LECTURE - 16

- **Recall**: $R : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \times \mathcal{T}^1(M) \longrightarrow \mathcal{T}^1(M)$.

$$R(X, Y) Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

$$Rm(X, Y, Z, W) := \langle R(X,Y)Z, W \rangle.$$

Locally we write $R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{l} \cdot \partial_l$

$$Rm(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl} = g_{pl} \cdot R_{ijk}{}^{p}.$$

- **DIFFERENTIAL BIANCHI**.

Prop 16.1 $\forall\ X, Y, Z, W, V \in \mathcal{T}^1(M)$,

$$\nabla_W Rm(X, Y, Z, V) + \nabla_Z Rm(X, Y, V, W) + \nabla_V Rm(X, Y, W, Z) = 0.$$

In local coordinates if $\nabla_m R_{ijkl} := (ijkl)^{th}$ component of $\nabla_{\partial_m} Rm$, then

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0.$$

Pf: Let $\{x^i\}$ be normal coordinates at $p \in M$.

Then $g_{ij;k}(p) = 0\ \forall\ i, j, k$. Also $[\partial_i, \partial_j] = 0$.

Then at p,

$$\nabla_m R_{klij} = \nabla_m \langle R(\partial_k, \partial_l)\partial_i, \partial_j \rangle.$$

$$= \langle \nabla_m R(\partial_k, \partial_l)\partial_i, \partial_j \rangle \text{ since } \nabla_m(\partial_j)(p) = 0.$$

$$= \langle \nabla_m [\nabla_k \nabla_l \partial_i - \nabla_l \nabla_k \partial_i], \partial_j \rangle.$$

$$= \langle \nabla_m \nabla_k \nabla_l \partial_i - \nabla_m \nabla_l \nabla_k \partial_i , \partial_j \rangle.$$

Similarly at p.

$$\nabla_k R_{lmij} = \langle \nabla_k \nabla_l \nabla_m \partial_i - \nabla_k \nabla_m \nabla_l \partial_i , \partial_j \rangle$$

$$\nabla_l R_{mkij} = \langle \nabla_l \nabla_m \nabla_k \partial_i - \nabla_l \nabla_k \nabla_m \partial_i , \partial_j \rangle$$

So $\nabla_m R_{klij} + \nabla_k R_{lmij} + \nabla_l R_{mkij}$.

$$= \langle R(\partial_m, \partial_k) \nabla_l \partial_i + R(\partial_k, \partial_l) \nabla_m \partial_i + R(\partial_l, \partial_m) \nabla_k \partial_i , \partial_j \rangle.$$

$= 0$, since $\nabla_l \partial_i (p) = \nabla_m \partial_i (p) = \nabla_k \partial_i (p) = 0$.

- COMMUTATION FORMULAE

- NOTATION: Let ∇ be a connection on $E \to M$. If $\{e_\alpha\}$ is a local trivializing basis for E & $\{x^i\}$ local coordinates on M, then

$$\nabla_i s^\alpha = \alpha^{th} \text{ component of } \nabla_{\partial_i} s.$$

i.e $\nabla_{\partial_i} s = \nabla_i s^\alpha \cdot e_\alpha$. or $\iff d_\nabla s = \nabla_i s^\alpha \cdot dx^i \otimes e_\alpha$.

- Recall that Levi-Civita connection induces a connection on T^*M by

$$\nabla_i \alpha_j = \partial_i \alpha_j - \Gamma_{ij}^k \cdot \alpha_k$$

One can then check (since $\nabla g = 0$) that

$$\nabla_X \alpha = (\nabla_X \alpha^\#)^\flat$$

2). ∇ naturally induces a connection on $\Lambda^p T^*M$ by

$$\nabla_X \omega(X_1, \dots X_p) = X\omega(X_1, \dots, X_p) - \sum_{i=1}^{p} \omega(X_1 \dots, \nabla_X X_i, \dots, X_p).$$

<u>Prop</u> 16.2. Let $\xi = \xi^i \partial_i \in \mathcal{T}^1(U)$ & $\alpha = \alpha_j dx^j \in \mathcal{T}_1(U)$ for some $U \subset_{open} M$ w/ coordinates $\{x^i\}$. Then.

(1) $[\nabla_i, \nabla_j]\xi^l = R_{ijk}{}^l \xi^k$.

(2) $[\nabla_i, \nabla_j]\alpha_k = -R_{ijk}{}^l \alpha_l$.

<u>Pf</u>: 1) $[\nabla_i, \nabla_j](\xi^k \partial_k) = \xi^k [\nabla_i, \nabla_j](\partial_k)$

$$= \xi^k \cdot R_{ijk}{}^l \cdot \partial_l.$$

So $[\nabla_i, \nabla_j]\xi^l = \xi^k \cdot R_{ijk}{}^l$

2). Let $\xi = \alpha^\#$. So $\xi^m = g^{mk} \cdot \alpha_k$ or $\alpha_k = g_{mk} \cdot \xi^m$.

Then

$$[\nabla_i, \nabla_j]\alpha_k = [\nabla_i, \nabla_j] g_{mk} \xi^m$$

$$= g_{mk} R_{ijl}{}^m \xi^l$$

$$= R_{ijlk} \xi^l$$

$$= -R_{ijkl}\,\xi^{l} = -R_{ijk}{}^{m}\,g_{ml}\,\xi^{l}$$

$$= -R_{ijk}{}^{m}\,\alpha_m.$$

Rk: We can define $R_{ij}{}^{k}{}_{l} = g^{km}R_{ijml}$. Then

$$[\nabla_i, \nabla_j]\,\alpha_k = R_{ij}{}^{l}{}_{k}\,\alpha_l.$$

Since $R_{ij}{}^{k}{}_{l} = -R_{ijk}{}^{l}$.

Cor 1.6.3 If $T \in \mathcal{T}^r_s(U)$ & $\omega \in \Lambda^p(U)$. Then

① $$[\nabla_k, \nabla_l]\,T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = R_{kli}{}^{i_1}\,T^{i i_2 \cdots i_r}_{j_1 \cdots j_s} + \cdots R_{kli}{}^{i_r}\,T^{i_1 \cdots i_{r-1} i}_{j_1 \cdots j_s}$$
$$- R_{klj_1}{}^{j}\,T^{i_1 \cdots i_r}_{j j_2 \cdots j_s} - \cdots - R_{klj_s}{}^{j}\,T^{i_1 \cdots i_r}_{j_1 \cdots j}.$$

② $$[\nabla_i, \nabla_j]\,\omega_{k_1 \cdots k_p} = -R_{ijk_1}{}^{k}\,\omega_{k k_2 \cdots k_p}$$
$$\cdots - R_{ijk_p}{}^{k}\,\omega_{k_1 \cdots k_{p-1} k}.$$

- **SECTIONAL CURVATURE**

Def^n: Let $X_p, Y_p \in T_pM$ be linearly independent. We define the sectional curvature of $\{X_p, Y_p\}$ by

$$K_p(X_p, Y_p) := \frac{Rm(X_p, Y_p, Y_p, X_p)}{|X_p|^2|Y_p|^2 - \langle X_p, Y_p\rangle^2}$$

Rk: In terms of the curvature operator

$$K_p(X_p, Y_p) = \frac{\langle Q(X_p \wedge Y_p),\, X_p \wedge Y_p\rangle}{\|X_p \wedge Y_p\|^2}$$

Prop 16.4: The quantity $K_p(X_p, Y_p)$ only depends on the 2-plane $\Pi_p = \text{span}\langle X_p, Y_p\rangle \subset T_pM$.

Pf: Let $X_p' = aX_p + bY_p$, $Y_p' = cX_p + dY_p$.

Then by using the symmetries of Rm & linearity

$$Rm(X_p', Y_p', Y_p', X_p') = (ad-bc)^2 Rm(X_p, Y_p, Y_p, X_p)$$

Also $|X_p'|^2 |Y_p'|^2 - \langle X_p', Y_p'\rangle^2$

$$= (ad-bc)^2 \left[|X_p|^2 |Y_p|^2 - \langle X_p, Y_p\rangle^2 \right].$$

So the prop is proved.

Def^n: Given any 2-plane $\Pi_p \subset T_pM$, we define

$$K_p(\Pi_p) := K_p(X_p, Y_p), \text{ where}$$

$\Pi_p = sp\{X_p, Y_p\}$

We can recover Rm from K. To see this, we first need the foll. definition.

Def^n: Let V be a f.d v.space/$\mathbb{R}$. A multilinear form $F: V\times V\times V\times V \longrightarrow \mathbb{R}$ is said to be <u>curvature like</u> if:

(a) $F(x, y, z, w) = -F(y, x, z, w)$

$$= -F(x, y, w, z)$$

$$= F(z, w, x, y).$$

(b) (Bianchi) $F(x, y, z, w) + F(y, z, x, w) + F(z, x, y, w) = 0.$

Lemma 16.5: If F is a curvature-like 4-form on V s.t. $\forall$ linearly independent $x, y \in V$, $F(x,y,y,x) = 0$. Then $F \equiv 0$.

Pf: Claim 1: $\forall x, y, z \in V$, $F(x,y,z,x) = \underset{=0}{F(x,y,x,z)}$

Pf: $0 = F(x, y+z, y+z, x) = F(x, y, y+z, x)$
$$+ F(x, z, y+z, x)$$
$$= F(x,y,y,x) + F(x,y,z,x)$$
$$+ F(x,z,y,x) + F(x,z,z,x).$$
$$= 2F(x,y,z,x)$$

So $F(x,y,z,x) = 0$.

Claim 2: $F(x,y,z,w) = F(y,z,x,w)$.

Pf: By Claim 1

$$0 = F(x+z, y, x+z, w) = \overset{0 \leftarrow \text{Claim 1}}{F(x, \cancel{y}, x, z)} + F(x,y,z,w) + F(z,y,x,w) + \overset{0}{F(z,\cancel{y},z,w)}$$

So $F(x,y,z,w) = -F(z,y,x,w)$
$$= F(y,z,x,w).$$

Now Bianchi $\Rightarrow$

$0 = F(x,y,z,w) + F(y,z,x,w) + F(z,x,y,w)$
$$= 3F(x,y,z,w)$$

So $F(x,y,z,w) = 0$.

Prop. 16.6: Let F be a curvature-type 4-form on T_pM s.t

$$K(u \wedge v) = \frac{F(u,v,v,u)}{|u \wedge v|^2}$$

Then $F(u,v,x,y) = Rm(u,v,x,y)$.

Pf: $\Delta(u,v,x,y) = F(u,v,x,y) - Rm(u,v,x,y)$ is again curvature-like. Moreover, $\Delta(u,v,v,u) = 0 \ \forall u,v$. So Lemma $\Longrightarrow \Delta(u,v,x,y) \equiv 0$.

Rk: 1) In fact it is possible to write a formula for Rm in terms of K. ( ).

2) Next lecture we will give a more geometric meaning to sectional curv.

• RICCI & SCALAR CURVATURE

Def^n: The Ricci curvature is a $(2,0)$ tensor $Rc : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \longrightarrow C^\infty(M)$ defined as

$$Rc(X,Y) = tr(u \longrightarrow R(u,X)Y)$$

Locally. $R_{ij} := Rc(\partial_i, \partial_j) = tr(\partial_k \longrightarrow R_{kij}{}^{l}\partial_l)$

$$= R_{kij}{}^{k}$$

So $\boxed{R_{ij} = R_{kij}{}^{k} = R_{kijl}\, g^{kl}}$

Rk: $R_{ij} = R_{kijl}\, g^{kl} = R_{jlki}\, g^{kl} = R_{ljik}\, g^{lk} = R_{ji}$. So Rc is symmetric.

Prop 16.7: If $\{\frac{X}{|X|}, e_2, \dots, e_n\}$ is an o.n. basis for T_pM, then

$$Rc(X) := Rc(X,X) = \left(\sum_{i=2}^{n} K(X, e_i)\right)|X|^2$$

Pf: $$Rc(X,X) = \sum_{i=2}^{n} Rm(e_i, X, X, e_i) + \overset{0}{\cancel{Rm(X,X,X,X)}}$$

$$= \sum_{i=2}^{n} K(e_i, X)\,|X|^2$$

Def^n: The Ricci curvature of g in the direction of X is defined to be $Rc(X)/|X|^2$.

Def^n: The scalar curvature is defined to be

$$S = tr_g\, Rc$$

i.e. $$S = g^{ij} R_{ij}.$$

Prop 16.8 (Contracted Bianchi).

$$\operatorname{div} Rc := \nabla^j R_{ij} = g^{jk} \nabla_k R_{ij} = \frac{1}{2} \nabla_i S.$$

Pf: $$g^{jk} \nabla_k R_{ij} = g^{jk} \nabla_k R_{lijm}\, g^{lm}$$

$$= g^{lm} g^{jk} \nabla_k R_{lijm}$$

$$= - g^{lm} g^{jk} \nabla_l R_{ikjm} - g^{lm} g^{jk} \nabla_i R_{kljm}$$

$$= - g^{lm} g^{jk} \nabla_l R_{kimj} + g^{lm} g^{jk} \nabla_i R_{klmj}$$

$$= - g^{lm} \nabla_l R_{im} + \nabla_i S$$

$$= - (\mathrm{div}\, Rc)_i + \nabla_i S .$$

So $(\mathrm{div}\, Rc)_i = \frac{1}{2} \nabla_i S$

<u>Def</u>^{n}: (M,g) is said to be <u>Einstein</u> if $Rc = \lambda g$ for some $\lambda \in \mathbb{R}$.

<u>Cor</u> 16.9 If $Rc = \lambda g$ for some $\lambda \in C^\infty(M)$, & $n \geq 3$ then (M,g) is Einstein & $S = n\lambda$ is a const.

<u>Pf</u>: $Rc = \lambda g \Rightarrow S = n\lambda$. So $Rc = Sg/n$

Taking divergence: $\nabla^i R_{ij} = \frac{\nabla^i S \, g_{ij}}{n}$

$$= \frac{\nabla_i S}{n}$$

By contracted Bianchi, $\frac{\nabla_i S}{n} = \frac{\nabla_i S}{2}$

$n \geq 3 \Rightarrow \nabla_i S = 0$ or $S = \text{const}$.

So $\lambda = \text{const}$

$\Rightarrow (M,g)$ is Einstein

<u>Rk</u>: Not true if $n = 2$. For, if $\{e_1, e_2\}$ is an o.n.b. of T_pM, then

$$Rc_p(e_1, e_1) = Rc_p(e_2, e_2) = \lambda(p) \text{ say.}$$

Also
$$Rc(e_1, e_2) = Rm(e_1, e_1, e_2, e_1) + Rm(e_2, e_1, e_2, e_2)$$
$$= 0.$$

So $\quad Rc_p = \lambda(p)\, g_p.$

But of course need not be const.