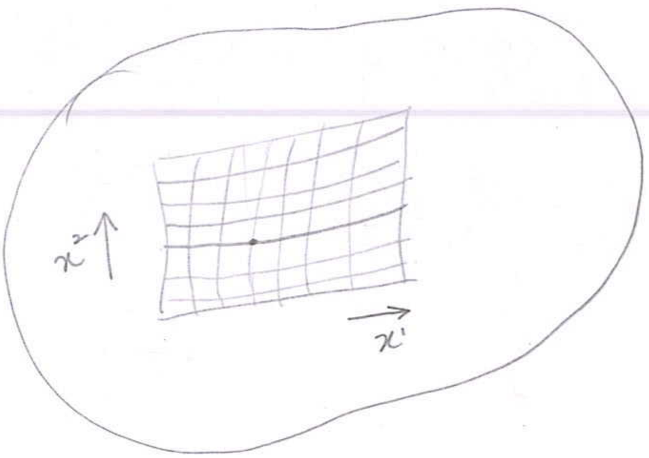
LECTURE - 15

• LOCAL PARALLEL VECTOR FIELDS

Question: Given a Riemannian mfld (M, g), $p \in M$ & $\xi_p \in T_pM$, can ξ_p be extended locally to a parallel v.f?

Note that this is always true if (M, g) is locally isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$.

• Strategy: For simplicity, sps $n = 2$.



Choose local coordinates (x^1, x^2) near p s.t $p = (0,0)$.

Step 1: $\|$- translate Z_p along X^1-axis to cons. $Z(x^1, 0) \in T_{(x^1,0)}M$.

Step 2: For each x^1, $\|$-transport $Z(x^1, 0)$ along the curve (x^1, t).

Clearly, $\nabla_{\partial_2} Z \equiv 0$. But apriori $\nabla_{\partial_1} Z(x^1, x^2) = 0$ only if $x^2 = 0$.

Sps: $\nabla_{\partial_2} \nabla_{\partial_1} Z = \nabla_{\partial_1} \nabla_{\partial_2} Z$, then $\nabla_{\partial_1} Z$ would be a parallel v.f along $\gamma_{x_1}(t) = (x^1, t)$ w/ $\nabla_{\partial_1} Z(\gamma_{x_1}(0)) = 0$.

By uniqueness of ‖-transport $\Rightarrow \nabla_{\partial_1} Z \equiv 0$

So only obstruction is the commutativity of second covariant derivatives.

A measure of this obstruction is given by the Riemann-curvature tensor.

- <u>CURVATURE ENDOMORPHISM</u>.

<u>Def^n</u>: The (Riemann) <u>curvature</u> <u>endomorphism</u> is the map $R : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \times \mathcal{T}^1(M) \longrightarrow \mathcal{T}^1(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

<u>Rk</u>: In particular $\nabla_{\partial_i} \nabla_{\partial_j} Z - \nabla_{\partial_j} \nabla_{\partial_i} Z = 0 \Longleftrightarrow$

$R(\partial_i, \partial_j) Z = 0$.

<u>Prop 15.1</u>: $R \in \mathcal{T}^1_3(M)$.

<u>Pf</u>: We need to check R is $C^\infty(M)$ multi-linear.

Let $f \in C^\infty(M)$,

$$\begin{aligned} R(fX, Y)Z &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z \\ &= f \nabla_X \nabla_Y Z - \nabla_Y f \nabla_X Z - \nabla_{f[X,Y]} Z \\ &\quad + \nabla_{Y(f)X} Z \\ &= f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\ &= f R(X,Y) Z. \end{aligned}$$

A similar calculation shows that $R(X, fY)Z = f R(X,Y)Z$ & $R(X,Y)(fZ) = f R(X,Y)Z$.

<u>Def^n</u>: The (Riemann) <u>curvature</u> <u>tensor</u> is defined by

$$Rm(X, Y, Z, W) = g(R(X,Y)Z, W).$$

i.e. $Rm = R^\flat$.

• (Coordinate description). If $\{x^i\}$ are local coordinates, then we can write

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{l}\, \partial_l.$$

$$Rm(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl}.$$

Then by definition $\boxed{R_{ijkl} = R_{ijk}{}^{p}\, g_{pl}}$

As tensors, we can write

$$R = R_{ijk}{}^{l}\, dx^i \otimes dx^j \otimes dx^k \otimes \partial_l.$$

$$Rm = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l.$$

<u>Rk</u>: (CAUTION): There is no standard convention for definition of R & Rm. Our convention for R is the same as Lee & is widely accepted, but is $-$ve of the def^n in Gallot et. al. Convention for Rm, also from Lee is less standard. Later we will define sectional, Ricci & scalar curvatures; for which the convention is universal.

Prop 15.2: (Naturality). If $\varphi : (M, g) \longrightarrow (\tilde{M}, \tilde{g})$ is a local isometry, then

$$\varphi^* \widetilde{Rm} = Rm$$

$$\tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_* R(X,Y)Z.$$

Cor 15.3: If $\varphi : (\mathcal{U}, g) \longrightarrow (V, g_{\mathbb{R}^n})$ is an isometry

Then $Rm \equiv 0$, $R \equiv 0$.

Pf: In $\mathbb{R}^n$, $R_{g_{\mathbb{R}^n}}(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$

$$= 0.$$

So $R_{g_{\mathbb{R}^n}} \equiv 0$.

Then Prop $\Longrightarrow$ $R \equiv 0$ & hence $Rm \equiv 0$.

- FLAT MANIFOLDS.

Prop 15.4: A Riem. mfld. (M, g) is locally isometric to Euclidean space (i.e $\forall\, p \in M$, $\exists$ nbd. $p \in \mathcal{U} \subset_{op} M$, $V \subset_{open} \mathbb{R}^n$ & an isom. $\varphi_p : (\mathcal{U}, g) \rightarrow (V, g_{\mathbb{R}^n})$ if & only if $R \equiv 0$.

Pf: $\Longrightarrow$ by Cor 15.3.

$\Longleftarrow$: Sps $R \equiv 0$. Let $p \in M$.

Claim!: $\exists$ nbd $\mathcal{U}$ of p & ortho-normal v. fields $\{E_1, \dots, E_n\}$ on $\mathcal{U}$ s.t $\nabla_X E_i \equiv 0$ $\forall$ $X \in \mathcal{T}^1(\mathcal{U})$ & $\forall$ i.

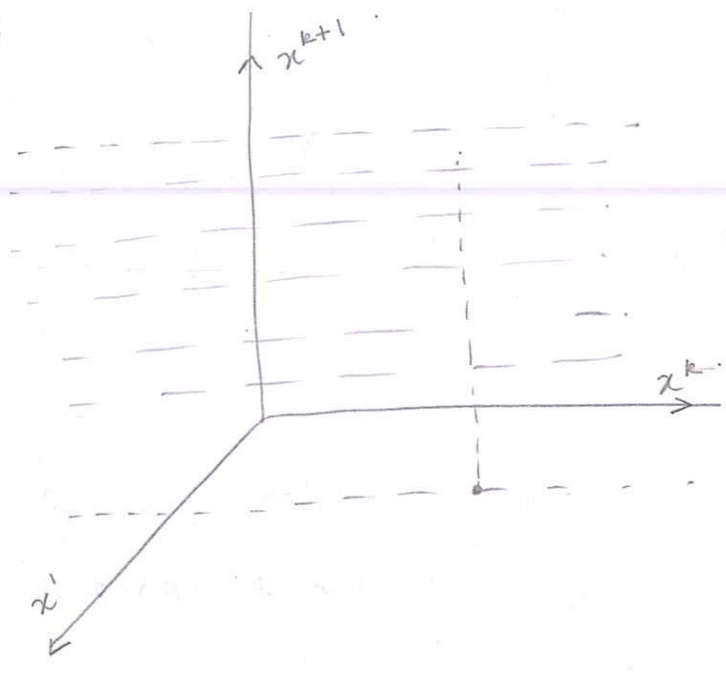
Assuming the claim, by the symmetry of the connection,

$$[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0 .$$

Frobenius $\Longrightarrow$ $\exists$ coordinates $\{y^i\}$ on $\tilde{U} \subset\subset U$ s.t. $E_i = \partial/\partial y^i$. Then the metric tensor in these coordinates is $g_{ij} = g(E_i, E_j) = \delta_{ij}$.

So $\{y^i\} : \tilde{U} \longrightarrow \mathbb{R}^n$ give an isometry.

Pf of Claim 1



Let $E_1|_p, \cdots, E_n|_p$ be an ortho-normal frame at T_pM. W.l.o.g we assume image of U is a cube.

$C_\varepsilon = \{\{x^i\} \mid |x^i| < \varepsilon \ \forall i\}$.

Step 1 : $\parallel$- translate $E_i|_p$ along x^1- axis

Step 2 : $\parallel$- translate the resulting E_i along x^2- axis i.e. along the curve $(x^1, t, 0, \cdots, 0)$.

Step 3 : Succesively $\parallel$- transport along lines $\parallel$ to

x^3 through x^n- axes.

Existence + Uniqueness for ODEs $\Longrightarrow$ $\{E_i\}$ are <u>smooth</u> vector fields on U.

Also $\{E_i\}$ form an o.n.b of T_xM $\forall\, x \in U$.
since $\|$- transport preserves g.

Fix $j \in \{1, \dots, n\}$. By construction $\nabla_{\partial_1} E_j = 0$ on x^1- axis, $\nabla_{\partial_2} E_j = 0$ on (x^1, x^2)- plane. More generally. Let $S_k \subset C_\varepsilon$ be the k- dim slice.

$$S_k = \{x^{k+1} = \dots = x^n = 0\} \subseteq C_\varepsilon$$

<u>Claim 2.</u> $\nabla_{\partial_i} E_j = 0$ on S_k $\forall\, i \in \{1, \dots, k\}$.

Assuming this, we are done, since $S_n = C_\varepsilon$.

<u>Pf of Claim 2</u>: We proceed by induction. $k = 1$ is true by construction. Sps it is true for some $k = p$, i.e $\nabla_{\partial_1} E_j = \dots = \nabla_{\partial_p} E_j = 0$ on S_p.

On S_{p+1}, $\nabla_{\partial_{p+1}} E_j = 0$ by construction. For $i \le p$, since $[\partial_i, \partial_{p+1}] = 0$,

$$\nabla_{\partial_{p+1}} \nabla_{\partial_i} E_j = \nabla_{\partial_i} \nabla_{\partial_{p+1}} E_j + R(\partial_{p+1}, \partial_i) E_j$$

$$= 0$$

So $\nabla_{\partial_i} E_j$ is a $\|$- vector field along lines $\|$ to x^{p+1}- axis. i.e along lines $(x^1, \dots, x^p, t, 0, \dots 0)$

But $\nabla_{\partial_i} E_j \Big|_{t=0} = 0$, since $\nabla_{\partial_i} E_j \Big|_{S_p} \equiv 0$. Uniqueness of $\|$-transport $\Longrightarrow \nabla_{\partial_i} E_j \Big|_{S_{p+1}} \equiv 0$. Done!

· PROPERTIES OF R & Rm.

Prop 15.5 (1st Bianchi identity). $\forall\, X, Y, Z, W \in \mathcal{T}^1(M)$,

(a) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.

$(\Longleftrightarrow R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l = 0)$.

(b) $Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0$.

$(\Longleftrightarrow R_{ijkl} + R_{jkil} + R_{kijl} = 0)$.

Pf: Clearly (a) $\Rightarrow$ (b). To prove (a)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$$

$$= (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$
$$+ (\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X)$$
$$+ (\nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y)$$

$$= \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y]$$
$$- \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y - \nabla_{[X,Y]} Z$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$$

$= 0$ by Jacobi identity.

Prop 15.6 (Symmetries of Rm). Let $X, Y, Z, W \in \mathcal{T}^1(M)$. Then

$$Rm(X,Y,Z,W) \overset{(a)}{=} -Rm(Y,X,Z,W)$$
$$\overset{(b)}{=} -Rm(X,Y,W,Z)$$
$$\overset{(c)}{=} Rm(Z,W,X,Y).$$

In local coordinates

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$$

Pf: (a) Follows from $R(X,Y)Z = -R(Y,X)Z$

(b) Claim: $Rm(X,Y,Z,Z) = 0$.

If true, then $0 = Rm(X,Y,Z+W,Z+W)$

$$= Rm(X,Y,Z,W) + Rm(X,Y,W,Z)$$

and we are done

Pf of Claim: Compute

$$Rm(X,Y,Z,Z) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, Z \rangle.$$

$$\langle \nabla_X \nabla_Y Z, Z \rangle = X \langle \nabla_Y Z, Z \rangle - \langle \nabla_Y Z, \nabla_X Z \rangle$$
$$= \frac{1}{2} XY |Z|^2 - \langle \nabla_Y Z, \nabla_X Z \rangle.$$

Similarly

$$\langle \nabla_Y \nabla_X Z, Z \rangle = \frac{1}{2} YX|Z|^2 - \langle \nabla_X Z, \nabla_Y Z \rangle.$$

$$\langle \nabla_{[X,Y]} Z, Z \rangle = \frac{1}{2}[X,Y]|Z|^2$$

So, $Rm(X, Y, Z, Z) = 0$.

(c) Apply Bianchi 4 time (Exercise).

<u>Cor 15.7</u>: (Curvature operator) $\exists\,!$ linear self adj. operator $Q: \Lambda^2 TM \longrightarrow \Lambda^2 TM$ s.t

$$\langle Q(u \wedge v), w \wedge z \rangle = Rm(u, v, z, w).$$

<u>Rk</u> 1) The Q is called the <u>curvature operator</u>. Note that any one of R, Rm, Q determines the other 2.

2) One can identify $\Lambda^2 TM = \{$2-dim sub-sp of $TM\}$.

<u>Pf</u>: Consider the map $B: \Lambda^2 T_pM \times \Lambda^2 T_pM \longrightarrow \mathbb{R}$

$$B(u_p \wedge v_p, w_p \wedge z_p) = Rm_p(u_p, v_p, z_p, w_p).$$

Then Prop 15.6 $\Longrightarrow$ B is well defined & symmetric.

Since, $B(w_p \wedge z_p, u_p \wedge v_p) = Rm_p(w_p, z_p, v_p, u_p)$

$$= Rm_p(v_p, u_p, w_p, z_p)$$

$$= Rm_p(u_p, v_p, z_p, w_p)$$

linear alg $\Rightarrow$ $\exists$ a linear map. $P: \Lambda^2 TM \to (\Lambda^2 TM)^*$ s.t.

$$B(u_p \wedge v_p, w_p \wedge z_p) = P(u_p \wedge v_p)(w_p \wedge z_p).$$

$\langle , \rangle$ gives an iso $(\Lambda^2 TM)^* \cong \Lambda^2 TM$. So $\exists$ a Q s.t

$$B(u_p \wedge v_p, w_p \wedge z_p) = \langle Q(u_p \wedge v_p), w_p \wedge z_p \rangle$$

B symmetric $\Rightarrow$ Q is self adjoint.
Also Q is smooth since everything in the construction is smooth & has smooth inverses.