LECTURE – 11

• Recall: Our aim is to prove the foll.

Thm 10.4: For any $p \in M$, $\exists$ nbd W & an $\varepsilon > 0$ s.t. $\forall\, q_1, q_2 \in W$ $\exists !$ geodesic $\gamma : [0,1] \longrightarrow M$ s.t $\gamma(0) = q_1$, $\gamma(1) = q_2$ & $L(\gamma) < \varepsilon$. Moreover if $\omega : [0,1] \longrightarrow M$ $\in \Omega(q_1, q_2)$, then

$$L(\omega) \geq L(\gamma).$$

w/ equality $\iff$ $\omega = \gamma \circ \alpha$.

The key ingredient is the foll.

Lemma 10.6 (Gauss' Lemma). Let U_q be a geodesic ball around q. Then for any $v \in T_qM$, $\gamma_v(t) = \exp_q tv$ is g- orthogonal to the geodesic spheres

$$S_{q,\delta} = \{\exp_q w \in U_q \mid w \in T_q M,\ \|w\| = \delta\}.$$

as long as $\gamma_v(t) \in U_q$.

• If U_q is a geodesic ball centered at q w/ normal coordinates $\{x^i\}$, we have

$$r_q(x) := \left(\sum (x^i)^2\right)^{1/2}$$

$$\frac{\partial}{\partial r_q} := \frac{x^i}{r_q} \frac{\partial}{\partial x^i}.$$

Cor 11.1 1) $\nabla r_q = \partial/\partial r_q$ on $U_q / \{q\}$.

2). If $\omega : [a,b] \longrightarrow U_q/\{q\}$ is admissible. Then.

$$L(\omega) \geq |r_q(\omega(b)) - r_q(\omega(a))|.$$

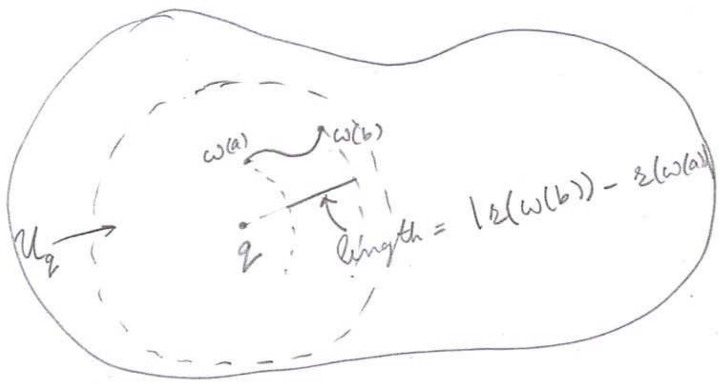
w/ equality if and only if $\omega(t) = \exp_q(tv)$ for some fixed $v \in T_qM$.

(3): For any $q' \in U_q$ w/ $\exp_q v = q'$, $\gamma_v(t) = \exp_q tv$ is <u>the</u> minimal geodesic from q to q'.

(4) $\forall\ x \in U_q$, $r_q(x) = d(q, x)$.

<u>Rk</u>: 1) We had seen earlier that $\|\partial/\partial r\| = 1$. So $|\nabla r_q| \equiv 1$.

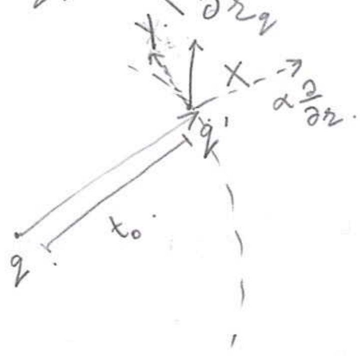
2). 2) $\Longrightarrow$ that shortest path joining two concentric spherical shells is a radial geodesic.



<u>Pf</u>: 1) Recall that $\nabla r_q \doteq (dr_q)^{\#}$ i.e.

$$X(r_q) = \langle \nabla r_q, X \rangle \quad \forall\ X \in \mathcal{T}^1(U_q).$$

<u>Goal</u>: To show $X(r_q) = \left\langle \frac{\partial}{\partial r_q}, X \right\rangle$.



Sps we compute at a point $q' = \exp_q t_0 v_0$.
where $|v_0| = 1$. If $\gamma_0(t) = \exp_q(t v_0)$.

Then Th^m $\Rightarrow \left.\frac{\partial}{\partial r_q}\right|_{q'} = \gamma_0'(t_0)$.

Given $X \in T_{q'}M$, we can write $X = \alpha \left.\frac{\partial}{\partial r}\right|_{q'} + Y$.
where $Y \in T_{q'}S_{q'}$, here $S_{q'} = \exp_q(\partial B(\vec{0}_q, t_0))$.

<u>Note</u>: $dr\left(\frac{\partial}{\partial r}\right) = 1$ & $dr(Y) = 0$ since Y is tangential to the surface $r = t_0$ & so

$$dr(X) = \alpha.$$

On the other hand, Gauss-lemma $\Rightarrow \left\langle \frac{\partial}{\partial r_q}, Y \right\rangle = 0$,

Th^m $\Rightarrow \left|\frac{\partial}{\partial r_q}\right|^2 = 1$. So

$$\left\langle \frac{\partial}{\partial r_q}, X \right\rangle = \alpha.$$

Done!

2). Let $\omega'(t) = \alpha(t) \left.\frac{\partial}{\partial r_q}\right|_{\omega(t)} + Y(t)$, where $Y(t)$ is tangential to the geodesic sphere at $\omega(t)$. Let
$r(t) = r_q(\omega(t))$.

<u>Claim</u> $\alpha(t) = r'(t)$.

<u>Pf</u>: $r'(t) = dr_q(\omega'(t)) = \langle \nabla r_q, \omega'(t) \rangle \overset{(1)}{=} \left\langle \left.\frac{\partial}{\partial r_q}\right|_{\omega(t)}, \omega'(t) \right\rangle$

Gauss Lemma $\Longrightarrow \left.\frac{\partial}{\partial r}\right|_{\omega(t)} \perp Y(t)$. So $r'(t) = \alpha(t)$.

Since decomposition of ω' is orthogonal

$$|\omega'(t)|^2 = |\alpha(t)|^2 + |Y(t)|^2 \geq |\alpha(t)|^2 = |r'(t)|^2 .$$

So $\displaystyle L(\omega) = \int_a^b |\omega'(t)| \geq \int_a^b |r'(t)|\,dt \geq \left| \int_a^b r'(t)\,dt \right|.$

$$= |r(b) - r(a)| .$$

$$= |r_q(\omega(b)) - r_q(\omega(a))|$$

Also equality if and only if $Y(t) \equiv 0$ & $r'(t) \neq 0$.
So ω is a radial path, which after re-parametrization will be $\exp_q(tv)$ for some $v \in T_q M$.

3). Follows from 2). In fact if $\omega : [0,1] \longrightarrow M$ is any path from q to q'.

Claim: $\forall\ \delta > 0$, $L(\omega|_{[\delta,1]}) = L(\gamma_v|_{[\delta,1]}) \Longleftrightarrow \omega|_{[\delta,1]} = \gamma_v|_{[\delta,1]} \circ \alpha$.

This follows trivially from 2).

4) Follows trivially from 3) since if $x = \exp_q v$.

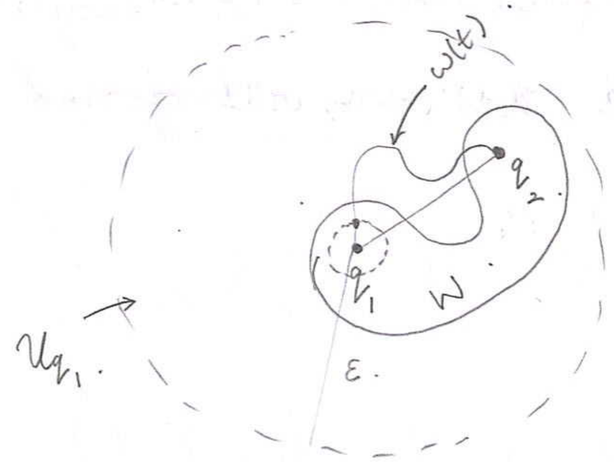
$$L(\gamma_v) = |v| = r_q(x).$$

Pf of Th^m 10.3. Let W & $\varepsilon > 0$ as in Pf of Lemma. Then $\forall\ q_1, q_2 \in W$. $\exists\,!$ geodesic $\gamma(t) = \exp_{q_1}(tv)$ for some $|v| < \varepsilon$ s.t $\gamma(1) = q_2$. Let

$$\mathcal{U}_{q_1} := \{ \exp_{q_1} w \mid w \in T_q M,\ |w| < \varepsilon \}.$$

Let $\omega : [0,1] \longrightarrow M$, $\omega(0) = q_1$, $\omega(1) = q_2$.

Note that $W \subset \mathcal{U}_{q_1}$ by Lemma 10.3.



Let $1 \gg \delta > 0$. Then $\omega(t)$ connec ts point lying on a spherical shell of radius δ to one lying on a spherical shell of radius $r = |v|$. Then if $\omega(t) \in \mathcal{U}_q$, by Cor (2) $\Longrightarrow$ $L(\omega) \geqslant r - \delta$.

If $\omega(t)$ goes outside $\mathcal{U}_q$, then again by Cor (2)
$L(\omega) \geqslant \varepsilon - \delta$. So in both case $L(\omega) \geqslant r - \delta$.
Eq $\Longleftrightarrow$ equality in Cor(2).
$\delta \longrightarrow 0 \Rightarrow L(\omega) \geqslant r$.

Done!

<u>Cor 11.2</u>: Let $p \in M$ & $S_p^{n-1} = \{ v \in T_pM \mid |v| = 1 \}$. Let $\varepsilon > 0$ s.t. $\exp_p w$ is defined $\forall$ $|w| < \varepsilon$, and for $r < \varepsilon$, denote

$$S_{p,r} = \{ \exp_p w \mid |w| = r,\ w \in T_pM \}$$

$$\mathcal{U}_{p,r} = \{ \exp_p tw \mid |w| = r \quad |t| < 1 \}$$

Then 1) $g\big|_{\mathcal{U}_{p,\delta}}(q) = dr^2 + h_{r,w}$ where $h_{r,w}$ is the metric induced by g on $S_{p,r}$, at $w \in S_{p,r}$.

2) If $\varepsilon << 1$, then $\mathcal{U}_{p,r} = B(p,r) := \{ q \in M \mid d(p,q) < r \}$ $\forall$ $r < \varepsilon$.

<u>Pf</u>: 1) Follows from the fact that for any $q = \exp_p W$, w/ $|W| = r$. $\in \mathcal{U}_{p,\varepsilon_\lambda}$, any $X \in T_q M$ has an orthogonal decomposition

$$X = \alpha \cdot \frac{\partial}{\partial r} + Y, \quad Y \in T_q S_{p,r}.$$

So for any X, $g_q(X, X) = \alpha^2 + h_{r,W}(Y, Y)$.

$$= \left(dr^2 + h_{r,W}\right)(X, X).$$

2). By Cor 11.1 (4), $\mathcal{U}_{p,r} \subset B(p, r)$.

Conversely, let $q \in B(p, r)$. If $\omega : [0, 1] \longrightarrow M$ <u>any</u> path in $\Omega(p, q)$, then if $q \notin \mathcal{U}_{p,r}$, let

$$t_0 = \sup\{t \mid \omega(t) \in \mathcal{U}_{p,r}\}.$$

By Cor 11.1 (2), $\forall\ t < t_0$.

$$L(\omega) \geq L(\omega|_{[0,t]}) \geq r(\omega(t)) \xrightarrow{t \to t_0} r$$

So $\quad d(p, q) \geq r$.

Since $q \in B(p, r)$ this is a contradiction unless $q \in \mathcal{U}_{p,r}$.

· GEODESIC COMPLETENESS

Def^n: A Riemannian manifold (M, g) is said to be geodesically complete if every geodesic $\gamma(t)$ exists for all time $t \in \mathbb{R}$.

Th^m 11.3 (Hopf – Rinow) Let (M, g) be a Riemannian manifold w/ induced distance function d.
TFAE.

(1) (M, d) is a complete metric sp.

(2) (M, g) is geodesically complete.

(3) $\forall\, p \in M$, $\exp_p$ is defined on all of T_pM.
$(\Leftrightarrow\ \mathcal{E} = TM)$.

Moreover each of the above implies

(4) Any two points $p, q \in M$ can be joined by at least one minimal geodesic.

Rk 1) We say (M, g) is complete if any (and hence all) of (1) – (3) are satisfied.

2) Converse of (4) is not true. e.g $B(0,1) \subset \mathbb{R}^n$ has the property (4) but is not complete.

3) Uniqueness in (4) need not hold.

Pf (1) $\Longrightarrow$ (2). Proceed by contradiction. Let $\gamma : [0,b) \to M$ be a geodesic s.t. there is no extension to $[0, b+\varepsilon)$ for any $\varepsilon > 0$. Let $t_i \nearrow b$ & set $q_i = \gamma(t_i)$.
Sps γ is parametrized by arc-length. Then.

$$L(\gamma|_{[t_i, t_j]}) = |t_i - t_j|.$$

So

$$d(q_i, q_j) \le |t_i - t_j|.$$

Hence $\{q_i\}$ is a Cauchy seq^n & (1) $\Rightarrow$ $\exists$ $q \in M$. s.t. $q_i \longrightarrow q$. By Th^m 8.3, $\exists$ nbd W of q & $\varepsilon > 0$ s.t. $\forall$ $x \in W$, $\exp_x v$ is defined $\forall$ $v \in T_x M$. s.t. $|v|_x < \varepsilon$. Let $j >> 1$ s.t. $q_j \in W$ & $t_j > b - \varepsilon$.
Let $v_j = \gamma'(t_j)$. Then $\sigma_j(t) = \exp_{q_j}(t v_j)$. exists (at least) for time $t \in [0, \varepsilon)$. By uniqueness of geodesics

$\sigma_j(t) = \gamma(t + t_j)$. But then, γ has an extension $\tilde{\gamma}$ to $[0, b+\varepsilon)$ by.

$$\tilde{\gamma}(t) := \begin{cases} \gamma(t), & 0 \le t < b. \\ \sigma(t - t_j), & t_j \le t < t_j + \varepsilon. \end{cases}$$

Contradiction!

(2) $\Longrightarrow$ (3). Trivial!