in \mathbb{R} \cap (0,1)$. Recall that in Example-2 in Lecture-21, we proved the following identity: For $0 < a < 1$,

$$\int_0^\infty \frac{v^{-s}}{1+v}\, dv = \frac{\pi}{\sin \pi s}.$$

We now rewrite

$$\Gamma(1-s) = \int_0^\infty e^{-x} x^{-s}\, dx = t \int_0^\infty e^{-vt} (vt)^{-s}\, dv,$$

where we made the change of variables $x = vt$. Note that the above formula for $\Gamma(1-s)$ is valid for all $t \geq 0$. Now we compute

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t} t^{s-1} \Gamma(1-s)\, dt \\
&= \int_0^\infty e^{-t} t^{s-1} \Big(t^{1-s} \int_0^\infty e^{-vt} v^{-s}\, dv \Big)\, dt \\
&= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s}\, dt\, dv \\
&= \int_0^\infty \frac{v^{-s}}{1+v}\, dv \\
&= \frac{\pi}{\sin \pi s}.
\end{aligned}$$

□

26.4 The Riemann Zeta function

For $Re(s) > 1$, we define the *zeta function* by the infinite series

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s},$$

where as before, $n^s = e^{s \ln(n)}$. This time by comparison test for series, since $|n^s| = n^{\mathrm{Re}(s)}$, this is a convergent series for $\mathrm{Re}(s) > 1$, and by the Weierstrass M-test defines a holomorphic function on $Re(s) > 1$.

Theorem 26.4.1. *The zeta function above satisfies the following properties*

1. *For $Re(s) > 1$, we have the identity,*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t - 1} t^{s-1}\, dt.$$

That is, the zeta function (upto a factor of $\Gamma(s)$) is the Mellin transform of $(e^t-1)^{-1}$.

2. *$\zeta(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $z = 1$ and holomorphic on $\mathbb{C} \setminus \{1\}$. Moreover, we have*

$$\mathrm{Res}_{z=1} \zeta(s) = 1.$$

3. *$\zeta(s) = 0$ whenever $s = -2n$ for some $n \in \mathbb{N}$. These are the so called* trivial zeroes *of the zeta function.*

The Riemann hypothesis conjectures that in fact *all* other zeroes (the so called non-trivial zeroes) lie on the line $Re(s) = 1/2$.

Proof. For the first identity, we observe that

$$\frac{\Gamma(s)}{n^s} = \frac{1}{n}\int_0^\infty e^{-x}(x/n)^{s-1}\,dx = \int_0^\infty e^{-nt}t^{s-1}\,dt.$$

where we changed variables $x = nt$ in the second equality. Summing up we obtain

$$\zeta(s)\Gamma(s) = \int_0^\infty \Big(\sum_{n=1}^\infty e^{-nt}\Big)t^{s-1}\,dt = \int_0^\infty \Big(\frac{1}{1-e^{-t}} - 1\Big)t^{s-1}\,dt = \int_0^\infty \frac{1}{e^t - 1}t^{s-1}\,dt.$$

Now let $f(s) = \int_0^\infty \frac{1}{e^t-1}t^{s-1}\,dt$. We can write

$$f(s) = \int_0^\infty \varphi(t)t^{s-2},$$

where $\varphi(t) = t/(e^t - 1)$. From the Tayor expansion of e^t we can see that $\varphi(t)$ is smooth on $[-1, 1]$, and so by Theorem 26.2.1 with $z = s-2$, $f(s)$ is holomorphic on $\mathrm{Re}(s-2) > -1$ or equivalently on $\mathrm{Re}(s) > 1$, and admits a meromorphic extension with simple poles at $\mathrm{Re}(s) = 1, 0, -1, -2, \cdots$. But $\Gamma(s)$ itself has simple poles at $s = 0, -1, -2, \cdots$, and hence the meromorphic extension $\zeta(s) = f(s)/\Gamma(s)$ will have a pole only at $s = 1$. When $s = 1$, $s - 2 = -1$, and so from Theorem 26.2.1, $\mathrm{Res}_{s=1}f(s) = \varphi(0)$. But

$$\frac{t}{e^t - 1} = \frac{t}{t + t^2/2 + \cdots} = \frac{1}{1 + t/2 + \cdots},$$

and so $\varphi(0) = 1$, and hence $\mathrm{Res}_{s=1}f(s) = 1$. But since $\Gamma(1) = 1$, we then have that $\mathrm{Res}_{s=1}\zeta(s) = 1$. It follows from Problem-7 in Assignment-4 that

$$\varphi(t) = \frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n=1}^\infty (-1)^{n-1}\frac{B_n}{(2n)!}z^{2n},$$

where B_n is the n^{th} Bernoulli number. In particular, $\varphi^{(2n+1)}(0) = 0$, for all $n = 1, 2, \cdots$, and hence

$$\mathrm{Res}_{s=-2n}f(s) = \mathrm{Res}_{z=-2n-2}f(z) = \frac{\varphi^{(2n+1)}}{(2n+1)!} = 0.$$

So $f(s)$ has a removable singularity at $s = -2n$. But since $\Gamma(s)$ has a simple pole at $s = -2n$, it follows that $\zeta(-2n) = 0$ for all $n \in \mathbb{N}$. □

Example 26.4.1. *Let us calculate $\zeta(0)$. First, note that if two functions $f(z)$ and $g(z)$ have a simple pole at $z = 0$, then $h(z) = f(z)/g(z)$ has a removable singularity at $z = 0$. Moreover the extension, which we also denote by $h(z)$, satisfies $h(0) = Res_{z=0}f(z)/Res_{z=0}g(z)$. We apply this to the meromorphic extension of*

$$f(s) = \int_0^\infty \frac{1}{e^t - 1}t^{s-1}\,dt,$$

and $g(s) = \Gamma(s)$. For $\mathrm{Re}(s) > 1$, we re-write

$$f(s) = \int_0^1 \frac{t}{e^t - 1}t^{s-2}\,dt + \int_1^\infty \frac{1}{e^t - 1}t^{s-1}\,dt.$$

The second part, by the argument above is an entire function. So the residue comes from the meromorphic extension of the first integral. We apply Theorem 26.2.1 with $\varphi(t) = t/(e^t - 1)$ *and* $z = s - 2$. *To find the residue at* $s = 0$ *we apply the second part of Theorem 26.2.1 (with* $n = 2$, *since* $z = -2$ *is the same as* $s = 0$*)*

$$\mathrm{Res}_{s=0} f(s) = \frac{\varphi'(0)}{1!}.$$

But we can write down the Taylor expansion of

$$\varphi(t) = \frac{t}{e^t - 1} = \frac{t}{t + t^2/2 + \cdots} = \frac{1}{1 + t/2 + \cdots} = 1 - \frac{t}{2} + \textit{higher order terms},$$

and so $\varphi'(0) = -1/2$. *On the other hand the residue of the Gamma function is given by Theorem 26.3.1, and we see that* $\mathrm{Res}_{s=0}\Gamma(s) = 1$, *and putting everything together, we obtain that*

$$\zeta(0) = -\frac{1}{2}.$$

Example 26.4.2. *By working a bit harder, we can compute* $\zeta(-1)$. *Once again applying Theorem 26.2.1, this time with* $n = 3$ *(since* $s = -1$ *is* $z = -3$*) we have*

$$\mathrm{Res}_{s=-1} f(s) = \frac{\varphi^{(2)}(0)}{2!}.$$

Computing the next term in the Taylor expansion,

$$\varphi(t) = \frac{1}{1 + t/2 + t^2/6 + O(t^3)} = 1 - \frac{t}{2} - \frac{t^2}{6} + \frac{t^2}{4} + O(t^3) = 1 - \frac{t}{2} + \frac{t^2}{12} + O(t^3),$$

and so $\mathrm{Res}_{s=-1} f(s) = 1/12$. *On the other hand,* $\mathrm{Res}_{s=-1}\Gamma(s) = (-1)^1/1 = -1$, *and hence*

$$\zeta(-1) = -\frac{1}{12}.$$

Remark 26.4.3. Recall that for $Re(s) > 1$,

$$\zeta(s) = \sum_{n=0}^{\infty} n^{-s}.$$

We can then formally (and formally is the key word here) “plug in" $s = -1$, and write

$$1 + 2 + \cdots \text{“} = \text{”} \zeta(-1) = -\frac{1}{12}. \tag{26.3}$$

Of course this does not make any “real" sense since $1 + 2 + 3 \cdots$ is a divergent series and $\zeta(-1)$ is a finite number since $\zeta(s)$ is holomorphic at $s = -1$. The equation (26.3) is found in one of Ramanujan's notebooks. Apparently Ramanujan had stumbled upon a way of summing up certain divergent series, and being unaware of analytic continuation, used the rather crude notation that seems to suggest that the sum of all natural numbers is not only a finite number but also negative!

There is a particularly misleading video posted by numpherphile, an otherwise decent youtube math channel, on this “astounding" identity - `https://www.youtube.com/watch?v=w-I6XTVZXww`, which gives a “derivation" of the above “identity" using other misleading identities such as (26.1). Following the barrage of criticism that this video received, other channels made better videos. For instance this video - `https://www.youtube.com/watch?v=jcKRGpMiVTw` by mathlogger clarifies the identity using Cesaro summability. There is also a very beautiful video on the analytic continuation of the zeta function by 3Blue1Brown - `https://www.youtube.com/watch?v=sD0NjbwqlYw`.

We end this lecture, with the beautiful functional equation of Riemann's which we state without proof.

Theorem 26.4.2. *[Functional equation]*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

Example 26.4.4. *As an application we can calculate* $\zeta(2)$*. Namely,*

$$\zeta(2) = 4\pi\zeta(-1)\lim_{s\to 2}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s).$$

We can compute the limit directly by using the Residue of $\Gamma(1-s)$ *at* $s=2$*. Alternately, by Theorem 26.3.2*

$$\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = \frac{\sin \pi s}{2\cos(\pi s/2)}\Gamma(1-s) = \frac{\pi}{2\Gamma(s)\cos \pi s/2},$$

and so

$$\zeta(2) = -\frac{\pi^2}{3\Gamma(2)(-2)} = \frac{\pi^2}{6}.$$

This gives a third derivation of the famous Basel-Euler identity:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

and is a good place to end.

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Lecture 27

Prime numbers and the Riemann hypothesis

The Riemann hypothesis is one of the most famous unsolved problems in all of mathematics. In the previous chapter we saw that the zeta function has zeroes at all the negative even integers $s = 0, -2, -4, \cdots$. Any zero that is not one of these, is called the a non-trivial zero of the Riemann zeta function. The following is Riemann's conjecture:

Conjecture 27.0.1. *All the non-trivial zeroes of the zeta function lie on the line*

$$s(t) = \frac{1}{2} + it.$$

To explain the significance of the Riemann hypothesis, we need to introduce the prime counting function. Recall that a positive integer p is called a prime if it's only positive factors are 1 and p. We then let

$$\pi(x) := \text{ number of primes less than or equal to } x.$$

For $x > 1$, we also let

$$Li(x) = \text{p.v.} \int_0^x \frac{dt}{\log t} := \lim_{\varepsilon \to 0^+} \Big(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \Big) \frac{dt}{\log t}.$$

Theorem 27.0.2. *Riemann hypothesis implies the following estimate on $\pi(x)$:*

$$\pi(x) = Li(x) + O(\sqrt{x} \log x).$$

What does the zeta function have to do with the primes? The following observations goes back to Euler. Indeed, what we call the Riemann zeta function, was first studied in detail by Euler himself.

Proposition 27.0.3. *For $\Re(s) > 1$,*

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

The proof is an elementary consequence of the unique factorisation theorem, and is left as an exercise.

27.1 The prime number theorem

Theorem 27.1.1.

$$\lim_{x\to\infty} \frac{\pi(x)}{x\log x} = 1.$$

Lecture 28

Elliptic functions

28.1 Doubly periodic functions

We have already encountered many holomorphic functions that *periodic*. A function $f : \mathbb{C} \to \mathbb{C}$ is periodic if there exists a $\omega \neq 0$ such that $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}$. Examples include most notably the exponential function which is periodic with a period of $2\pi\sqrt{-1}$. In fact, as we saw earlier, this can be taken as a definition of the real number π. In this chapter we will be interested in functions $f : \mathbb{C} \to \mathbb{C}$ that are *doubly periodic*, that is,

$$f(z + \omega_1) = f(z) = f(z + \omega_2)$$

for some $\omega_1, \omega_2 \in \mathbb{C}^*$ and all $z \in \mathbb{C}$. At the end of this lecture, we shall give an application of such functions to the evaluation of certain real valued integrals that arise in computing lengths of arcs on ellipses - hence the name elliptic functions. But first, we note that the study of such doubly periodic functions falls naturally into two cases - either $\omega_2 = \lambda\omega_1$ for some $\lambda \in \mathbb{R}$ or not. We claim that the first case is not very interesting. Indeed we have the following:

Proposition 28.1.1. *Let f be a meromorphic function on $\mathbb{C}$. Suppose there exists $\omega \in \mathbb{C}^*$ and $\lambda \in \mathbb{R}^*$ and $\lambda \neq 1$ such that*

$$f(z + \omega) = f(z + \lambda\omega) = f(z),\ \forall z \in \mathbb{C},$$

then we have the following dichotomy:

1. *If $\lambda \in \mathbb{Q}$, then f is periodic with a single period.*

2. *If λ is irrational, then f is a constant.*

In view of the above we restrict to the second case, that is, ω_1 and ω_2 are linearly independent as vectors in $\mathbb{R}^2$. It is then clear that any *holomorphic* function which is doubly periodic will be forced to be a constant by Liouville's theorem. Indeed the values that f takes are determined by the *fundamental parallelogram* of f:

$$P_0 = \{a\omega_1 + b\omega_2 \mid 0 \leq a < 1,\ 0 \leq b < 1\}.$$

So f itself is bounded, and hence constant. So the upshot is that to get something interesting, we need to consider doubly periodic meromorphic functions whose periods are linearly independent. Such functions are called *elliptic functions*. Any translate $P = P_0 + h$, $h \in \mathbb{C}$ is called *a* period parallelogram of f.

28.2 The canonical basis

Suppose now f is an elliptic function with bases ω_1 and ω_2. Then clearly for any integers m and n, $m\omega_1 + n\omega_2$ is a period for f. Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the lattice generated by ω_1 and ω_2.

28.3 Poles and zeroes

Proposition 28.3.1. *Let f be an elliptic function with fundamental period $P_0 = \langle 1, \tau \rangle$. Then the number of poles and zeroes of f in P_0 (counted with multiplicity) are equal. We call this number the order ν_f of the elliptic function. Consequently, for any $c \in \mathbb{C}$, the equation*

$$f(z) = c$$

has exactly ν_f solutions when counted with multiplicity.

Proof. Without loss of generality (for instance by perturbing P_0 to a neighbouring period parallelogram) we may assume that there are no zeroes or poles on ∂P_0. If we denote the number of poles and zeroes by N_∞ and N_0, by the residue theorem we have

$$\frac{1}{2\pi i} \int_{\partial P_0} \frac{f'(z)}{f(z)}\, dz = N_0 - N_\infty.$$

□

A natural question is what integers can appear as orders elliptic functions? We have the following elementary observation.

Proposition 28.3.2. *For any elliptic function f, $\nu_f \geq 2$.*

28.4 Weierstrass $\wp$ function

A natural question now arises on the existence of such functions. In this section we give an explicit (and very classical) construction of an elliptic function of order two.

Lecture 29

Modular forms

29.1 The modular group

Part V

Analytic continuation

Lecture 30

Analytic continuation along curves

30.1 An example

Suppose we have an analytic function defined in some disc D, which we assume to be centred at $z_0 = 0$. Suppose we want to extend it to an analytic function at some $z_1 \notin D$. A naive idea would be to join z_1 to the origin by a path, say a straight line, and to attempt to extend the analytic function along the the straight line by using power series expansions centred at points on the straight line. If one is lucky, this procedure can be carried out until one reaches the desired point z_1. To illustrate this, consider analytic function $f : \mathbb{D} \to \mathbb{C}$ on the standard unit disc:

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

Suppose we want to extend this to the point $z = -1$. We consider a point close to $z = -1$. In this case $z_0 = -1/2$ will work. We try to expand the function $f(z)$ as a power series around $z = -1/2$. To do this, we recall that whenever $|z| < 1$,

$$f(z) = \frac{1}{1-z}.$$

In a neighbourhood of $z_0 = -1/2$ we write this as

$$\begin{aligned} f(z) &= \frac{1}{3/2 - (z + 1/2)} \\ &= \frac{2}{3}\frac{1}{1-w}, \end{aligned}$$

where $w = \frac{2}{3}(z + 1/2)$. But then as long as $|w| < 1$, we can expand the above and obtain

$$f(z) = \frac{2}{3}\sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1}\left(z + \frac{1}{2}\right)^n.$$

The right hand side is a power series which converges as long as

$$\left|z + \frac{1}{2}\right| < \frac{3}{2}.$$

Since $z = -1$ satisfies the above inequality we have the required extension of our original function. This is not surprising, since we already know that $f(z) = (1-z)^{-1}$, and the latter function is clearly analytic at -1 (indeed it is analytic everywhere except $z = 1$). But one can imagine that this process holds good for many other analytic functions.

On the other hand, note that there cannot be any such extension at $z = 1$. For if there were such an extension, say $F(z)$, then

$$F(1) = \lim_{x+0i \to 1} F(x + 0i) = \lim_{x \to 1} \frac{1}{1 - x} = \infty.$$

30.2 Formalizing the main ideas

30.3 The monodromy principle

30.4 A (very) surprising application - Picard's little theorem

Having set-up the beautiful theory of modular forms, and in particular having constructed an elliptic modular function, we now combine with the monodromy principle to obtain an immediate and stunning consequence. Recall that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$. This follows from the Casorati-Weierstrass theorem by analyzing the singularity at infinity. We have the following vast generalization.

Theorem 30.4.1 (Picard's little theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Then $f(\mathbb{C})$ can miss at most one point in $\mathbb{C}$. That is, if there exist $a \neq b$ such that $f(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$, then f is a constant.*

Of course we do have entire functions that miss exactly one point - the function e^z for instance. So this theorem is sharp.

Proof. □

Lecture 31

An exposition on Riemann surfaces

Appendix A

Problems

1. Find all possible solutions to the following equations:
   (a) $z^4 = i + 1$.
   (b) $z^n = 1$.
2. Determine all values of 2^i, i^i and $(-1)^{2i}$.
3. For what values of z is e^z equal to 2, $i/2$?
4. Show that there are complex numbers z satisfying
$$|z-a| + |z+a| = 2|c|$$
   if and only if $|a| \leq |c|$. If this condition is satisfied, what are the smallest and largest values of $|z|$.
5. (a) Let $z, w \in \mathbb{C}$ such that $z\bar{w} \neq 1$. Then prove that if $|z|, |w| < 1$, then
$$\left|\frac{w-z}{1-z\bar{w}}\right| < 1.$$
   Moreover
$$\left|\frac{w-z}{1-z\bar{w}}\right| = 1$$
   if and only if either $|z| = 1$ or $|w| = 1$.
   (b) Now, fix a $w \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, and consider the mapping $\phi_w : \mathbb{D} \to \mathbb{C}$,
$$\phi_w(z) := \frac{w-z}{1-z\bar{w}}.$$
   Prove that ϕ_w has the following properties:
      i. ϕ_w is a holomorphic map of $\mathbb{D}$ into itself.
      ii. ϕ_w interchanges 0 and w. That is, $\phi_w(0) = w$ and $\phi_w(w) = 0$.
      iii. ϕ_w is a biholomorphism. **Hint.** Compute $\phi_w \circ \phi_w$.
6. Prove that $f(z)$ is holomorphic if and only if $\overline{f(\bar{z})}$ is holomorphic. How are the two complex derivatives related?
7. Find the radius of convergence of the following power series.

(a) $\sum_{n=1}^{\infty}(\log n)^2 z^n$.

(b) $\sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}z^n$.

(c) The Bessel function:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+r)!}\left(\frac{z}{2}\right)^{2n}.$$

8. For what values of $z \in \mathbb{C}$ is the series

$$\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$$

convergent. In the regions that the series does converge, is it absolutely and/or uniformly convergent?

9. Prove that the function $f(x+iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin, and yet, using only the definition, prove that the function is not complex differentiable at 0.

10. Suppose f is holomorphic on a region Ω. If any one of the following holds:

(a) $Re(f)$ is a constant,

(b) $Im(z)$ is a constant,

(c) $|f|$ is a constant,

prove that f is a constant function.

11. Let $\Omega \subset \mathbb{C}$ be open and $f = u + iv : \Omega \to \mathbb{C}$ be a smooth map.

(a) If h is a smooth map in a neighborhood of $f(p)$, then prove that

$$\frac{\partial h \circ f}{\partial z}(p) = \frac{\partial h}{\partial w}(f(p)) \cdot \frac{\partial f}{\partial z}(p) + \frac{\partial h}{\partial \bar{w}} f(p) \cdot \frac{\partial \bar{f}}{\partial z}(p)$$
$$\frac{\partial h \circ f}{\partial \bar{z}}(p) = \frac{\partial h}{\partial w}(f(p)) \cdot \frac{\partial f}{\partial \bar{z}}(p) + \frac{\partial h}{\partial \bar{w}} f(p) \cdot \frac{\partial \bar{f}}{\partial \bar{z}}(p)$$

(b) If f is holomorphic, prove that $\det J_f(z) = |f'(z)|^2$ for all $z \in \Omega$.

(c) Using the usual inverse function theorem from multivariable calculus, prove the following holomorphic inverse function theorem: If f is holomorphic at p with $f'(p) \neq 0$, then there there exists open sets U and V around p and $f(p)$ respectively, such that $f : U \to V$ has a holomorphic inverse $f^{-1} : V \to U$.

12. Let $\Omega \subset \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$ a holomorphic function satisfying $|f(z) - 1| < 1$ in Ω. Prove that for any closed regular curve γ in Ω,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0.$$

13. If $P(z)$ is a polynomial and $R > 0$ prove that

$$\int_{|z-a|=R} P(z)d\bar{z} = -2\pi i R^2 P'(a).$$

14. (a) Let $f(z)$ be an entire function such that $Re(f(z)) > -a$ for all $z \in \mathbb{C}$. Prove that f is a constant.

(b) Let $f(z)$ be an entire function such that $f(z)$ is real whenever $|z| = 1$. Prove that f is a constant.

(c) Let $f(z)$ be an entire function such that $|f(z)| \leq |z|^n$. Prove that $f(z) = cz^n$ for some constant $c \in \overline{\mathbb{D}}$.

(d) Let f be an entire function, and $\alpha, C > 0$ be constants. Suppose

$$|f(z)| \leq C(1 + |z|^\alpha)$$

for all $z \in \mathbb{C}$. Prove that $f(z)$ is a polynomial of degree at most $\lfloor \alpha \rfloor$, where $\lfloor \alpha \rfloor$ denotes the greatest integer less than or equal to α.

(e) Find all entire functions such that $|f(z)| > 1$ whenever $|z| > 1$.

15. Compute the following integrals. The circles are traversed once in the anti-clockwise direction.

(a) $\int_{|z|=1} e^z z^{-n} \, dz$.

(b) $\int_{|z|=2} \frac{dz}{1+z^2}$.

(c) $\int_{|z|=r} \frac{|dz|}{|z-a|^2}$, where $|a| \neq r$. **Hint.** First prove that $|dz| = -ir dz/z$ on the circle $|z| = r$.

16. Is there a holomorphic function f on a domain Ω such that there exists and integer $N \in \mathbb{N}$ and a complex number $z \in \Omega$ such that $|f^{(n)}(z)| > n!n^n$ for all $n > N$? Can you formulate a sharper theorem of the same kind?

17. Let $\Omega \subset \mathbb{C}$ be open, and let $f : \Omega \to \Omega$ be a holomorphic function. Suppose there exists a point $z_0 \in \Omega$ such that

$$f(z_0) = z_0, \ f'(z_0) = 1.$$

Prove that f is linear, that is, there exists $a, b \in \Omega$ such that $f(z) = az+b$. **Hint.** First prove that one can assume without loss of generality that $z_0 = 0$. If f is not linear, then $f(z) = z+a_n z^n + O(z^{n+1})$ for some $a_n \neq 0$ and $n > 1$. If f_k denote f composed with itself k-times, prove that $f_k(z) = z + ka_n z^n + O(z^{n+1})$. Now, apply Cauchy inequalities and let $k \to \infty$ to conclude the desired result.

18. Suppose $f : \mathbb{D} := D_1(0) \to \mathbb{C}$ is holomorphic. Prove that the diameter of the image $d := \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ satisfies the estimate

$$d \geq 2|f'(0)|,$$

and that equality holds if and only if the function is linear.

19. Let γ be a simple closed curve, and $a \notin \mathrm{Supp}(\gamma)$. The prove that

$$n(\gamma, a) = \begin{cases} \pm 1, \ a \in int(\gamma) \\ 0, \ a \in ext(\gamma). \end{cases}$$

20. Are there holomorphic functions $f(z)$ and $g(z)$ in a neighbourhood of 0 such that for $n = 1, 2 \cdots$ we have

(a) $f(1/n) = f(-1/n) = 1/n^2$,

(b) $g(1/n) = g(-1/n) = 1/n^3$?

21. Determine all holomorphic functions on the unit disc $\mathbb{D} := D_1(0)$ such that

$$f''\Big(\frac{1}{n}\Big) + f\Big(\frac{1}{n}\Big) = 0,$$

for all $n = 2, 3, \cdots$.

22. For any open set $\Omega \subset \mathbb{C}$ and any complex valued function, the L^2-norm on U is defined to be

$$||f||_{L^2(\Omega)} := \Big(\int_\Omega |f(x,y)|^2\, dx\, dy\Big)^{1/2}$$

if it is finite. We then define the space $H(\Omega)$ by

$$H(\Omega) := \{f \in \mathcal{O}(\Omega) \mid ||f||_{L^2(\Omega)} < \infty\}.$$

We equip it with the norm $||\cdot||_{L^2(\Omega)}$ defined above.

(a) Let $z_0 \in \Omega$ and $\overline{D_r(z_0)} \subset \Omega$. For any $0 < s < r$, prove that

$$\sup_{z \in D_s(z_0)} |f(z)| \leq \frac{1}{\sqrt{\pi}(r-s)} ||f||_{L^2(D_r(z_0))}.$$

Hint. Write $f(z)$ in terms of an integral on $\partial D_r(z_0)$ by the Cauchy integral formula and use polar coordinates.

(b) Prove that if $\{f_n\}$ is a sequence in $H(\Omega)$ that is Cauchy with respect to the $L^2(\Omega)$ norm, then $f_n \to f$ compactly on Ω.

(c) Hence prove that $H(\Omega)$ with the metric

$$d(f,g) := ||f-g||_{L^2(\Omega)}$$

is a complete metric space.

23. Suppose that f and g are holomorphic in a region containing $\overline{D_1(0)}$. Suppose f has a simple zero at 0 (ie. the order is one), and has no other zero in $\overline{D_1(0)}$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

Show that if ε is sufficiently small, then

(a) f_ε has a unique zero (counted with multiplicity) in $\overline{D_1(0)}$.

(b) Moreover, if that unique zero is p_ε, then $\varepsilon \to p_\varepsilon$ is a continuous function.

24. Find the branch points (including infinity) for the following functions. Also give a branch cut that will make the function a single valued holomorphic function on the complement of the cut.

(a) $\sqrt{z-1}$

(b) $\log(z^2+z+1)$

25. (a) Let $f : \mathbb{D} \to \mathbb{C}$ such that the functions $g = f^2$ and $h = f^3$ are holomorphic on $\mathbb{D}$. Prove that f is holomorphic. Is the statement true if either g is not holomorphic or h is not holomorphic? If so, give a proof. Else give counterexamples.

(b) Either prove or provide a counter-example to the following statement: If f is a continuous function on a connected open subset Ω such that f^2 is holomorphic. Then so is f.

26. (a) Let Q_R be the rectangle with vertices $(-R,0)$, $(R,0)$, (R,R) and $(-R,R)$. Compute the integral
$$\int_{\partial Q_R} \frac{dz}{(1+z^2)^{n+1}},$$
where Q_R has the anti-clockwise orientation.

    (b) Use this to prove that
$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{(2n)!}{4^n(n!)^2}\pi.$$

27. In each of the cases below, classify the isolated singularities, and in case of poles, compute the order.
    (a) $\frac{z^2-\pi^2}{\sin^2 z}$.
    (b) $\frac{1-\cos z}{\sin z}$.
    (c) $\frac{1}{e^z-1} - \frac{1}{z-2\pi i}$.
    (d) $\frac{1}{\cos(1/z)}$.

28. If f and g are entire functions such that $|f(z)| < |g(z)|$ for $|z| > 1$, then show that $f(z)/g(z)$ is a rational function.

29. Let $R(z)$ be a rational function such that $|R(z)| = 1$ for $|z| = 1$.
    (a) Show that α is a zero or a pole of order m, if and only if $1/\bar{\alpha}$ is a pole or zero of order m respectively. **Hint.** First show that
$$M(z) = R(z)\overline{R\Big(\frac{1}{\bar{z}}\Big)}$$
is a rational function such that $M(z) = 1$ on $|z| = 1$.

    (b) Let $\{\alpha_j\}_{j=1}^N$ be zeroes and poles of $R(z)$ of order m_j in the unit disc $|z| < 1$. Here $m_j > 0$ is α_j is a zero and $m_j < 0$ is it is a pole. Define
$$B(z) = \Big(\frac{z-\alpha_1}{1-z\bar{\alpha}_1}\Big)^{m_1}\Big(\frac{z-\alpha_2}{1-z\bar{\alpha}_2}\Big)^{m_2}\cdots\Big(\frac{z-\alpha_N}{1-z\bar{\alpha}_N}\Big)^{m_N}.$$
Show that $R(z) = \lambda B(z)$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$

30. Recall that a function is said to have a removable singularity (resp. pole or essential singularity) at infinity if the function $f(1/z)$ has a removable singularity (resp. pole or essential singularity) at $z = 0$.
    (a) Show that an isolated singularity (including at infinity) of $f(z)$ cannot be a pole for $\exp(f(z))$.
    (b) In particular, if f is a non-constant entire function, then $\exp(f(z))$ has an essential singularity at infinity.

31. Let f and g be entire functions such that $h(z) = f(g(z))$ is a non-constant polynomial. Prove that both $f(z)$ and $g(z)$ are polynomials.

32. Show that when $0 < |z| < 4$,
$$\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$
Using the Laurent series, evaluate
$$\int_{|z|=2} \frac{1}{4z-z^2}\,dz,$$

where the circle is given positive orientation.

33. Show that the Laurent series for $(e^z - 1)^{-1}$ at the origin takes the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1},$$

where the numbers B_k are the called the *Bernoulli numbers*. Calculate B_1, B_2, B_3.

34. Find a Laurent series that converges in the annulus $1 < |z| < 2$ to a branch of the function

$$f(z) = \log\Big(\frac{z(2-z)}{1-z}\Big).$$

35. Recall that we proved the identity

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \Big(\frac{1}{z-n} + \frac{1}{n}\Big).$$

Recall also, the definition of Bernoulli numbers from the previous assignment

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n}.$$

Finally, for any complex number $Re(s) > 1$, we define $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$.

(a) Prove that

$$\pi z \cot \pi z = 1 - \sum_{n=1}^{\infty} \zeta(2n) z^{2n}.$$

(b) Prove that

$$\zeta(2n) = 2^{2n-1} \frac{B_n}{(2n)!} \pi^{2n},$$

In particular, from your answers in the previous assignment, you should be able to calculate $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$. **Hint.** First observe that

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1}.$$

36. Let $\Omega \subset \mathbb{C}$ be an open set containing the closure $\overline{D}_r(0)$ of the disc of radius r centred at the origin. Suppose $f : \Omega \to \mathbb{C}$ be is a holomorphic function with zeroes $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $D_r(0)$ with multiplicities $m_1, \cdots, m_n$ respectively, and no zero on $\partial D_r(0)$. For any entire function $\varphi : \mathbb{C} \to \mathbb{C}$, show that

$$\frac{1}{2\pi i} \int_{|z|=r} \varphi(z) \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{n} m_j \varphi(\alpha_j).$$

37. Let f be a function that is holomorphic on the annulus $A_{R,\infty}(0)$. The residue of $f(z)$ at infinity is defined to be

$$\mathrm{Res}_{z=\infty} f(z) = -\frac{1}{2\pi i} \int_{|z|=r} f(z)\, dz,$$

where $r > R$. Note that by Cauchy's theorem, the definition is independent of r. The reason for the negative sign is that morally, one would like to define the residue at infinity, in the same way as for a point in $\mathbb{C}$, namely via an integral on a small circle around the point with positive orientation. But a small circle with positive orientation around the point at infinity is a large circle in $\mathbb{C}$ with negative orientation, and hence the negative sign in the above expression.

(a) Prove that

$$\mathrm{Res}_{z=\infty} f(z) = -\mathrm{Res}_{z=0} \frac{1}{z^2} f\Big(\frac{1}{z}\Big).$$

(b) If f is holomorphic in $\mathbb{C} \setminus \{p_1, \cdots, p_n\}$. Then prove that

$$\mathrm{Res}_{z=\infty} f(z) + \sum_{k=1}^{n} \mathrm{Res}_{z=p_k} f(z) = 0.$$

38. Show that if f is an injective entire function, then it must be linear. That is, $f(z) = az + b$, for some $a, b \in \mathbb{C}$ with $a \neq 0$. **Hint.** First show that $f(z)$ cannot have an essential singularity at infinity.

39. Let f be a non-constant holomorphic map defined in an open set Ω containing the unit disc $\mathbb{D}$ centred at the origin.

    (a) Suppose $|f(z)| = 1$ whenever $|z| = 1$, then show that $f(\Omega)$ contains the unit disc.

    (b) Show that if $|f(z)| \geq 1$ whenever $|z| = 1$, and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, the prove that $f(\Omega)$ contains the unit disc.

40. Show that there is no holomorphic function on $\mathbb{D}$ that extends continuously to $\partial\mathbb{D}$ such that $f(z) = 1/z$ for all $z \in \partial\mathbb{D}$.

41. In each of the cases below, calculate the total number of solutions (with multiplicity) in the regions indicated.

    (a) $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ in $|z| < 1$.

    (b) $cz^n = e^z,\ |c| > e$ in $|z| < 1$.

42. Compute the following real-variable integrals using the residue theorem.

    (a) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\, dx,\ a \in \mathbb{R}$.

    (b) $\int_0^\infty \frac{x \sin x}{x^2+a^2}\, dx,\ a \in \mathbb{R}$.

    (c) $\int_0^\infty \frac{x^{1/3}}{x^2+1}\, dx$.

    (d) $\int_0^\infty \frac{\log x}{1+x^2}\, dx$.

    (e) $\int_0^\infty \log(1+x^2) \frac{dx}{x^{1+\alpha}}, 0 < \alpha < 2$.

43. (a) Find a fractional linear transformation that takes the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$ to $w_1 = 1$, $w_2 = i$ and $w_3 = -1$.

    (b) Find the fractional linear transformation that maps $z_1 = -i$, $z_2 = 0$ and $z_3 = i$ to $w_1 = -1$, $w_2 = i$ and $w_3 = 1$. What curve does the x-axis transform into?

44. Find a conformal map from the wedge

$$W = \{z \in \mathbb{C} \mid -\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}\}$$

onto the unit disc $\mathbb{D} = \{z \mid |z| \leq 1\}$.

45. (a) Show that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half disc $\{z = re^{i\theta} \mid r < 1,\ 0 < \theta < \pi\}$ to the upper half plane. **Hint.** The equation $w = f(z)$ reduces to the quadratic $z^2 + 2wz + 1 = 0$ which has two distinct roots if $w \neq \pm 1$, which is certainly the case when w lies in the upper half plane.

(b) Use this to show that $g(z) = \sin z$ maps the infinite strip $\{z = x + iy \mid -\pi/2 < x < \pi/2,\ y > 0\}$ conformally onto the upper half plane.

46. If a Mobius transformation takes two concentric circles of radii $r < R$ to two other concentric circles with radii $s < S$. Prove that $r/R = s/S$. **Hint.** First argue that by pre-composing and post-composing, one can assume that both the pairs of concentric circles have centres at 0 and $s = r = 1$, and that the inner circle is sent to the inner circle. After this has been arranged, argue that 0 and ∞ will have to be fixed points.

47. (Schwarz-Pick theorem) If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map from the unit disc to itself, prove the following.

    (a) For any $z_1 \neq z_2$,
    $$\Big|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\Big| \leq \Big|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\Big|.$$

    (b) For any $z \in \mathbb{D}$,
    $$\frac{|f'(z)|}{1 - |z|^2} \leq \frac{1}{1 - |z|^2}.$$

    (c) In either of the two inequalities, when (and only when) can equality occur?

48. Let $\mathbb{H} := \{z = x + iy \mid y > 0\}$ be the upper half plane, and let $PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\pm I$ be the set of 2×2 *real* matrices modded out by the equivalence relation $A \sim -A$. Then prove that $\mathrm{Aut}(\mathbb{H})$ is isomorphic to $PSL(2, \mathbb{R})$

49. (a) Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions which converge compactly to $f : \Omega \to \mathbb{C}$ which is not identically zero. If z_0 is a root of $f(z)$ of multiplicity 'm', then show that for all sufficiently small $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that f_n has exactly 'm' roots (counting multiplicity) in $D_\varepsilon(z_0)$ for all $n > N$.

    (b) From this conclude that if $f_n : \Omega \to \mathbb{C}$ is injective, and the sequence converges compactly to a non constant function $f : \Omega \to \mathbb{C}$, then f is injective.

    (c) Let $f_n : \mathbb{D} \to \mathbb{D}$ be a sequence of holomorphic functions such that $f_n(0) \to 1$. Then show that $f_n \to 1$ compactly on $\mathbb{D}$. **Hint.** First show that you get convergence after passing to a subsequence.

50. For $\mathrm{Re}(s) > 0$, consider the function
    $$f(s) = \int_0^1 \frac{t^s}{1 - \cos\sqrt{t}}\, dt,$$
    and let
    $$\Lambda(s) = \frac{f(s)}{\Gamma(s)}.$$

    (a) Show that $f(s)$ is holomorphic on $\mathrm{Re}(s) > 0$ and can be extended to a meromorphic function on all of $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \cdots$.

    Hint. What is the behavior of $1 - \cos\sqrt{t}$ near $t = 0$. Use this to compare with the model case discussed in the lecture.

    (b) Find
    $$\mathrm{Res}_{s=-1} f(s).$$

(c) Use the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

to show that the Gamma function has no zeroes in $\mathbb{C}$. **Caution:** If Γ has a root at s, $\Gamma(1-s)$ might have a pole at $1-s$. So you need to rule out this case.

(d) Show that $\Lambda(s)$ is an entire function, and evaluate $\Lambda(-1)$.

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