LECTURE - 10

· Recall :

Thm 9.4 (1^{st} variation formula) Let $\gamma : [a,b] \to M$ be any admissible curve, Γ a variation w/ variation field $V(t) := \partial_s \Gamma(0,t)$. Then

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}(\gamma_s) = -\int_a^b \langle V, D_t \dot{\gamma} \rangle\, dt + \langle V(b), \dot{\gamma}(b-) \rangle$$
$$- \langle V(a), \dot{\gamma}(a+) \rangle - \sum_{k=1}^{n-1} \langle V(a_k), \Delta_k \dot{\gamma} \rangle .$$

where $\Delta_k \dot{\gamma} := \dot{\gamma}(a_k+) - \dot{\gamma}(a_k-)$.

Cor 10.1 1) The critical points of $\mathcal{E} : \Omega(p,q) \to \mathbb{R}$, namely curves $\gamma(t)$ s.t

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}(\gamma_s) = 0$$

for any proper variation $\Gamma(s,t)$ of γ, are (smooth) geodesics.

2) If γ is parametrized w/ arc-length, then for any proper variation, $\Gamma(s,t)$,

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}(\gamma_s) = \frac{d}{ds}\Big|_{s=0} \mathcal{L}(\gamma_s).$$

3) A curve $\gamma \in \Omega(p,q)$ is critical point of $\mathcal{L}$ if and only if $\exists$ a re-parametrization $\tilde{\gamma} = \gamma \circ \alpha$ s.t $\tilde{\gamma}$ is a geodesic.

Rk : If γ is a geodesic, and $\tilde{\gamma} = \gamma \circ \alpha$ is a re-parametrization, then $\tilde{\gamma}$ will still be a critical pt of $\mathcal{L}$

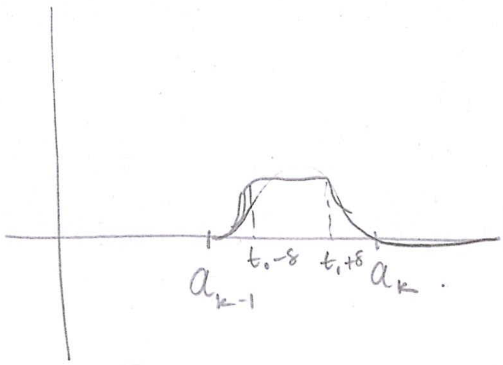
but <u>not</u> necessarily of $\mathcal{E}$. This is one of the reasons it is easier to work w/ $\mathcal{E}$ rather than $\mathcal{L}$.

<u>Pf</u>: 1) Let γ be a critical point of $\mathcal{E}$. We first claim that γ is a "broken" geodesic.

<u>Claim 1</u> $D_t \dot\gamma = 0$ in $[a_{k-1}, a_k]$

<u>Pf</u>: Sps $D_t \dot\gamma(t_0) \neq 0$ for some $t_0 \in (a_{k-1}, a_k)$ & $\delta > 0$ s.t.

$$D_t \dot\gamma(t) \neq 0 \quad \forall\, t \in (t_0 - \delta, t_0 + \delta) \subset (a_{k-1}, a_k).$$



Let $\varphi \in C^\infty(\mathbb{R})$ s.t. $\varphi \geq 0$ on $\mathbb{R}$, $\varphi \equiv 0$ on $\mathbb{R} \setminus (a_{k-1}, a_k)$ & $\varphi \equiv 1$ on $(t_0 - \delta, t_0 + \delta)$.

Define $V(t) = \varphi D_t \dot\gamma$ &

$$\Gamma(s,t) = \exp_{\gamma(t)}(s V(t))$$

Note $V(a_j) = 0 \quad \forall\, j = 0, 1, \dots, n$. So

$$\gamma \text{ critical} \xRightarrow{1^{st} \text{ variati}} 0 = -\int_{a_{k-1}}^{a_k} \varphi\, |D_t \dot\gamma|^2 dt$$

$$< -\int_{t_0-\delta}^{t_0+\delta} \varphi\, |D_t \dot\gamma|^2$$

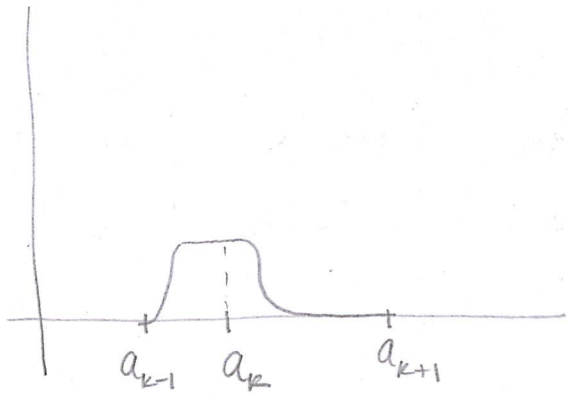
$$= -\int_{t_0-\delta}^{t_0+\delta} |D_t \dot\gamma|^2 < 0$$

Contradiction!

<u>Claim 2</u> $\Delta_k \dot\gamma = 0 \quad \forall\, k = 1, 2, \dots, n-1$.

<u>Pf</u>: Again, using a bump function, construct a v.f. V along γ s.t. $V(a_k) = \Delta_k \dot\gamma$ & $V(a_j) = 0$ $\forall\, j \neq k$.

Use the foll. bump function



Then 1^{st} variation + Claim 1 $\Longrightarrow -|\Delta_k \dot{r}|^2 = 0$. Done!

To complete proof of 1, since $r(a_k+) = r(a_k-)$, by uniqueness of geodesics, $r|_{[a_k, a_{k+1}]}$ is a cont. of the geodesic $r|_{[a_{k-1}, a_k]}$ & so r is smooth.

2). Sps r s.t $|\dot{r}(t)| = 1 \ \forall t$ & Γ is a variation of r. Then

$$\frac{d}{ds}\Big|_{s=0} L(r_s) = \frac{d}{ds}\Big|_{s=0} \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |\dot{r}_s(t)|\, dt$$

$$= \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} \frac{1}{2|\dot{r}(t)|} \cdot \frac{d}{ds}\Big|_{s=0} |\dot{r}_s(t)|^2\, dt \quad \left(\text{since } f' = \frac{(f^2)'}{2f}\right)$$

$$= \frac{d}{ds}\Big|_{s=0} \mathcal{E}(r_s) \quad \text{since } |\dot{r}(t)| = 1.$$

3). Sps r is a critical point of L. Let $\tilde{r} = r \circ \alpha$ be an arc-length reparametrization. Then since $L(r) = L(\tilde{r})$, $\tilde{r}$ is also a critical point. This is because if $\tilde{\Gamma}$ is a variation of $\tilde{r}$, then $\Gamma(s,t) = \tilde{\Gamma}(s, \alpha^{-1}(t))$ is a variation of r w/

$\gamma_s := \Gamma(s,t) = \tilde{\gamma}_s \circ \alpha^{-1}$. So $\mathcal{L}(\gamma_s) = \mathcal{L}(\tilde{\gamma}_s)$.

γ critical $\Rightarrow$ $\frac{d}{ds}\Big|_{s=0} \mathcal{L}(\tilde{\gamma}_s) = \frac{d}{ds}\Big|_{s=0} \mathcal{L}(\gamma_s) = 0$.

Then 2) $\Rightarrow$ $\tilde{\gamma}$ is critical for $\mathcal{E}$.

1) $\Rightarrow$ $\tilde{\gamma}$ is a geodesic.

Conversely, if some re-parametrization $\tilde{\gamma} = \gamma \circ \alpha$ is a geodesic. Then by a linear change of parametrization we can assume $|\tilde{\gamma}'(t)| = 1$.

Then 1) & 2) $\Rightarrow$ $\tilde{\gamma}$ is a critical pt for $\mathcal{L}$.

Again by above reason γ is also critical for $\mathcal{L}$.

<u>Cor 10.2</u>: Let $\gamma \in \Omega(p,q)$ s.t. $\mathcal{L}(\gamma) = d(p,q)$ & $|\dot{\gamma}(t)| =$ const. Then γ is a geodesic.

<u>Pf</u>: W.l.o.g sps $|\dot{\gamma}(t)| = 1$. Then Cor 10.1 (2) $\Rightarrow$ γ is critical pt. of $\mathcal{E}$. Then 1) $\Rightarrow$ γ is a geodesic.

So $\boxed{\text{Minimizing Curves are geodesics}}$.

<u>Ques</u>: What about the converse.

Not true in general. e.g. The greater arc of the great circle connecting two <u>non</u> antipodal points on S^2.

• Local Behavior of Geodesics

Lemma 10.3. For each $p \in M$, $\exists$ a nbd W & $\varepsilon > 0$ s.t $\forall q_1, q_2 \in W$, $\exists!$ a geodesic $\gamma : [0,1] \to M$ s.t $\gamma(0) = q_1$, $\gamma(1) = q_2$.

Pf: Let V be an open nbd of $(p, \vec{0}_p) \in TM$ s.t $\phi|_V$ is a local diffeo to an open nbd of $(p,p) \in M \times M$. Recall

$$\phi : \mathcal{E} \longrightarrow M \times M$$
$$v \longrightarrow (\pi(p); \exp_{\pi(v)}(v)).$$

By possibly shrinking V, we can assume $\exists\, \varepsilon > 0$, U open
$V = \{(q, v) \mid q \in U$ open nbd of p & $\|v\| < \varepsilon\}$.

Let W open nbd of p s.t $W \times W \subset \phi(V)$.

Sps $q_1, q_2 \in W$. Then $\exists!\, v \in T_{q_1}M$ s.t
$\phi(q_1, v) = (q_1, q_2)$ i.e $\exp_{q_1} v = q_2$.

Let $\gamma(t) = \exp_{q_1}(tv)$. Then $\gamma(0) = q_1$, $\gamma(1) = q_2$.

And $|\gamma'(0)| = |v| < \varepsilon$. Since γ is a geodesic $|\dot\gamma(t)| < \varepsilon$ $\forall t \in [0,1]$. So $L(\gamma) < \varepsilon$.

Rk: γ might not be contained completely in W. By refining the argument, one can choose W to be "convex" i.e $\forall q_1, q_2$ $\exists!$ minimizing geod. connecting q_1 to q_2. This is spelled out in one of the assignment problems.

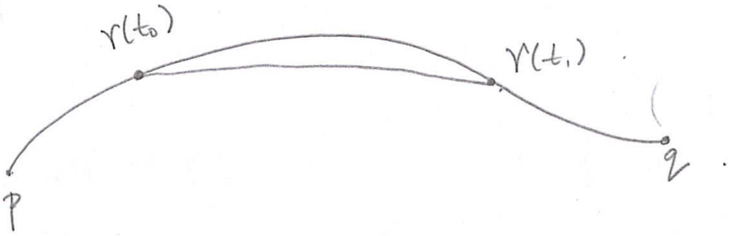
Th^m 10.4 For any $p \in M$, $\exists$ nbd W & $\varepsilon > 0$ s.t $\forall q_1, q_2 \in W$ $\exists!$ geodesic $\gamma : [0,1] \to M$ s.t. $\gamma(0) = q_1$, $\gamma(1) = q_2$, $L(\gamma) < \varepsilon$. Moreover, if $\omega : [0,1] \to M$ s.t $\omega \in \Omega(q_1, q_2)$. Then

$$L(\gamma) \leq L(\omega).$$

with equality holding if & only if $\omega = \gamma \circ \alpha$. for some α

Def^n: A geodesic $\gamma : [a,b] \to M$ is called minimal if $\forall$ $\omega \in \Omega(\gamma(a), \gamma(b))$, $L(\gamma) \leq L(\omega)$.

Rk: If $\gamma : [a,b] \to M$ is minimal, then $\forall$ $a \leq t_0 < t_1 \leq b$, $\gamma|_{[t_0, t_1]}$ is also minimal.



Cor 10.5. Let $\gamma : I \to M$ be a geodesic. For any $t_0 \in I$, $\exists$ $\delta_0 = \delta(t_0)$, s.t $\gamma|_{[t_0 - \delta_0, t_0 + \delta_0]}$ is a minimal geodesic. i.e

$\boxed{\text{Geodesic are locally length minimizing}}$

Pf: Apply Th^m 10.4 w/ $p = \gamma(t_0)$. Let $\delta_0 > 0$ s.t $[t_0 - \delta_0, t_0 + \delta_0] \subset W$ & $L(\gamma) < \varepsilon$. Let $q_1 = \gamma(t_0 - \delta_0)$, $q_2 = \gamma(t_0 + \delta_0)$

Th^m 10.4 $\Longrightarrow$ $\exists$ geodesic $\tilde{\gamma} : [0,1] \to M$ s.t. $\tilde{\gamma}(0) = q_1$ & $\tilde{\gamma}(1) = q_2$. which is minimal.

By uniqueness of Th^m 10.4, $\gamma \equiv \tilde{\gamma}$ on $[t_0-\delta, t_0+\delta]$

They key ingredient in the proof of Th^m 10.4 is Gauss' Lemma.

<u>Def^n</u>: For any $q \in M$, we say $\mathcal{U}_q$ is a <u>geodesic ball</u> if it is a diffeomorphic of some $B(0,\varepsilon) \subset T_qM$ under $\exp_q$. i.e $\mathcal{U}_q = \exp_q(B(0,\varepsilon))$

Here $B(0,\varepsilon) = \{ v \in T_qM \mid |v|_{g_q} < \varepsilon \}$.

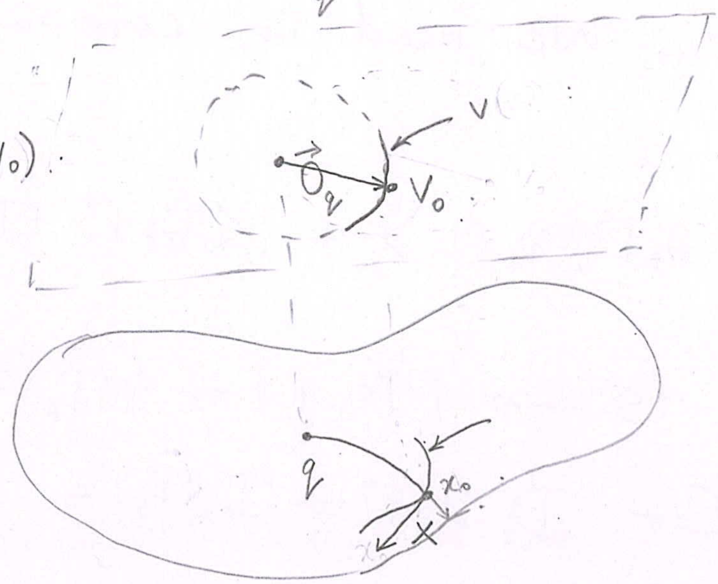
<u>Lemma 10.6</u> (Gauss Lemma). Let $\mathcal{U}_q$ be a geodesic ball around q. Then the geodesics $\gamma_v(t) = \exp_q(tv)$ are g-orthogonal to the geodesic sphere

$$S_{q,\delta} = \{ \exp_q v \in \mathcal{U}_q \mid v \in T_qM, \ |v|_{g_q} = \delta \}.$$

<u>Pf</u>: Let $t \to v(t)$ be a curve in T_qM s.t $|v(t)|_{g_q} = \delta$ (& so $\exp_q v(t) \in S_{q,\delta}$); sps $v_0 = v(0)$

$\exp_q v_0 = x_0 \in S_{q,\delta}$.

$\gamma(t) = \exp_q(tv_0)$.



$t \in (-\varepsilon, \varepsilon)$.

$\omega(t) = \exp_q v(t)$

$X = \omega'(0) \in T_{x_0}M$.

<u>Claim</u>. $\gamma'(1) \perp X$

Pf: Let $\Gamma(s,t) := \exp_q(s V(t))$, $0 \le s \le 1$ & $-\varepsilon < t < \varepsilon$

Claim 2: If $f(s,t) := \langle \partial_s \Gamma(s,t), \partial_t \Gamma(s,t) \rangle : [0,1] \times (-\varepsilon, \varepsilon) \to \mathbb{R}$.

Then $\partial f / \partial s \equiv 0$.

Assuming this, we note:

$$\partial_s \Gamma(1,0) = \left.\frac{d}{ds}\right|_{s=1} \exp_q(s v_0) = \dot{r}(1)$$

$$\partial_t \Gamma(1,0) = \left.\frac{d}{dt}\right|_{t=0} \exp_q(V(t)) = \omega'(0) = X.$$

$$\partial_s \Gamma(0,0) = \left.\frac{d}{ds}\right|_{s=0} \exp_q(s v_0) = \dot{r}(0) = v_0.$$

$$\partial_t \Gamma(0,0) = \left.\frac{d}{dt}\right|_{t=0} \exp_q(0) = 0.$$

$$\text{Claim 2} \overset{f(0,0)=f(1,0)}{\Longrightarrow} \langle \dot{r}(t), X \rangle = \langle v_0, 0 \rangle = 0.$$

Pf of Claim 2: We need to compute $\partial f / \partial s$ & $\partial f / \partial t$.

$$\frac{\partial f}{\partial s} = \langle D_s \partial_s \Gamma, \partial_t \Gamma \rangle + \langle \partial_s \Gamma, D_s \partial_t \Gamma \rangle.$$

For each t, $s \longrightarrow \Gamma(s,t) = \exp_q(s v(t)) = r_t(s)$.

a geodesic. So $D_s \partial_s \Gamma = D_s \dot{r}_s = 0$.

By symmetry lemma

$$\langle \partial_s \Gamma, D_s \partial_t \Gamma \rangle = \langle \partial_s \Gamma, D_t \partial_s \Gamma \rangle.$$

$$= \frac{1}{2} \frac{\partial}{\partial t} \langle \partial_s \Gamma, \partial_s \Gamma \rangle .$$

But $\partial_s \Gamma = \dot{\gamma}_t(s)$. For each t,

$$|\dot{\gamma}_t(s)| = |\dot{\gamma}_t(0)| = |v(t)| = \delta .$$

So $$\frac{\partial}{\partial t} \langle \partial_s \Gamma, \partial_s \Gamma \rangle = 0 .$$

Hence $\partial f / \partial s = 0$.

<u>Rk</u>: In fact, for any t,

$$\partial_t \Gamma(0, t) = \frac{d}{dt} \exp_{p_q} 0 = 0 .$$

So $f(s,t)$ is independent of (s,t). (and not just independent of s) i.e $f(s,t) \equiv 0$.