· Recall: (M, g) is called geodesically complete if every geodesic is defined on all of $\mathbb{R}$.

Our aim is to prove the foll.

Th^m 11.3 (Hopf - Rinow) Let (M, g) be a Riemannian mfld w/ induced distance d. TFAE

1) (M, d) is a complete metric sp.

2) (M, g) is geodesically complete

3) $\forall$ $p \in M$, $\exp_p$ is defined on all of T_pM.

Moreover each implies

4) $\forall$ $p, q \in M$, $\exists$ a minimal geodesic connecting p & q.

Pf: Last class we proved $1) \Rightarrow 2) \Longrightarrow 3)$.

Today: we show: $3) \Longrightarrow 4)$ & $3) + 4) \Rightarrow 1)$.

$3) \Rightarrow 4)$: Let $p, q \in M$, $d(p, q) = r$. Let U_p be a geodesic ball around p of radius ε i.e.

$$U_p = \{ \exp_p v \mid |v| < \varepsilon \}$$

where U_p is diffeomorphic to $B(0, \varepsilon) \subset T_pM$.

Let $\delta < \varepsilon$, $S_{p,\delta} = \{ \exp_p w \mid |w| = \delta \} \subset U_p$ the corresponding "spherical shell" of radius δ & $p_0 \in S_{p,\delta}$ s.t. $d(p_0, q) = \inf_{x \in S_{p,\delta}} d(x, q)$.

Such a p_0 exists since $S_{p,\delta}$ is compact.

Let $v_0 \in T_pM$ s.t $p_0 = \exp_p \delta v_0$ (in particular $|v_0| = 1$)

Let $\gamma(t) = \exp_p t v_0$, $t \in [0, r]$. Note $L(\gamma) = r$.

Claim 1: $\gamma(r) = q$

Claim 2: $d(\gamma(t), q) = r - t \quad (*)_t$.

Claim 2 at $t = r \Longrightarrow \gamma(r) = q$. So we have found a geodesic connecting p & q s.t $L(\gamma) = d(p, q)$. Hence we are done.

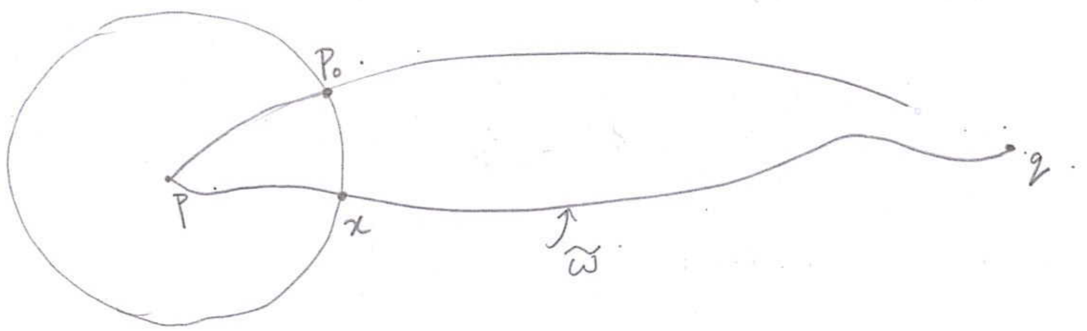
Pf of Claim 2:

Step 1 $(*)_t$ holds at $t = \delta$.

Pf: Note $\gamma(\delta) = p_0$. Sps $(*)_\delta$ fails. Then since $d(p, p_0) = \delta$, Δ-ineq $\Longrightarrow d(p, q) < d(p, p_0) + d(p_0, q)$.
Let $\omega(t)$ be a path from p to q s.t

$$L(\omega) < d(p, p_0) + d(p_0, q).$$

Sps $\omega(t)$ intersects $S_{p,\delta}$ at x, then letting $\tilde{\omega}$ be the part of ω from x to q, $L(\omega) \geq \delta + L(\tilde{\omega})$.



Then $d(p, p_0) + d(p_0, q) > L(\omega)$.

$$\geqslant \delta + L(\tilde{\omega})$$

$$= d(p, p_0) + L(\tilde{\omega})$$

So $L(\tilde{\omega}) < d(p_0, q)$ & hence $d(x, q) < d(p_0, q)$ contradicting the choice of p_0.

Step 2: Let $T = \sup\{t \in [\delta, r] \mid (*)_t \text{ holds}\}$.

If $t_k \nearrow T$ s.t $(*)_{t_k}$ holds then

$$d(\gamma(T), q) = \lim_{k\to\infty} d(\gamma(t_k), q)$$

$$= \lim_{k\to\infty} r - t_k$$

$$= r - T$$

So $(*)_T$ holds

$\boxed{\text{Goal: } T = r}$

Sps not, i.e sps $T < r$. As before, if we set $p' = \gamma(T)$, we have a spherical shell S' of radius δ' around p' & a $p_0' \in S'$ s.t.

$$d(p_0', q) = \min_{x\in S'} d(x, q).$$

Claim 3: $\gamma(T + \delta') = p_0'$.

Sps, this is true. Then note that as in Step 1.

$$d(p', q) = d(p', p_0') + d(p_0', q) \qquad (*)$$

$$= \delta' + d(p_0', q)$$

On the other hand since $(*)_T$ holds.

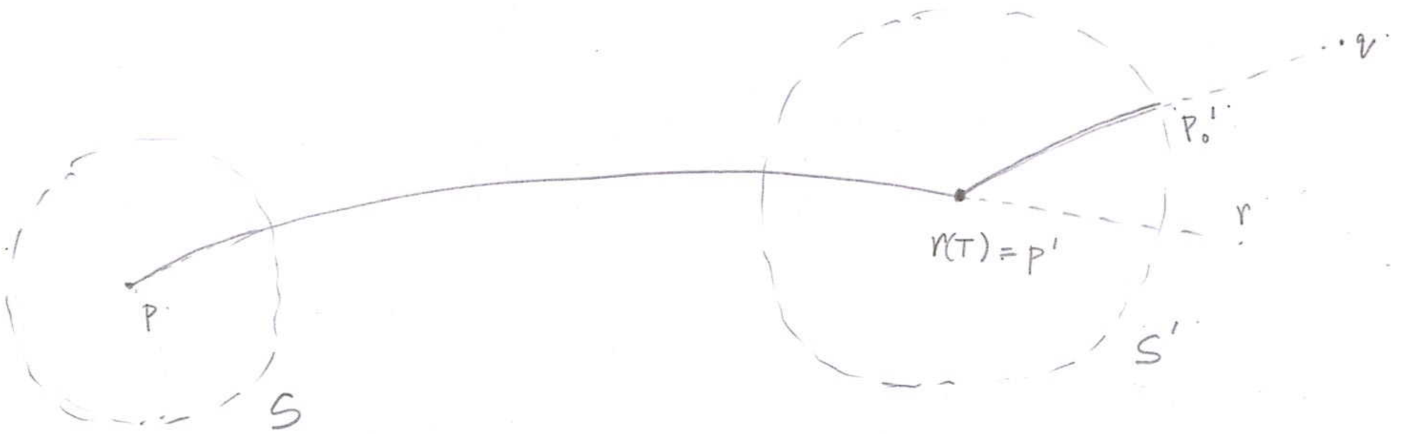
$$d(p', q) = d(r(T), q) = r - T$$

So Claim 3 $\Rightarrow d(r(T+\delta'), q) = d(p_0', q) = r - (T+\delta')$.

Hence $(*)_{T+\delta'}$ holds, contradicting T being supremum, & we are done.

<u>Pf of Claim 3</u>: First, note that as above, by $(*)$ & $(*)_T$

$$d(p, p_0') \geq d(p, q) - d(p_0', q) = T + \delta'. \quad (**)$$



Let $\sigma(t) = \exp_{p'} tv'$ be the minimal radial geodesic from p' to p_0', where $|v'| = 1$. Consider

$$\tilde{r}(t) = \begin{cases} r(t), & 0 \leq t \leq T \\ \sigma(t-T), & T \leq t \leq T+\delta' \end{cases}$$

$$L(\tilde{r}) = L(r|_{[0,T]}) + \delta'$$

$$= T + \delta' \quad \text{since } |r'(t)| = |v| = 1.$$

So by (**), $\tilde{\gamma}$ is an admissible path realizing the distance $d(p, p_0')$. Hence by Cor 10.2 since $|\dot{\tilde{\gamma}}(t)| = 1$, we have that $\tilde{\gamma}$ is a minimal <u>geodesic</u>. By uniqueness of geodesics $\tilde{\gamma} \equiv \gamma$ $\forall t$, and hence $\gamma(T+\delta') = \tilde{\gamma}(T+\delta')$

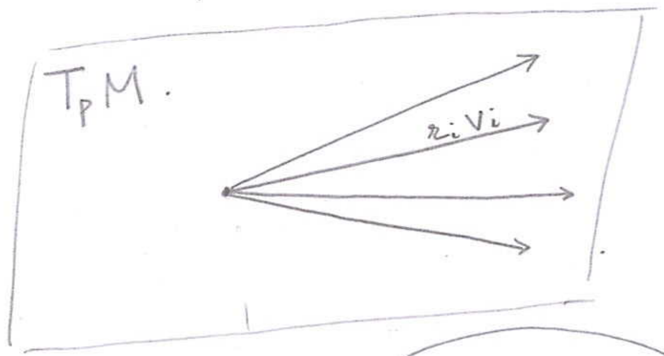
$= \sigma(\delta') = p_0'$, and Claim 3 is proved.

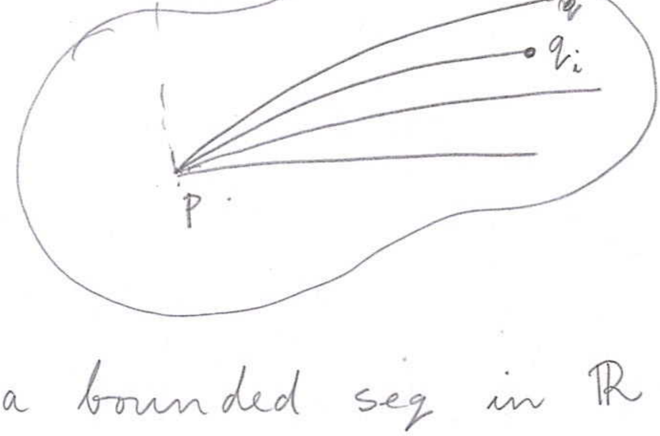
• <u>$(3) + (4) \Rightarrow (1)$</u>: Let $\{q_i\}$ be Cauchy. $\forall i$ let $\gamma_i(t)$ be a unit speed length minimizing geodesic from p to q_i. We can write

$$\gamma_i(t) = \exp_p t \cdot v_i : [0, r_i] \longrightarrow M.$$

where $v_i \in T_pM$, $|v_i| = 1$, $r_i = d(p, q_i)$.

$$\Rightarrow q_i = \exp_p r_i v_i$$





$\{r_i\}$ is a bounded seq in $\mathbb{R}$ (since Cauchy seq^n are bounded), and v_i is bounded in $T_pM \approx \mathbb{R}^n$. Moreover $g_p\big|_{T_pM}$ is equivalent to $g_{\mathbb{R}^n}$. So

Bolzano - Weierstrass $\Longrightarrow$ $\exists$ sub-seq i_k s.t.

$$\begin{cases} r_{i_k} \longrightarrow r \\ V_{i_k} \longrightarrow V \end{cases}$$

3) $\Longrightarrow$ $\exp_p rV$ is defined & continuity of exp map $\Longrightarrow$ $q_{i_k} = \exp_p r_{i_k} V_{i_k} \longrightarrow \exp_p rV$.

$\{q_i\}$ Cauchy $\Longrightarrow$ $q_i \longrightarrow \exp_p rV =: q$.

So every Cauchy seq^n converges & hence (M,d) is complete.

<u>Cor 12.1</u>: (M, g) is complete $\Longleftrightarrow$ (M, d) has the Bolzano - Weierstrass property i.e <u>every</u> bounded set has compact closure. In particular all balls $B(p, r) := \{x \in M \mid d(p, x)\}$ are relatively compact.

<u>Pf</u>: $\Longleftarrow$ is true in general for metric spaces

<u>$\Longrightarrow$</u>: We show closed & bounded sets are compact. Let $K \subset M$ be closed & bounded, and let $\{q_i\}$ infinite seq in K. Let $r_k = d(q_i, P)$ for some <u>fixed</u> $p \in M$. K bounded $\Longrightarrow$ $\{r_i\}$ is bounded. (M, g) complete $\Longrightarrow$ $q_i = \exp_p r_i V_i$ for some $|V_i| = 1$, $V_i \in T_p M$.

As before, $\exists$ sub-seq s.t

$$\begin{cases} r_{i_k} \longrightarrow r \in \mathbb{R} \\ V_{i_k} \longrightarrow V \in T_pM \end{cases}$$

Cont + (M, g) complete $\Longrightarrow$ $q_{i_k} = \exp_p V_{i_k} \longrightarrow q$.

K closed $\Rightarrow$ $q \in K$.

This is true $\forall$ sequences $\{q_{i_k}\}$ So K is compact.