LECTURE - 18

- Let V be an n-dim $\mathbb{R}$-vector sp.

  $GL(V)$ = $\mathbb{R}$-linear automorphisms of V.

Then $GL(V) \curvearrowright T^r_s(V) = \bigotimes^r V \otimes \bigotimes^s V$ by

$$\gamma \cdot (\xi_1 \otimes \cdots \otimes \xi_r \otimes \varphi^1 \otimes \cdots \otimes \varphi^s)$$

$$= \gamma\xi_1 \otimes \cdots \otimes \gamma\xi_r \otimes {}^t\gamma^{-1}\varphi^1 \otimes \cdots \otimes {}^t\gamma^{-1}\varphi^s$$

where ${}^t\gamma \in GL(V^*)$ is characterized by

$$ {}^t\gamma(\varphi)(\xi) = \varphi(\gamma\xi)$$

- If q is a non-degenerate form on V, we can identify $V \approx V^*$

- $O(q) := \{\gamma \in GL(V) \mid q(\gamma x, \gamma y) = q(x, y)\}$

  We have an isomorphism $\flat : V \longrightarrow V^*$ s.t

$$\langle x^\flat, y\rangle = q(x, y)$$

natural pairing $\langle , \rangle : V \times V^* \longrightarrow \mathbb{R}$.

Under this identification

$$O(q) = \{\gamma \in GL(V) \mid {}^t\gamma^{-1} = \gamma\}$$

& $V \approx V^*$ as $O(q)$-modules

From now on we focus only on V^* (later we will apply this discussion to $V = T_pM$)

- $\Lambda^2 V^* = \{\omega : V \times V \longrightarrow \mathbb{R} \mid \omega(x, y) = -\omega(y, x)\} \subset T^0_2 V$.

  $S^2 V^* = \{\varphi : V \times V \longrightarrow \mathbb{R} \mid \varphi(x, y) = \varphi(y, x)\}$.

Def^n: The quadratic form q induces a trace operator $tr_q : S^2V^* \longrightarrow \mathbb{R}$ defined by

$$tr_q(\varphi) := \sum_i \varphi(e_i, e_i).$$

where e_i is a q-orthonormal frame.

We define

$$S^2_0 V^* := \ker(tr_q).$$

- Def^n: Let E be a v.s. A representation ρ of a group G on E is a group homomorphism $\rho : G \to GL(E)$. We say ρ is irreducible if for any sub-space $W \subset E$, $\rho_g(W) \subset W \Rightarrow W = \{0\}$ or E. i.e $\nexists$ non-trivial ρ-invariant sub-space.

Rk: 1). If W is a ρ-invariant sub-space, then ρ induces a representation $\rho|_W : G \longrightarrow GL(W)$.

2). Clearly the trivial representation $GL(E) \curvearrowright E$ is irreducible. In fact if q is a n.d quadratic form on E, then $O(q) \curvearrowright E$ is also irreducible. But $O(q) \curvearrowright \otimes^2 E$ need not be irreducible.

3) FACT: Any fd representation of $O(n)$ is decomposible into irreducible orth. components.

Prop 18.1: The irreducible, orthogonal decomoposition of $\otimes^2 V^*$ is given by.

$$\bigotimes^2 V = \Lambda^2 V^* \oplus S_0^2 V^* \oplus \mathbb{R} q .$$

"Outline of the Pf": For any $B \in \bigotimes^2 V$, we can write.

$$\Lambda^2 B(x,y) = \frac{1}{2}\left(B(x,y) - B(y,x)\right)$$

$$S^2 B(x,y) = \frac{1}{2}\left(B(x,y) + B(y,x)\right)$$

$$S_0^2 B = S^2 B - \frac{tr_q B}{n} \cdot q$$

Note $\Lambda^2 B \in \Lambda^2 V^* \Longrightarrow tr_q \Lambda^2 B = 0$ for any q.

So $tr_q B = tr_q S^2 B$, hence $tr_q S_0^2 B = 0$. Clearly.

$$B = \Lambda^2 B + S_0^2 B + \frac{tr_q B}{n} \cdot q .$$

One can show that this is q- orthogonal & $O(q) \circlearrowright$ $\Lambda^2 V^*$, $S_0^2 V^*$, $\mathbb{R} q$ are irreducible representations.

- ALGEBRAIC CURVATURE TENSORS.

- Recall that the curv. tensor lies inside the subspace $S^2 \Lambda^2 T^* M = \{$ symm- bilinear forms on $(\Lambda^2 T^* M)^* = \Lambda^2 TM \}$ via

$$R(x \wedge y, w \wedge z) = Rm(x, y, z, w).$$

i.e $R = Q^\flat$.

Def^n: The Biancchi map is a map $b : \bigotimes^4 V^* \longrightarrow \bigotimes^4 V^*$ defined by

$$b(R)(x, y, z, w) = \frac{1}{3}\left(R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w)\right).$$

We can think of $S^2\Lambda^2 V^* \subset \otimes^4 V^*$, as

$$S^2\Lambda^2 V^* = \{ R \in \otimes^4 V \mid R \text{ satisfies curv. like symmetries} \}$$

Lemma 18.2 1) b is $GL(V^*)$ equivariant i.e $b(\gamma \cdot R) = \gamma \cdot b(R)$

2) b is idempotent i.e $b^2 = b$

3) $b : S^2\Lambda^2 V^* \longrightarrow S^2\Lambda^2 V^*$

4) We have a $GL(V^*)$ equi-variant decomp

$$S^2\Lambda^2 V^* = \ker b \oplus \operatorname{Im}(b)$$

Pf: 1), 4) & 2) are trivial. For 3), we have to check that if R satisfies the symmetrie then $b(R)$ also does. This is easy to check

Rk: In fact $\operatorname{Im}(b)\big|_{S^2\Lambda^2 V^*} = \Lambda^4 V^*$. To see this sps $R \in S^2\Lambda^2 V^*$. Then $b(R) \in S^2\Lambda^2 V^*$. So

$$b(R)(x,x,z,w) = b(R)(x,y,z,z) = 0. \text{ Also}$$

$$b(R)(x,y,x,w) = \frac{1}{3}\big(R(y,x,x,w) + R(x,x,y,w) + R(x,y,x,w)\big)$$

$$= 0$$

So $b(R)(x,y,z,w) = 0$ whenever 2 entries are same and so $b(R) \in \Lambda^4 V^*$.

Def^n We define the vector sp of *algebraic curvature tensors* by

$$\mathcal{C}V^* := \ker b\big|_{S^2\Lambda^2 V^*}$$

Rk: If $n = 2$ or 3, then $\Lambda^4 V^* = \{0\}$, and so by earlier Rk & Lemma 18.2(4), $\mathcal{C}V^* = S^2\Lambda^2 V^*$.

Example: Consider

$$q \owedge q\,(x, y, z, w) := 2\big(q(x, w)\,q(y,z) - q(x,z)\,q(y, w)\big).$$

Then one can show that $q \owedge q \in \mathcal{C}V^*$

More generally,

Def^n: The Kulkarni-Nomizu product is defined for $h, k \in S^2V^*$ by

$$h \owedge k\,(x, y, z, w) = \frac{1}{2}\Big[h(x,w)\,k(y,z) + h(y,z)\,k(x,w) - h(x,z)\,k(y,w) - h(y,w)\,k(x,z)\Big].$$

Prop 18.3: ① $\owedge : S^2V^* \times S^2V^* \longrightarrow \mathcal{C}V^*$ & is symmetric & bi-linear

② $q \owedge q$ is the identity on $\Lambda^2 V^*$ via the identification of $\mathrm{End}(\Lambda^2V^*) \simeq \Lambda^2V^* \otimes \Lambda^2 V \simeq_q \otimes^2 \Lambda^2 V^* \subset \otimes^4 V^*$.

Pf: ① is simply a computation.

For ② note that a $T \in \mathrm{End}(\Lambda^2V^*)$ defines an element in $B_T \otimes^2 \Lambda^2 V^*$ by

$$B_T(x, y, z, w) = T\big((x \wedge y)^\flat\big)(w \wedge z)$$

where $\flat : \Lambda^2 V \longrightarrow \Lambda^2 V^*$ s.t. $(x\wedge y)^\flat(w \wedge z) = q(x \wedge y, w \wedge z)$. If $T = id$, then

$$B_{id}(x \wedge y, w \wedge z) = q(x \wedge y, w \wedge z) = \begin{vmatrix} q(x,w) & q(x,z) \\ q(y,w) & q(y,z) \end{vmatrix}$$

$$= q \owedge q\,(x, y, z, w).$$

· **Def^n** The <u>Ricci</u> <u>contraction</u> is the map $C : S^2\Lambda^2V^* \longrightarrow S^2V^*$ defined by

$$C(R)(x, y) = \operatorname{tr}_q R(\cdot, x, y, \cdot).$$

<u>Rk</u>: Given x, y, $(u, v) \longmapsto R(u, x, y, v)$ means that we can think of $R(\cdot, x, y, \cdot) \in S^2V^*$. So tr_q is defined as before.

· Consider the map $q \owedge \cdot : S^2V^* \longrightarrow \mathcal{C}V^*$ given by

$$k \longmapsto q \owedge k.$$

<u>Lemma 18.4</u> $\forall\, R \in \mathcal{C}V^*$ & $\forall\, k \in S^2V^*$, we have

$$\langle R, q \owedge k \rangle = 4 \langle C(R), k \rangle.$$

Here $\langle\,,\rangle$ is the quadratic forms on $\mathcal{C}V^*$ & S^2V^* by q.

<u>Pf</u>: If $\{e^i\}$ is a q-orthonormal basis on V^*, then $\{e^{i_1} \otimes e^{i_2} \otimes e^{i_3} \otimes e^{i_4}\}$ & $\{e^{i_1} \otimes e^{i_2}\}$ resp form o.n.b on $\bigotimes^4 V^*$ & $\bigotimes^2 V^*$. Then

$$(q \owedge k)_{pqrs} = \frac{1}{2}\left[\delta_{ps} k_{qr} + \delta_{qr} k_{ps} - \delta_{pr} k_{qs} - \delta_{qs} k_{pr}\right].$$

and so if we let $C(R) = R_{ij}\, e^i \otimes e^j$, $R_{ij} = R_{kijk}$, then

$$\langle R, q \owedge k \rangle = \underline{R_{pqrs}}\,(\delta_{ps} k_{qr} + \delta_{qr} k_{ps} - \delta_{pr} k_{qs} - \delta_{qs} k_{pr})$$

$$= \frac{1}{2}\left(R_{qr}\, k_{qr} + R_{ps}\, k_{ps} - R_{pqps}\, k_{qs} - R_{pqrq}\, k_{pr}\right.$$

$$= 4 \langle c(R), k \rangle.$$

<u>Lemma 18.5</u>: For any $k \in S^2 V^*$,

$$c(q \owedge k) = (n-2)\, k + (\mathrm{tr}_q k)\, q.$$

<u>Pf</u>: Again using the o.n.f

$$2c(q \owedge k)_{qr} = (q \owedge k)_{pqrp}$$

$$= \frac{1}{2}\left[n \cdot k_{qr} + \delta_{qr}\, k_{pp} - 2k_{qr}\right].$$

$$= (n-2)\, k_{qr} + \frac{\mathrm{tr}_q k}{2} \cdot \delta_{qr}.$$

Done!

<u>Lemma 18.6</u>: The map $q \owedge \cdot : S^2 V^* \longrightarrow \mathcal{C} V^*$ is injective if $n \geq 3$.

<u>Pf</u>: Sps k s.t $q \owedge k = 0$. Then

$$0 = \langle q \owedge k, q \owedge k \rangle \overset{\text{Lem } 18.4}{=\!=} 4 \langle c(q \owedge k), k \rangle.$$

$$\overset{\text{Lem}}{\underset{18.5}{=}} 4\left((n-2)|k|^2 + \frac{(\mathrm{tr}_q k)^2}{2}\right).$$

So $|k|^2 = 0$ or $k \equiv 0$.

Thm 18.7: If $n \geq 4$, the $O(q)$-module $\mathcal{C}V^*$ has the following decomposition into (unique) irreducib sub-spaces.

$$\mathcal{C}V^* = \mathcal{U}V^* \oplus \mathcal{L}V^* \oplus \mathcal{W}V^*, \qquad (*)$$

where

$$\mathcal{U}V^* = \mathbb{R}\, q \owedge q$$

$$\mathcal{L}V^* = q \owedge (S_0^2 V^*)$$

$$\mathcal{W}V^* = \ker(c).$$

Pf: Let $R \in \mathcal{C}V^*$. We let $S(R) := \mathrm{tr}_q\, c(R)$;

$$c(R)_0 := c(R) - \frac{Sq}{n} \in S_0^2 V^* \quad \&$$

$$W(R) = R - \frac{c(R)_0 \owedge q}{n-2} - \frac{S}{2n(n-1)}\, q \owedge q. \qquad (*)$$

$$= R - A \owedge q,$$

where $A := \dfrac{1}{n-2}\left(c(R) - \dfrac{S}{2(n-1)}\, q \right)$

Claim 1: $c(W) = 0$

Pf: $\mathrm{tr}_q\, A = \dfrac{1}{n-2}\left(S - \dfrac{Sn}{2(n-1)} \right) = \dfrac{S}{2(n-1)}$

Lemma 18.5 $\Longrightarrow$

$$c(W) = c(R) - (n-2)A - (n-2)\,\mathrm{tr}_q A \cdot q$$

$$= 0.$$

Claim 2 The decomposition in (**) is orthogonal.

Pf. 1) $\left\langle \frac{c(R)_0 \owedge q}{n-2}, \frac{S}{2(n-1)n} q \owedge q \right\rangle$

$$\overset{\text{Lem 18.4}}{=} \quad 4 \left\langle \frac{c(c(R)_0 \owedge q)}{n-2}, \frac{S}{2(n-1)n} q \right\rangle$$

Lemma 18.5 $\Rightarrow$ $c(c(R)_0 \owedge q) = (n-1)c(R)_0 + \mathrm{tr}_q\, c(R_0) q$

$$= (n-1) c(R)_0$$

So $\left\langle \frac{c(R)_0 \owedge q}{n-2}, \frac{S}{2(n-1)n} q \owedge q \right\rangle$

$$= 2 \frac{(n-1)S}{n(n-2)^2} \langle c(R)_0, q \rangle$$

$$= 2 \frac{(n-1)S}{n(n-2)} \mathrm{tr}_q\, c(R)_0 = 0$$

2) $\langle W, q \owedge q \rangle = \langle R, q \owedge q \rangle - \frac{S}{2(n-1)n} \langle q \owedge q, q \owedge q \rangle$

Now, $\langle q \owedge q, q \owedge q \rangle = 4 \langle c(q \owedge q), q \rangle$

$$= 4 \langle (n-2) q + n q, q \rangle$$

$$= 8(n-1)n$$

$\langle R, q \owedge q \rangle = 4 \langle c(R), q \rangle$

$$= 4 \left\langle c(R)_0 + \frac{S}{n} q, q \right\rangle = 4S$$

$\Rightarrow \langle W, q \owedge q \rangle = 0$.

3) $\langle W, c(R)_\circ \owedge q \rangle = \langle R, c(R)_\circ \owedge q \rangle$

$$- \frac{1}{n-2} \langle c(R)_\circ \owedge q, c(R)_\circ \owedge q \rangle$$

$$\langle R, c(R)_\circ \owedge q \rangle = 4 \langle c(R), c(R_\circ)_\circ \rangle$$

$$= 4 \langle c(R)_\circ + \frac{S}{n} q, c(R)_\circ \rangle$$

$$= 4 |c(R)_\circ|^2$$

$$\langle c(R)_\circ \owedge q, c(R)_\circ \owedge q \rangle = 4 \langle c(c(R)_\circ \owedge q), c(R)_\circ \rangle$$

But $c(c(R)_\circ \owedge q) = (n-2) c(R)_\circ + tr_q c(R)_\circ \cdot q$

$$= 4(n-2) c(R)_\circ$$

So $\langle c(R_\circ)_\circ \owedge q, c(R)_\circ \owedge q \rangle = 4(n-2) |c(R)_\circ|^2$.

$\Rightarrow \langle W, c(R)_\circ \owedge q \rangle = 0$

Claim 3: $\mathcal{U}V^*$, $\mathcal{L}V^*$ & $\mathcal{W}V^*$ are inv. under $O(q)$-action

Pf: $\mathcal{U}V^*$, $\mathcal{L}V^*$ are invariant since

$$c(r \cdot R) = r \cdot c(R) \quad \forall\ r \in O(q).$$

$\mathcal{W}V^*$ i also inv. since $R \in \ker c$

then $c(r \cdot R) = r \cdot c(R) = 0$, so $r \cdot R \in \ker c$.

Claim 4: $O(q)$ restricts to irreducible rep. on $\mathcal{U}V^*$, $\mathcal{L}V^*$ & $\mathcal{W}V^*$

"Outline of Pf": It follows from inv. theory that the only $O(q)$ - inv quadratic form on $\otimes^4 V^*$ is 3-d & spanned by $q(R,R)$, $q(c(R),c$ & $(\mathrm{tr}_q C(R))^2$. Sps $W \subset WV^*$ is an inv sub-sp then so is $W^\perp \subset WV^*$. Consider the quadratic form

$$B(R,R) = q(\pi^\perp R, \pi^\perp R) \quad \text{on} \quad WV^*,$$

where $\pi^\perp : WV^* \longrightarrow W^\perp$ is the q-orth. proj.

Then
$$\begin{aligned} B(R,R) &= a\,q(R,R) + b\,q(c(R), c(R)) \\ &\qquad + c[\mathrm{tr}_q\, C(R)]^2 \\ &= a q(R,R) \quad \text{since} \quad R \in \ker c. \end{aligned}$$

If $W^\perp \neq 0$ (i.e $W \neq WV^*$), then letting $R \in W^\perp$, we see $a = 1$. But then if $R \in W$, we have $q(R,R) = 0$. So $R = 0$. i.e $W \neq WV^* \Longrightarrow W = \{0\}$ & vice-versa. So $O(q)$ is irreducible on WV^*. The other 2 are the same.

Def^n: 1) WV^* is called the set of Weyl-type tensors, and for any $R \in \mathcal{C}V^*$, $W(R)$ is called the corresponding Weyl tensor. If $R = Rm$, we simply write $W(R)$ as W.

2). The tensor A is called the Schouten tensor.

Rk On (M^n, g) orientable.

1) $n = 2$, $Rm = \frac{\overset{\text{Gauss}}{K}}{2} g \owedge g = \frac{S}{4} g \owedge g$.

2) $n = 3$, one can show that $W = 0$, so that

$$Rm = \left(Ric - \frac{S}{4} g \right) \owedge g .$$

3) $n > 4$, the decomposition $(*)$ is also into $SO(n)$ irr. components. But when $n = 4$, W is not $SO(4)$ irreducible & we further get

$$W = W^+ + W^-$$

4). (M, g) is said to be locally conformally flat if $\forall\, p \in M$, $\exists$ a nbds $p \in U \subset M$, $V \subset_{open} \mathbb{R}^n$ & an diffeo $\varphi : U \to V$ s.t $g = e^{2u} \cdot \varphi^* g_{\mathbb{R}^n}$, for some $u \in C^\infty(U)$.

FACT: If $n \geq 4$, $W \equiv 0$ on $M \iff (M, g)$ is locally conf. flat

Cor 18.8: Let (M^n, g) be a Riemannian mfld w/ $n \geq 4$. Then $\exists$ a pointwise orth. decomp

$$Rm = W + \frac{1}{n-2} E \owedge g + \frac{S}{2(n-1)n} g \owedge g .$$

where $E := Ric - \frac{S}{n} g$, $g^{ij} W_{ij} = 0$.

Moreover

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|Ric|^2 - \frac{2}{(n-1)(n-2)} S^2.$$

Pf: $$|Rm|^2 = |W|^2 + \frac{|E \owedge g|^2}{(n-2)^2} + \frac{S^2}{4n^2(n-1)^2}|g \owedge g|^2.$$

Now $$|E \owedge g|^2 = \langle E \owedge g, E \owedge g \rangle$$

$$= 4\langle tr_g (E \owedge g), E \rangle$$

$$= 4\langle (n-2) E + tr_g E \cdot g, E \rangle \quad (***)$$

$$\overset{(tr_g E = 0)}{=} 4(n-2)|E|^2.$$

Also by (***) $$|g \owedge g|^2 = 4\langle (n-2) g + n g, g \rangle$$

$$= 8(n-1)n.$$

So $$|Rm|^2 = |W|^2 + \frac{4|E|^2}{n-2} + \frac{2S^2}{n(n-1)}$$

But $$|E|^2 = \left\langle Rc - \frac{S}{n} g, Rc - \frac{S}{n} g \right\rangle$$

$$= |Rc|^2 + \frac{S^2}{n} - \frac{2S}{n}\langle Rc, g \rangle$$

$$= |Rc|^2 - \frac{S^2}{n}$$

So $$|Rm|^2 = |W|^2 + \frac{4|Rc|^2}{n-2} - \left[\frac{4}{n(n-2)} - \frac{2}{n(n-1)} \right] S^2.$$

$$= |W|^2 + \frac{4|Rc|^2}{n-2} - \frac{2}{(n-1)(n-2)} S^2.$$