LECTURE - 19

• HESSIAN OF THE ENERGY.

• Recall : 1) Given an admissible curve $\gamma : [a,b] \to M$ $\in \Omega(p,q)$, the length & energy are defined as :

$$\mathcal{L}(\gamma) = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |\dot{\gamma}(t)|\, dt$$

$$\mathcal{E}(\gamma) = \frac{1}{2} \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |\dot{\gamma}(t)|^2\, dt$$

where $\gamma|_{[a_{k-1}, a_k]}$ is smooth.

2). If $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$ is an admissible variation, then for $\gamma_s := \Gamma(s, \cdot)$ & $V_s := \partial_s \Gamma(s, \cdot)$ we have

$$\frac{d}{ds} \mathcal{E}(\gamma_s) = - \int_a^b \langle \partial_s \Gamma, D_t \dot{\gamma}_s \rangle + \langle V_s(b), \dot{\gamma}_s(b-) \rangle$$

$$- \langle V_s(a), \dot{\gamma}_s(a+) \rangle - \sum_{k=1}^{m} \langle V_s(a_k), \Delta_k \dot{\gamma}_s \rangle$$

where $\Delta_k W := W(a_k+) - W(a_k-)$.

Def^n: An admissible 2-parameter variation of γ is a map $\Gamma : (-\varepsilon, \varepsilon)^2 \times [a,b] \longrightarrow M$ s.t.

1) $\exists$ a sub-division $a = a_0 < a_1 < \cdots < a_m = b$ s.t. Γ is smooth on $(-\varepsilon, \varepsilon)^2 \times [a_{k-1}, a_k]$ $\forall k$.

2). $\gamma_0(t) := \Gamma(0,0,t) = \gamma(t)$.

We set $\gamma_{s_1, s_2} := \Gamma(s_1, s_2, \cdot)$.

We denote $W_i(t) = \partial_{s_i} \Gamma(0,0,t)$ for $i = 1, 2$.

We say Γ is <u>proper</u> if $\Gamma(s_1, s_2, a) = \gamma(a)$ & $\Gamma(s_1, s_2, b) = \gamma(b)$ $\forall (s_1, s_2) \in (-\varepsilon, \varepsilon)^2$.

In particular $W_i(a) = 0$, $W_i(b) = 0$.

<u>Ex</u>: If W_1, W_2 are v.f along γ, then

$$\Gamma(s_1, s_2, t) := \exp_{\gamma(t)}(s_1 W_1(t) + s_2 W_2(t))$$

is an example of such a variation. If $W_i(a) = 0$, $W_i(b) = 0$, then Γ is proper.

<u>Thm 19.1</u> (Second variation formula). Let Γ be an admissible 2-parameter variation of a geodesic γ w/ variation vector fields $W_i(t)$, $i=1,2$. Then

$$\left.\frac{\partial^2}{\partial s_1 \partial s_2}\right|_{(s_1,s_2)=(0,0)} \mathcal{E}(\gamma_{s_1,s_2}) = \int_a^b \Big(\langle D_t W_1, D_t W_2 \rangle - Rm(W_1, \dot\gamma, \dot\gamma, W_2) \Big) dt$$

$$= -\sum_{k=1}^{n-1} \langle \Delta_k D_t W_1, W_2(a_k) \rangle$$

$$- \int_a^b \Big\langle \frac{D^2 W_1}{dt^2} + R(W_1, \dot\gamma)\dot\gamma, W_2 \Big\rangle$$

<u>Pf</u>: Let $E_k(s_1, s_2) = \mathcal{E}\left(\gamma_{s_1,s_2}\big|_{[a_{k-1}, a_k]}\right)$.

Then

$$\frac{\partial E_k}{\partial s_1} = \int_{a_{k-1}}^{a_k} \left\langle \frac{D}{\partial s_1} \partial_t \Gamma, \partial_t \Gamma \right\rangle$$

$$= \int_{a_{k-1}}^{a_k} \left\langle \frac{D}{\partial t} \partial_{s_1} \Gamma, \partial_t \Gamma \right\rangle \quad \text{(Symmetry lemma)}.$$

Differentiating again

$$\frac{\partial^2 E_k}{\partial s_1 \partial s_2} = \int_{a_{k-1}}^{a_k} \langle D_{s_2} D_t \partial_{s_1} \Gamma, \partial_t \Gamma \rangle + \int_{a_{k-1}}^{a_k} \langle D_t \partial_{s_1} \Gamma, D_t \partial_{s_1} \Gamma \rangle$$

For the first term, if we put $\partial_{s_i} \Gamma = S_i$, $\partial_t \Gamma = T$,

$$D_{s_2} D_t \partial_{s_1} \Gamma = D_t D_{s_2} \partial_{s_1} \Gamma + R(S_2, T) S_1.$$

So, $$\int_{a_{k-1}}^{a_k} \langle D_{s_2} D_t \partial_s \Gamma, \partial_t \Gamma \rangle = \int_{a_{k-1}}^{a_k} \langle D_t D_{s_2} S_1, T \rangle - \int Rm(S_2, T, T, S_1)$$

$$= \int_{a_{k-1}}^{a_k} \frac{\partial}{\partial t} \langle D_{s_2} S_1, T \rangle - \int_{a_{k-1}}^{a_k} \langle D_{s_2} S_1, D_t T \rangle - \int Rm(S_2, T, T, S_1)$$

$$= \langle D_{s_2} S_1(a_k -), T(a_k -) \rangle - \langle D_{s_2} S_1(a_{k-1}+), T(a_{k-1}+) \rangle - \int_{a_{k-1}}^{a_k} \langle D_{s_2} S_1, D_t T \rangle - \int_{a_{k-1}}^{a_k} Rm(S_2, T, T, S_1)$$

Plugging in $(s_1, s_2) = 0$, since γ is a geodesic.
$D_t T\big|_{(0,0)} = D_t \dot{\gamma} = 0$. Also, $S_1(0,0) = W_1$, $S_2(0,0) = W_2$.
So,

$$\frac{\partial^2 E}{\partial s_1 \partial s_2}\Big|_{(0,0)} = \Delta + \int_a^b \langle D_t W_1, D_t W_2 \rangle - Rm(W_1, \dot{\gamma}, \dot{\gamma}, W_2)$$

<u>Claim 1</u>: $\Delta \equiv 0$

<u>Pf</u>: Note $\Delta = \langle D_{s_2} S_1(b-), T(b-) \rangle - \langle D_{s_2} S_1(a+), T(a+) \rangle$

$$+ \sum_{k=1}^{n-1} \langle D_{s_2} S_1(a_k+), T(a_k+) \rangle - \langle D_{s_2} S_1(a_k-), T(a_k-) \rangle$$

Since $S_1(s_1, s_2, a) \equiv 0$ & $S_1(s_1, s_2, b) \equiv 0$. 1^{st} 2 terms are zero. Also at $t = a_k$, $D_{s_2} S_1 = D_{s_2} \partial_{s_1} \Gamma$ depends only on values of Γ on $t = a_k$. So $D_{s_2} S_1$ is cont. $\forall$ (s_1, s_2, t). Also at $(s_1, s_2) = (0,0)$ $T = \dot{\gamma}$. Since γ is an unbroken geodesic, $\dot{\gamma}(a_k+) = \dot{\gamma}(a_k-)$. So, $\Delta \equiv 0$.

For the 2^{nd} formula, we have

<u>Claim 2</u>: $$\int_a^b \langle D_t W_1, D_t W_2 \rangle = - \int \left\langle \frac{D^2 W_1}{dt^2}, W_2 \right\rangle - \sum_{k=1}^{n} \langle \Delta_k D_t W_1, W_2(a_k) \rangle.$$

Pf: $$\int_{a_{k-1}}^{a_k} \langle D_t W_1, D_t W_2 \rangle = \int_{a_{k-1}}^{a_k} \frac{\partial}{\partial t} \langle D_t W_1, W_2 \rangle - \int_{a_{k-1}}^{a_k} \left\langle \frac{D^2 W_1}{dt^2}, W_2 \right\rangle.$$

$$= \langle D_t W_1(a_k -), W_2(a_k -) \rangle - \langle D_t W_1(a_{k-1}+), W_2(a_k+) \rangle - \int_{a_{k-1}}^{a_k} \left\langle \frac{D^2 W_1}{\partial t^2}, W_2 \right\rangle.$$

Again $\partial_{s_2} \Gamma$ is cont. at $t = a_k$. So $W_2(a_k+) = W_2(a_k^-)$

So adding up, we get the desired result.

Prop 19.2) If γ is a unit speed geodesic & Γ is a 2-parameter admissible prop. variation, then

$$\left. \frac{\partial^2 L(\gamma_{s_1, s_2})}{\partial s_1 \partial s_2} \right|_{(0,0)} = \int_a^b \langle D_t W_1, D_t W_2 \rangle - \langle D_t W_1, \dot{\gamma} \rangle \langle D_t W_2, \dot{\gamma} \rangle - Rm(W_1, \dot{\gamma}, \dot{\gamma}, W_2)).$$

$$= \int_a^b \langle D_t W_1^\perp, D_t W_2^\perp \rangle - Rm(W_1^\perp, \dot{\gamma}, \dot{\gamma}, W_2^\perp).$$

where $W_i^\perp$ is the component of W_i in $N\gamma(t)$

Pf: We only prove the final equality. Writing $W = W^T + W^\perp$ we have $W^T = \langle W, \dot{\gamma} \rangle \dot{\gamma}$.

So $D_t W^T = \langle D_t W, \dot{\gamma} \rangle \dot{\gamma} = (D_t W)^T$

$D_t W^\perp = (D_t W)^\perp$.

So $\langle D_t W_1, D_t W_2 \rangle = \langle (D_t W_1)^T, (D_t W_2)^T \rangle$

$$+ \langle D_t W_1^\perp, D_t W_2^\perp \rangle.$$

$$= \langle D_t W_1, \dot{\gamma} \rangle \langle D_t W_2, \dot{\gamma} \rangle$$

$$+ \langle D_t W_1^\perp, D_t W_2^\perp \rangle.$$

Also since $Rm(\dot{\gamma}, \dot{\gamma}, \cdot, \cdot) = Rm(\cdot, \cdot, \dot{\gamma}, \dot{\gamma})$, and $DW_i^T \parallel \dot{\gamma}$, we have $Rm(W_1, \dot{\gamma}, \dot{\gamma}, W_2) = Rm(W_1^\perp, \dot{\gamma}, \dot{\gamma}, W_2^\perp)$

<u>Def^n</u>: 1) We define the "<u>tangent sp</u>" to $\Omega(p,q)$ at $\gamma \in \Omega(p,q)$ by

$$T_\gamma \Omega(p,q) := \{ W \in \mathcal{J}(\gamma) \mid W(a) = 0, \ W(b) = 0 \}$$

2) The <u>Hessian</u> of $\mathcal{E} : \Omega(p,q) \longrightarrow \mathbb{R}$ at a geodesic γ is defined by $\mathrm{Hess}_\gamma \mathcal{E} : T_\gamma \Omega \times T_\gamma \Omega \longrightarrow \mathbb{R}$.

$$\mathrm{Hess}_\gamma \mathcal{E}(W_1, W_2) := \int_a^b \langle D_t W_1, D_t W_2 \rangle - Rm(W_1, \dot{\gamma}, \dot{\gamma}, W_2)$$

$$\shortparallel$$

$$I_\gamma(W_1, W_2) \leftarrow \text{index form}.$$

<u>Prop 19.3</u> I_γ is a symmetric, bilinear form on $T_\gamma \Omega(p,q)$

<u>Cor 19.4</u>: Let γ be a minimal geodesic between p and q. Then $I_\gamma(W, W) \geq 0 \ \forall \ W \in T_\gamma \Omega$

<u>Pf</u>: Let Γ be any variation w/ $\partial_s \Gamma(0,t) = W(t)$ (e.g $\Gamma(s,t) = \exp_{\gamma(t)} s W(t)$.

Let $\gamma_s(t) = \Gamma(s, \cdot)$. Then Cauchy Schwartz & γ being minimal $\Rightarrow$

$$\mathcal{E}(\gamma_s) \geq \frac{1}{2(b-a)} L(\gamma_s)^2 \geq \frac{1}{2(b-a)} L(\gamma)^2 = \mathcal{E}(\gamma).$$

So $\mathcal{E}(\Gamma(s, \cdot))$ has a minimum at $s = 0$. So.

$$0 \leq \left.\frac{d^2}{ds^2}\right|_{s=0} \mathcal{E}(\gamma_s) = I_\gamma(W, W).$$

· JACOBI EQUATION:

Def^n: The null space of a geodesic γ is defined by

$$\text{Null}_\gamma := \{ W \in T_\gamma \Omega \mid I_\gamma(W, W') = 0 \ \forall W' \}$$

Prop 19.5: $J \in \text{Null}_\gamma$ if and only if J is smooth & satisfies

$$\frac{D^2}{dt^2} J + R(J, \dot\gamma)\dot\gamma = 0. \quad (*)$$

Pf: $\Leftarrow$ by the second variation formula, since J is smooth (& hence $\Delta_k D_t J = 0$) we have $J \in \text{Null}_\gamma$.

$\Rightarrow$: Conversely sp $J \in \text{Null}_\gamma$, then choose a partition $a = a_0 < \cdots < a_m = b$ s.t $J|_{[a_{k-1}, a_k]}$ is C^∞. Let $\varphi : [a,b] \to [0,1]$ smooth s.t

(1) $\varphi(a_k) = 0 \ \forall k$

(2) $\varphi > 0$ otherwise.

and let

$$W := \begin{cases} \varphi(t)\left(\dfrac{D^2 J}{dt^2} + R(J, \dot{r})\dot{r}, \cancel{T}\right) & , t \neq a_k . \\ 0 & , \text{otherwise}. \end{cases}$$

Then $0 = - I_r(J, W) = \displaystyle\int_a^b \varphi(t) \left| \frac{D^2 J}{dt^2} + R_{\cancel{a}}(J, \dot{r})\dot{r}, \cancel{J} \right|^2$

So $\dfrac{D^2 J}{dt^2} + R_{\cancel{a}}(J, \dot{r})\dot{r}, \cancel{T} = 0$ for $t \neq a_k$.

Claim: J is smooth.

Pf: Now let $W' \in T_r\Omega$ s.t.

$$W'(a_k) = \Delta_k \frac{DJ}{dt}$$

Then

$$0 = - I_r(J, W') = \sum_k \left\| \Delta_k \frac{DJ}{dt} \right\|^2$$

So $\Delta_k \dfrac{DJ}{dt} = 0 \ \forall k$ i.e. $\dfrac{DJ}{dt}$ is cont at a_k.

From Th^m 20.1 (to be proved next time) any solution to $*$ on $[a_k, a_{k+1}]$ is completely determined by $J(a_k)$ & $DJ/dt\,(a_k)$. Hence J is smooth at a_k.

Def^n A

Defⁿ: A vector field J along γ is called a Jacobi field if

$$\frac{D^2 J}{dt^2} + R(J, \dot{\gamma})\,\dot{\gamma} = 0 .$$

Rk: We do not impose the restriction that $J(a) = 0$ & $J(b) = 0$. So $J \notin T_\gamma \Omega$. Prop can be reformulated as

$$J \in \mathrm{Null}_\gamma \iff J \text{ is a Jacobi field w/ } J(a) = 0,\ J(b) = 0 .$$