Lecture - 20

• Last Lecture : 1) If Γ is a 2-parameter proper var. of a geodesic $\gamma : [a,b] \longrightarrow M$, then

$$\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{(0,0)} \mathcal{E}(\gamma_{s_1,s_2}) = I_\gamma(W_1, W_2) := \int_a^b \Big(\langle D_t W_1, D_t W_2 \rangle - Rm(W_1, \dot\gamma, \dot\gamma, W_2)\Big) dt$$

$$= - \sum_{k=1}^{m-1} \langle \Delta_k D_t W_1, W_2(a_k) \rangle - \int_a^b \Big\langle \frac{D^2 W_1}{dt^2} + R(W_1, \dot\gamma)\dot\gamma , W_2 \Big\rangle dt$$

where
- $W_i = \partial_{s_i} \Gamma(0,0,t)$
- $a = a_0 < \cdots < a_m = b$, $\Gamma\big|_{(-\varepsilon,\varepsilon)^2 \times [a_{k-1}, a_k]} \in C^\infty$.
- $\gamma_{s_1,s_2}(\cdot) := \Gamma(s_1, s_2, \cdot)$.

2) $J \in \mathcal{T}(\gamma)$ is called a Jacobi field if

$$\frac{D^2 J}{dt^2} + R(J, \dot\gamma)\dot\gamma = 0 \quad \forall\, t \in [a,b].$$

3) $J \in Null_\gamma := \{ W \in T_\gamma \Omega(p,q) \mid I_\gamma(W,V) = 0 \ \forall\, V \}$.

$\Longleftrightarrow$ J is a Jacobi field s.t. $J(a) = 0$, $J(b) = 0$.

• Existence and Uniqueness of Jacobi Fields:

Th^m 20.1 Let $\gamma : I \to M$ be a geodesic, $a \in I$. & $p = \gamma(a)$. For any pair $X, Y \in T_pM$, $\exists\,!$ Jacobi field $J \in \mathcal{T}(\gamma)$ s.t

$$J(a) = X, \quad D_t J(a) = Y.$$

Pf: Let $\{E_i\}$ be an o.n.b for T_pM, and let $E_i(t)$ be the $\parallel$-extensions. Then $\{E_i(t)\}$ is an o.n.b for $T_{\gamma(t)}M$ $\forall t$. Writing $J(t) = J^i(t)E_i$, the Jacobi equation is

$$\begin{cases} \ddot{J}^i + R_{jkl}{}^i J^j \dot{\gamma}^k \dot{\gamma}^l = 0 \\ J^i(a) = X^i, \quad \dot{J}^i(a) = Y^i \end{cases}$$

Putting $\dot{J}^i \equiv V^i$, this can be converted into a linear system of 1^{st} order ODE w/ $2n$ unknown.

Hence $\exists$ a C^∞ sol^n on all of I.

Cor 20.2: Denote by

$$\mathcal{J}(\gamma) = \{J \in \mathcal{T}(\gamma) \mid J \text{ is a Jacobi field}\}.$$

where $\gamma: I \to M$ is a geodesic. Then $\mathcal{J}(\gamma)$ is a $2n$-dimensional real vector space.

Rk Trivial Jacobi fields: Given a geodesic $\gamma: [a,b] \to M$

i) $J_0(t) := \dot{\gamma}(t)$ satisfies

$$\frac{D^2 J_0}{dt^2} + R(J_0, \dot{\gamma}, \dot{\gamma}) = D_t(D_t \dot{\gamma}) + R(\dot{\gamma}, \dot{\gamma})\dot{\gamma}$$

$$= 0.$$

So J_0 is a Jacobi field. It is the unique one s.t

$$J_0(a) = \dot{\gamma}(a), \quad D_t J_0(a) = 0.$$

2) $J_1(t) = (t-a)\dot{r}(t)$ satisfies

$$\frac{D^2}{dt^2} J_1 + R(J_1, \dot{r})\dot{r} = D_t(\dot{r} + (t-a) D_t \dot{r}) + (t-a) R(\dot{r}, \dot{r})\dot{r}$$
$$= 0.$$

This is the unique Jacobi field s.t.

$$J_1(a) = 0, \; D_t J_1(a) = \dot{r}(a).$$

NOTATION From now on we assume $a = 0$, $b = 1$.

Def^n: A tangential vector field along r is a v.f s.t $V(t) = \lambda(t)\dot{r}(t)$ $\forall t$, where $\lambda(t) : [0,1] \to \mathbb{R}$.
We say it is a normal v.f if $V(t) \perp \dot{r}(t)$ $\forall t$.
We denote the space of tang. & normal vector fields (resp. Jacobi fields) by $\mathcal{J}^T(r)$ & $\mathcal{J}^\perp(r)$ (resp. $\mathcal{J}^T$ and $\mathcal{J}^\perp$).

Prop 20.3: Let $r : I \to M$ be a geodesic w/ $0 \in I$.
(a) Then $J \in \mathcal{J}^\perp(r) \iff J(0), D_t J(0) \perp \dot{r}(0)$.
(b) $\mathcal{J}^\perp(r)$ is a linear sub-sp. of $\mathcal{J}(r)$ of dim $= 2n-2$.
(c) $J \in \mathcal{J}^\perp(r) \iff \exists\, a, b \in I$ s.t. $J(a) \perp \dot{r}(a)$ & $J(b) \perp \dot{r}(b)$.
(d) Any $J \in \mathcal{J}(r)$ has a unique decomposition $J = J^T + J^\perp$ s.t $J^T \in \mathcal{J}^T(r)$ & $J^\perp \in \mathcal{J}^\perp(r)$.

Pf: Let $f(t) = \langle J(t), \dot{r}(t) \rangle$. Then

$$f'(t) = \langle D_t J, \dot{r} \rangle.$$

$$f''(t) = \langle D_t^2 J, \dot{r} \rangle$$
$$= -\langle R(J, \dot{r})\dot{r}, \dot{r} \rangle$$
$$= -Rm(J, \dot{r}, \dot{r}, \dot{r}) = 0.$$

So $f(t)$ is a linear function. So $f \equiv 0$ $\Longleftrightarrow$ $f(0) = \langle J(0), \dot{r}(0) \rangle$ & $f'(0) = \langle D_t J(0), \dot{r}(0) \rangle = 0$. This proves (a). Similarly $f \equiv 0 \Longleftrightarrow$ $f(a) = f(b) = 0$ for some $a, b \in I$, and hence (c) follows. (b) & (d) are then elementary consequences.

- VARIATIONS THROUGH GEODESICS:

Prop 20.4. Let $r : [0,1] \longrightarrow M$ be a geodesic. Then $J \in \mathcal{J}(r) \Longleftrightarrow \exists$ a smooth variation $\Gamma(s,t)$ of $r(t)$ s.t $\forall s$, $r_s(\cdot) := \Gamma(s, \cdot)$ is a geodesic & $J(t) = \partial_s \Gamma(0,t)$.

Pf: $\Leftarrow$ Let $S = \partial_s \Gamma$, $T = \partial_t \Gamma$. Since r_s is a geodesic $\forall s$, $D_t T \equiv 0$. Then

$$0 = D_s D_t T = D_t D_s T + R(S,T)T$$
$$= D_t D_t S + R(S,T)T \quad (\because D_s \partial_t \Gamma = D_t \partial_s \Gamma)$$

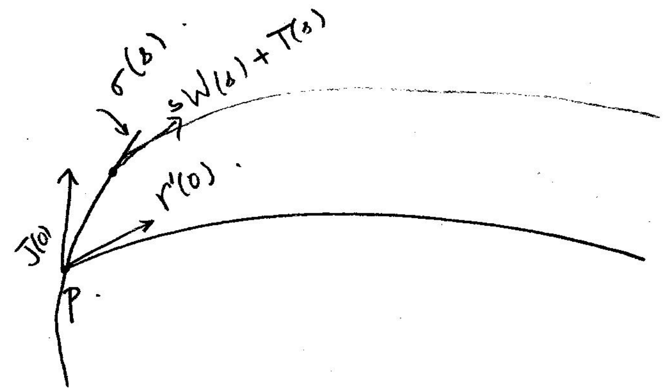
Evaluating at $s = 0$,

$$D_t^2 J + R(J, \dot{r})\dot{r} = 0$$

So $J \in \mathcal{J}(r)$.

$\Longrightarrow$ Conversely sps $J \in \mathcal{J}(\gamma)$. Let $\sigma : (-\varepsilon, \varepsilon) \to M$ s.t. $\sigma(0) = \gamma(0) = p$, and $\sigma'(0) = J'(0)$.

Let $X(s)$ & $W(s)$ be $\|$- transports of $\gamma'(0)$ & $D_t J(0)$ along $\sigma(s)$.



Consider $\Gamma(s,t) := \exp_{\sigma(s)} t\,(X(s) + sW(s))$.

Let $V(t) := \partial_s \Gamma(0,t)$.

Claim 1 $\quad V(0) = J(0)$.

Claim 2 $\quad D_t V(0) = D_t J(0)$.

Assuming these, note that $\Gamma(0,t) = \exp_p t\,\dot{\gamma}(0) = \gamma(t)$. and $\Gamma(s,\cdot)$ is a geodesic $\forall s$. So by first part $V \in \mathcal{J}(\gamma)$. But then by Claim 1 & 2 & uniqueness

$V(t) = J(t)$.

Pf of Claim 1: $\quad V(0) = \left.\dfrac{d}{ds}\right|_{s=0} \Gamma(s,0) = \left.\dfrac{d}{ds}\right|_{s=0} \sigma(s) = \sigma'(0)$

$= J(0)$.

Pf of Claim 2: By symmetry lemma

$$D_t \partial_s \Gamma = D_s \partial_t \Gamma.$$

At $t=0$, $\quad D_t \partial_s \Gamma(s,0) = D_s \left.\dfrac{d}{dt}\right|_{t=0} \Gamma(s,t)$.

$$= D_s\left(X(s) + s\,W(s)\right)$$

$$= W(s), \quad \text{since } D_s X = D_s W = 0.$$

So $D_t V(0) = D_t \partial_s \Gamma(0,0) = W(0) = D_t J(0).$

Done!

Cor 20.5 Let $p \in M$ & $\gamma(t) = \exp_p tv$, $0 \leq t \leq 1$ be a geodesic at p. Then for any $w \in T_pM$, the Jacobi field satisfying $J(0) = 0$ & $D_t J(0) = w$ is given by

$$J(t) = (d\exp_p)_{tv}(tw)$$

where we identify $T_{tv}T_pM \cong T_pM$.

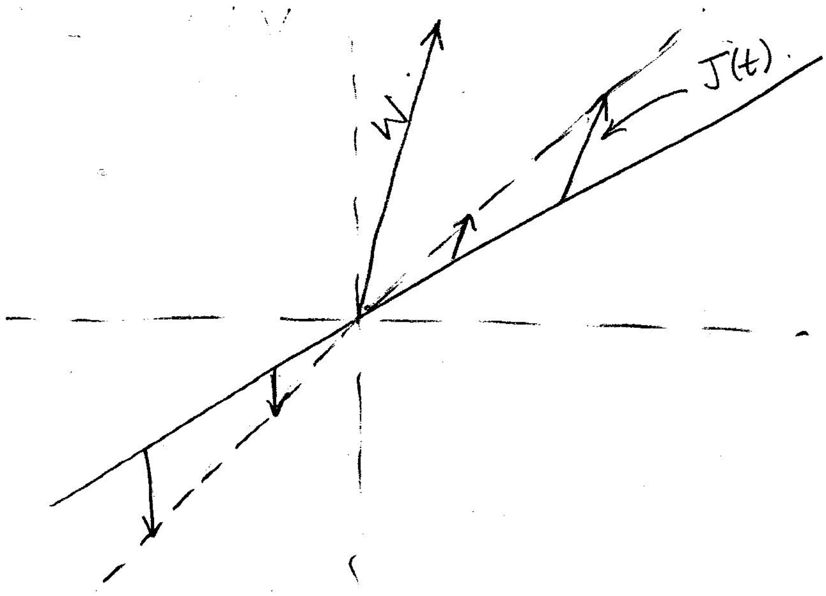
Pf: Consider $\Gamma(s,t) = \exp_p t(v + sw)$. Then Prop $\Rightarrow J(t) := \partial_s \Gamma(0,t)$ is a Jacobi field. $\Gamma(s,0) \equiv p$, so $J(0) = 0$. Also by above computation

$$D_t J(0) = w.$$ By definition for a fixed 't'

$$\partial_s \Gamma(0,t) = (d\exp_p)_{tv}(tw).$$

Cor 20.6: Let $p \in M$, and $\mathcal{U} \subset_o M$ be a normal neighbourhood w/ normal coordinates $\{x^i\}$. Let γ be a radial geodesic emanating from p. For any $W = W^i\partial_i \in T_pM$, the Jacobi field along γ s.t $J(0) = 0$ & $D_t J(0) = W$ is given by

$$J(t) = t\,W^i\partial_i \in T_{\gamma(t)}M \cong T_pM.$$



Pf: Sps $\gamma(t) = \exp_p tV$, where $V = V^i \partial_i$. Then

$$\Gamma(s,t) = \exp_p t(V + sW) = (t(V^1 + sW^1), \ldots, t(V^n + sW^n))$$

is a variation through geodesics. So $J(t) = \partial_s \Gamma(0,t)$ is a Jacobi field. But clearly

$$J(t) = t\, W^i \partial_i$$

Also, $J(0) = 0$, $D_t J(0) = W^i \partial_i = W$ since $\Gamma_{ij}^k(p) = 0$. So this is the Jacobi field we want.

• Local Characterization Of Space Forms.

Def^n: A Riemannian mfld (M,g) is called a space form if it has constant sectional curvature i.e $\exists\, K \in \mathbb{R}$ s.t $\forall\, p \in M$, $\forall$ 2-dim $\Pi_p \subset T_pM$, $\sec_g(\Pi_p) = K$. We write this as $\sec_g \equiv K$.

Prop 20.7: Let (M, g) be a space form with $\sec_g \equiv K$. Let

$$(\tilde{M}_K, \tilde{g}_K) = \begin{cases} (S^n_R, g_{S^n_R}) & \text{if } K = 1/R^2 > 0 \\ (\mathbb{R}^n, g_{\mathbb{R}^n}) & \text{if } K = 0 \\ (\mathbb{H}^n_R, g_{\mathbb{H}^n_R}) & \text{if } K = -1/R^2 < 0 \end{cases}$$

Then for any $p \in M$, $\tilde{p} \in \tilde{M}_K$, $\exists$ nbds $\mathcal{U}$ of p & $\tilde{\mathcal{U}}$ of $\tilde{p}$ & an isometry

$$\varphi : (\mathcal{U}, g) \longrightarrow (\tilde{\mathcal{U}}, \tilde{g}_K).$$

Lemma 20.8: Sps (M, g) is a sp. form w/ $\sec \equiv K$ & γ is a unit speed geodesic in M. The normal Jacobi fields along γ vanishing at $t = 0$ are precisely the vector fields

$$J(t) = u(t)\, E(t),$$

where $E(t)$ is <u>any</u> $\parallel$-normal v.f. along γ &

$$u(t) = \begin{cases} R \sin t/R, & K = 1/R^2 > 0 \\ t, & K = 0 \\ R \sinh t/R, & K = -1/R^2 < 0. \end{cases}$$

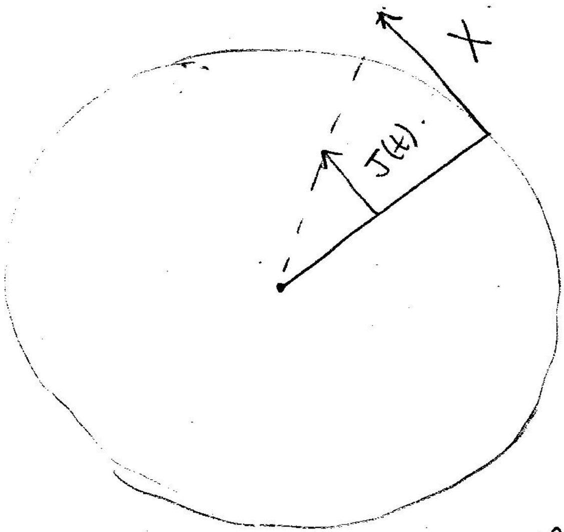
Lemma 20.9: Let (M, g) be a sp. form w/ $\sec \equiv K$. Let $\{x^i\}$ be normal coordinates at p on a nbd $\mathcal{U}$, and let $r := d(p, \cdot)$. $S_r := \{q \in \mathcal{U} \mid d(p, q) = r\}$. Then in polar coordinates $(r, \theta^1, \ldots, \theta^{n-1})$

$$g\big|_{\mathcal{U}} = dr^2 + A(r)\, g_{S^{n-1}}.$$

Pf: Gauss lemma $\Longrightarrow$ $g|_{\mathcal{U}} = dr^2 + h_r$, where h_r is the induced metric on S_r ($\because \frac{\partial}{\partial r} \perp S_r$)

Now, let $X \in T_q S_r$ & $\gamma = \exp_p(tv)$ s.t $\gamma(r) = q$.

Note: $|\dot{\gamma}(0)| = |v| = 1$



By Cor 20.6, the Jacobi field along γ s.t $J(0) = 0$ & $D_t J(0) = X/r$ is
$$J(t) = \frac{t}{r} X^i \partial_i .$$

Now $J \perp$ to γ at p & q. So $J \in \mathcal{J}^\perp(\gamma)$. Lemma: 20.8 $\Longrightarrow$ $J(t) = u(t) E(t)$ for some normal $E(t)$.

Then
$$D_t J(0) = \dot{u}(0) E(0) = E(0).$$

Hence, since $D_t E(t) = 0$ we have $|E(t)| = \text{const.}$, so
$$|X|^2 = |J(r)|^2 = |u(r)|^2 |E(r)|^2 = |u(r)|^2 |E(0)|^2$$
$$= |u(r)|^2 |D_t J(0)|^2$$

Since $|D_t J(0)| = X/r$ & since $g(p) = g_{\mathbb{R}^n}(p)$,
$$|X|^2 = \frac{|u(r)|^2}{r^2} |X|^2_{g_{\mathbb{R}^n}}$$

where $g_{S^{n-1}}$ is the standard metric on the round sphere. &

$$A(r) = \begin{cases} R^2 \sin^2\left(\frac{r}{R}\right), & K = 1/R^2 > 0. \\ r^2, & K = 0. \\ R^2 \sinh^2\left(\frac{r}{R}\right), & K = -1/R^2 < 0. \end{cases}$$

<u>Pf of Prop 20.7</u>: Immediate from Lemma 20.9 since in normal nbd, the metric has a unique formula.

<u>Pf of Lemma 20.8</u>: $\sec \equiv K \Rightarrow \forall\ X, Y, Z$

$$R(X, Y)Z = K(\langle Y, Z\rangle X - \langle X, Z\rangle Y).$$

So if J is normal Jacobi field, then since $\langle J, \dot{r}\rangle = 0$ & $|\dot{r}| = 1$,

$$0 = D_t^2 J + K(\langle \dot{r}, \dot{r}\rangle J - \langle \dot{r}, J\rangle \dot{r})$$

$$= D_t^2 J + KJ$$

If $\boxed{J(t) = u(t)E(t)}$ for some ∥- normal field then

$$u''(t) + Ku(t) = 0, \quad u(0) = 0$$

So u is given by a constant multiple of the st. forms (depending on K). The sp of such Jacobi fields has dim $= n-1$. But then if

$$\mathcal{J}_0^\perp(r) := \{J \in \mathcal{J}^\perp(r) \mid J(0) = 0\}$$

Clearly $\dim \mathcal{J}_0^\perp(r) = n-1$. So all must be of the above form.

Now $g_{\mathbb{R}^n} = dr^2 + r^2 g_{S^{n-1}}$, so

$$h_r(X, X) = \frac{|u(r)|^2}{r^2} \cdot r^2 g_{S^{n-1}}(X, X)$$

$$= |u(r)|^2 g_{S^{n-1}}(X, X).$$

Hence $h_r = A(r) g_{S^{n-1}}$, where $A(r) = |u(r)|^2$.