• SECOND FUNDAMENTAL FORM: Let $(M^n, g)$ be a Riemannian sub-manifold of $(\tilde{M}^m, \tilde{g})$.

Def$^n$: For any $p \in M$, we define the normal sp. to $M$ at $p$ by

$$N_p M := \{ v \in T_p \tilde{M} \mid \tilde{g}(v, u) = 0 \ \forall \ u \in T_p M \}.$$

Then $NM = \bigsqcup_p N_p M$ can be given the structure of a vector bundle over $M$, called the normal bundle. We obtain 2 orthogonal projection since $T\tilde{M} = NM \oplus TM$.

(1) Tangential projection: $\pi^T : T\tilde{M} \longrightarrow TM$.

(2) Normal projection: $\pi^\perp : T\tilde{M} \longrightarrow NM$.

We also denote

$$\mathcal{T}(\tilde{M}|_M) := \Gamma(M, T\tilde{M}|_M).$$

$$\mathcal{N}(M) := \Gamma(M, NM).$$

$\tilde{\nabla}, \nabla$ : Levi-Civita connection on $(\tilde{M}, \tilde{g})$ & $(M, g)$ resp.

Recall: $\nabla_X Y = (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^T$, where $\tilde{X}$ & $\tilde{Y}$ are arbitrary extension of $X$ & $Y$ to $\tilde{M}$.

Def$^n$: The second fundamental form of $M$ is defined to be a map $\mathrm{II} : \mathcal{T}(M) \times \mathcal{T}(M) \longrightarrow \mathcal{N}(M)$.

$$\mathrm{II}(X, Y) = (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^\perp$$

where $\tilde{X}, \tilde{Y}$ are arbitrary extensions of $X$ & $Y$ to $\tilde{M}$

Prop 17.1: $\mathrm{II}(X, Y)$ is

(a) well defined i.e independent of extension

(b). bi-linear over $C^\infty(M)$.

(c). Symmetric in $X, Y$.

Rk: These follow from the fact that

$$\boxed{\mathrm{II}(X, Y) = \tilde{\nabla}_X \tilde{Y} - \nabla_X Y}$$ Gauss formula.

Prop 17.2 (a) (Weingarten eq): If $X, Y \in \mathcal{T}(M)$, $N \in \mathcal{N}(M)$, then

$$\langle \tilde{\nabla}_X N, Y \rangle = - \langle N, \mathrm{II}(X, Y) \rangle.$$

Here $\langle , \rangle = \tilde{g}$

b) (Gauss eq$^n$): $\forall\ X, Y, Z, W \in T_pM$

$$\widetilde{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) - \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z) \rangle + \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W) \rangle$$

Pf (a) $0 = X\langle N, Y \rangle = \langle \tilde{\nabla}_X N, Y \rangle + \langle N, \tilde{\nabla}_X \tilde{Y} \rangle$

$$= \langle \tilde{\nabla}_X N, Y \rangle + \langle N, \nabla_X Y \rangle + \langle N, \mathrm{II}(X, Y) \rangle$$

$\nabla_X Y \in \mathcal{T}(M) \implies \langle N, \nabla_X Y \rangle = 0$

(3)

(b) $\widetilde{Rm}(X, Y, Z, W) = \langle \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{\widetilde{[X,Y]}} \tilde{Z}, \tilde{W} \rangle$

$$\langle \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{W} \rangle = \langle \tilde{\nabla}_{\tilde{X}} ( \nabla_Y Z + \mathrm{II}(Y, Z)), W \rangle$$

$$\overset{\text{Weingarten}}{=} \langle \tilde{\nabla}_{\tilde{X}} \nabla_Y Z, W \rangle - \langle \mathrm{II}(Y, Z), \mathrm{II}(X, W) \rangle$$

Again writing $\tilde{\nabla}_{\tilde{X}} = \nabla_X + \mathrm{II}(X, \cdot)$, since $\mathrm{II}(X, \cdot)$ is $\perp W$, we have

$$\langle \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{W} \rangle = \langle \nabla_X \nabla_Y Z, W \rangle - \langle \mathrm{II}(Y, Z), \mathrm{II}(X, W) \rangle$$

$|||^{ly}$

$$\langle \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z}, \tilde{W} \rangle = \langle \nabla_Y \nabla_X Z, W \rangle - \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W) \rangle$$

Also, $\langle \tilde{\nabla}_{[X,Y]} Z, \tilde{W} \rangle = \langle \nabla_{[X,Y]} Z, W \rangle$

This completes the proof.

HYPERSURFACES. From now on we assume $(M^n, g) \subset (\tilde{M}^{n+1}, \tilde{g})$ & $\tilde{M}$ is oriented.

Lemma 17.3: $M^n$ is oriented $\iff$ $\exists$ smooth section $\nu \in \mathcal{N}(M)$ s.t $|\nu| = 1$.

Def$^n$ Given a unit normal section $\nu \in \mathcal{N}(M)$, the scalar second fundamental form is defined to be $h : \mathcal{T}(M) \times \mathcal{T}(M) \longrightarrow \mathbb{R}$,

$$h(X, Y) = \langle \mathrm{II}(X, Y), \nu \rangle$$

Since $|\nu| = 1$, equivalently $\mathrm{II}(X, Y) = h(X, Y)\nu$

<u>Note</u>: Sign of $h$ depends on the choice of $\nu$.

<u>Def$^n$</u>: Given $\nu$, the <u>shape</u> operator is defined as $s \in \mathcal{T}^1_1(M)$ by.

$$\langle X, sY \rangle = h(X, Y).$$

<u>Rk</u>: 1). $h(X, Y)$ is symmetric & hence $s$ is <u>self adjoint</u> i.e.

$$\langle s(X), Y \rangle = \langle X, sY \rangle.$$

2) The Gauss formula & Weingarten & Gauss equation take form

$$\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \nabla_X Y + h(X, Y)\nu$$

$$\langle \widetilde{\nabla}_{\widetilde{X}} N, \widetilde{Y} \rangle = -h(X, Y) = -\langle X, sY \rangle.$$

$$\widetilde{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) - h(X, W)h(Y, Z) + h(X, Z)h(Y, W)$$

3). Since $\langle \nabla_{\widetilde{X}} \nu, \nu \rangle = \frac{1}{2}\widetilde{X}|\nu|^2 = 0$, Weingarten

$$\Longleftrightarrow \quad \boxed{\widetilde{\nabla}_X \nu = -sX}$$

4) If $(\widetilde{M}; \widetilde{g}) = (\mathbb{R}^{n+1}, g_{\mathbb{R}^{n+1}})$, then

$$Rm(X, Y, Z, W) = h(X, W)h(Y, Z) - h(X, Z)h(Y, W)$$

• <u>GAUSS CURVATURE</u>: Since $S : T_pM \to T_pM$ is a self adjoint operator, $\exists$ an o-n-b $\{E_i\}$ of eigenvectors of $S$, i.e. $SE_i = K_i E_i \; \forall i$.

<u>Defn</u>. 1) The eigenvalues $\{K_i\}$ are called the principle curvatures of $(M, g)$ & the corresponding eigen-spaces are called principle directions

2) The <u>Gaussian</u> <u>curvature</u> of $M$ is defined to be $K(M) = \det S = K_1 K_2 \cdots K_n$.

The <u>mean curvature</u> of $M$ is defined to be.

$$H(M) = \frac{\sum K_i}{n}.$$

<u>Rk</u> 1). $K_i$ change by a sign if $\nu \longleftrightarrow -\nu$. This also changes $H(M) \longleftrightarrow -H(M)$. A more intrinsic object is mean-curvature vector

$$\vec{H}(M) := H(M) \cdot \vec{\nu}.$$

The Gauss curv. <u>does not change</u> if $n \in 2\mathbb{N}$.

2) $M$ is called <u>minimal</u> if $H(M) = 0$. These are critical points of the "area functional"

$$M \longrightarrow \mathcal{A}(M) = \int_M dV_g .$$

<u>Th$^m$ 17.4</u> (Gauss' th$^m$ egregium) Let $M \subset \mathbb{R}^3$ be a hypersurface & $g$ the induced metric. For any $p \in M$ & <u>any</u> l.i $X, Y \in T_pM$,

$$K(M) = \frac{Rm(X, Y, Y, X)}{|X \wedge Y|^2} =: K(T_pM)$$

So Gauss Curv is an isometry invariant.

<u>Pf</u>: We have already proved that R.H.S is independent of the basis. Let $X = E_1$, $Y = E_2$ s.t $\{E_1, E_2\}$ o.n.b of $T_pM$ of eigenvectors i.e $SE_i = \kappa_i E_i$, $i = 1, 2$. In particular $h(E_i, E_j) = \kappa_i \delta_{ij}$. Gauss eq$^n$ $\Longrightarrow$

$$Rm(E_1, E_2, E_2, E_1) = h(E_1, E_1)\, h(E_2, E_2) - h(E_1, E_2)^2$$

$$= \kappa_1 \kappa_2 = K(M).$$

<u>Th$^m$ 17.5</u>: Let $(M, g)$ be an abstract Riem. mfld. & $\Pi \subset T_pM$ be a 2-plane. If $\tilde{\mathcal{U}}_p \subset T_pM$ is a normal, we let $S_\Pi = \exp_p(\tilde{\mathcal{U}}_p \cap \Pi)$. Then

$$K(\Pi) = K(S_\Pi).$$

where $K(S_\Pi)$ is the Gauss-curvature of $S_\Pi$ w/ induced metric $\tilde{g}$.

<u>Pf</u>: Let $X, Y$ o.n.basis for $\Pi$. Then

$$K(\Pi) = Rm(X, Y, Y, X)$$

$$K(S_\Pi) = \widetilde{Rm}(X, Y, Y, X).$$

Claim: $\mathrm{II}(X,Y) = 0 = \mathrm{II}(X,X) = \mathrm{II}(Y,Y)$.

If so, then Gauss eq $\Rightarrow$ $Rm(X,Y,Y,X) = \widetilde{Rm}(X,Y,Y,X)$

So Done!

Pf of Claim: Let $\gamma_X(t) = \exp_p(tX) \subset S_\Pi$. Then Gauss formula $\Rightarrow$

$$\widetilde{D}_t \dot{\gamma} = D_t \dot{\gamma} + \mathrm{II}(\dot{\gamma}, \dot{\gamma})$$

$\widetilde{D}_t \dot{\gamma} = 0$ since $\gamma$ is a geodesic in $\widetilde{M}$. Since on the right we have an orthogonal decomp.

Hence $\mathrm{II}(\dot{\gamma}, \dot{\gamma}) \equiv 0$. In particular, $\mathrm{II}(X,X) = 0$.

|||$^{ly}$ $\mathrm{II}(Y,Y) = 0$. Polarization $\Rightarrow$ $\mathrm{II}(X,Y) = 0$.

• CURVATURE OF CURVES. Let $(M^n, g) \subset (\widetilde{M}^m, \widetilde{g})$

Prop 17.6: Let $\gamma$ be a curve in $M$. For any $V(t) \in T_{\gamma(t)}M$, we have

$$\widetilde{D}_t V = D_t V + \mathrm{II}(\dot{\gamma}, V)$$

Pf: Let $\{E_i\}$ be a ||- v.f along $\gamma$ & $V(t) = V^i(t) E_i$.

Then Gauss formula $\Rightarrow$

$$\begin{aligned}
\widetilde{D}_t V &= \dot{V}^i E_i + V^i \widetilde{\nabla}_{\dot{\gamma}} E_i \\
&= \dot{V}^i E_i + V^i \nabla_{\dot{\gamma}} E_i + V^i \mathrm{II}(\dot{\gamma}, E^i) \\
&= D_t V + \mathrm{II}(\dot{\gamma}, V)
\end{aligned}$$

Cor 17.7: 1) If $\gamma$ is a geodesic in $M$, then

$$\tilde{D}_t \dot{\gamma} = \mathrm{II}(\dot{\gamma}, \dot{\gamma})$$

2) If $n = 2$, $\tilde{M} = \mathbb{R}^3$, then the

$$K_1(p) = \min_{v \in T_pM} \frac{\langle \ddot{\gamma}_v(0), \nu \rangle}{|v|^2}, \quad K_2 = \max_{v \in T_pM} \frac{\langle \ddot{\gamma}_v(0), \nu \rangle}{|v|^2}$$

where $\gamma_v(t) = \exp_p(tv)$.

Example: 1) $M = S^2_R \subset \mathbb{R}^3$. Let $p = N$, $v_0 = (1, 0, 0)$ $\in T_N S^2_R$. $\nu = (0, 0, 1)$. Then

$$\gamma_v(t) = (R\sin(t/R), 0, R\cos(t/R))$$

$$\dot{\gamma}_v = (\sin t/R, 0, \cos t/R)$$

$$\ddot{\gamma}_v(0) = (1/R, 0, 0)$$

So $\dfrac{\langle \ddot{\gamma}_v(0), \nu \rangle}{|v|^2} = \dfrac{1}{R}$

By symmetry $\forall\, v \in T_N S^2_R$, $\dfrac{\langle \ddot{\gamma}_v(0), \nu \rangle}{|v|^2} = \dfrac{1}{R}$

So $K_1 = K_2 = 1/R$. $K(S^2_R) \equiv 1/R^2$.

More generally if $M = S^n_R \subset \mathbb{R}^{n+1}$, if $\Pi_N = sp\{\partial_1, \partial_2\}$ where $N = (0, 0, \cdots, 1)$. Then for any $v \in \Pi_N$, $\gamma_v$ is a great circle in $(x^1, x^2, x^{n+1})$ space. So $S_\Pi \cong S^2_R$ & so $K(\Pi) = K(S_\Pi) = 1/R^2$.

By symmetry, $\forall\, p \in S^n_R$, $\forall\, \Pi \subset T_p S^n_R$,

$$K_p(\Pi) = 1/R^2.$$

(2) $M^n = \mathbb{H}^n_R := \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -x_0^2 + \dots + x_n^2 = -R^2,\ x_0 > 0\}$.

Assignment problem $\Longrightarrow$ $K_p(\Pi) = -1/R^2$ $\forall\, p \in \mathbb{H}^n_R$ & all $\Pi \subset T_p \mathbb{H}^n_R$.

Def$^n$: We say $(M, g)$ has <u>constant</u> sectional curvature $K$ if $K_p(\Pi) = K$ $\forall\, p \in M$, $\forall\, \Pi \subset T_p M$.

<u>Prop 17.8</u>: If $(M, g)$ has constant sectional curvature $K$, then

$$Rm(X, Y, Z, W) = K\big(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\big).$$

<u>Pf</u>: Let $F(X, Y, Z, W) = K\big(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle\big).$

Then $F(X, Y, Y, X) = K\big(|X|^2|Y|^2 - \langle X, Y\rangle^2\big)$
$= Rm(X, Y, Y, X)$.

Moreover $F$ is curvature like. So

$$F(X, Y, Z, W) = Rm(X, Y, Z, W).$$

<u>Rk</u>: Locally,

$$\boxed{\begin{aligned} R_{ijkl} &= K\, g_{il}\, g_{jk} - g_{ik}\, g_{jl} \\ R_{jk} &= K(n-1)\, g_{jk} \\ R &= K n(n-1). \end{aligned}}$$

<u>Prop 17.9</u> (Schur). Let $n \geq 3$. If $\exists$ $K \in C^\infty(M)$ s.t $\forall\, p \in M$, $\forall\, \Pi \subset T_pM$, $K_p(\Pi) = K(p)$. Then $K(p) \equiv$ const.

<u>Pf</u>: Again if $F(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle$

Then $Rm_p(X, Y, Z, W) = K(p) \cdot F_p$.

By Rk, $R_{ij} = K(p)(n-1)\, g_{ij}$.

$n \geq 3 \overset{\text{Cor 16.9}}{\Longrightarrow} K \equiv \text{const}.$